

Noncrossing partitions and intersections of shards

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Overview

The shard intersection order

Why shards?

Noncrossing partitions

Coxeter groups

A Coxeter group W is a group with a certain presentation:

Choose a finite generating set $S = \{s_1, \dots, s_n\}$.

For every $i < j$, choose an integer $m(i, j) \geq 2$.

Define:

$$W = \langle S \mid s_i^2 = 1, \forall i \text{ and } (s_i s_j)^{m(i,j)} = 1, \forall i < j \rangle.$$

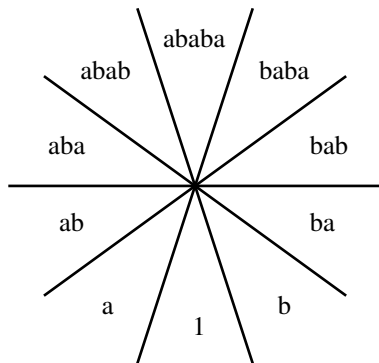
Finite Coxeter groups correspond to finite groups generated by reflections.

Not familiar with Coxeter groups? Two good classes of examples are the dihedral groups and the symmetric groups.

I'll use two running examples: the dihedral group $I_2(5)$ and the symmetric group S_4 .

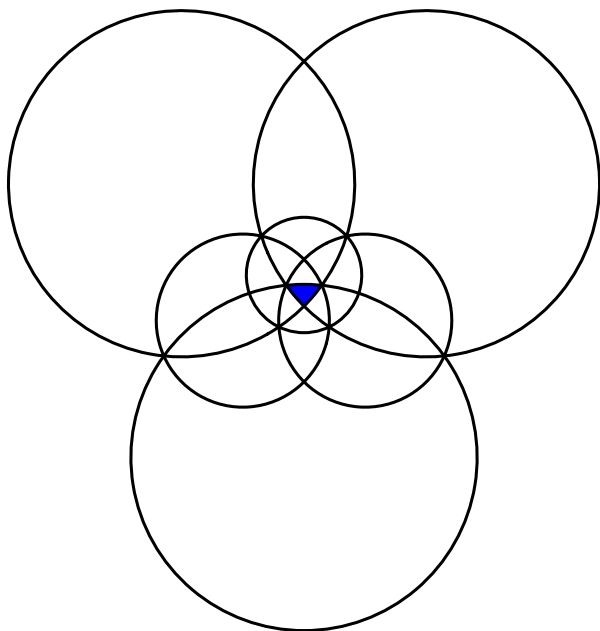
A dihedral example

The Coxeter group $I_2(5)$ given by $\langle \{a, b\} \mid a^2 = b^2 = (ab)^5 = 1 \rangle$ is the (dihedral) symmetry group of the regular pentagon.



Elements of the group are in bijection with “regions” cut out by the reflecting hyperplanes.

Symmetric group S_4 (symmetries of regular tetrahedron)



Regions $\leftrightarrow$
elements.

Blue region is 1.

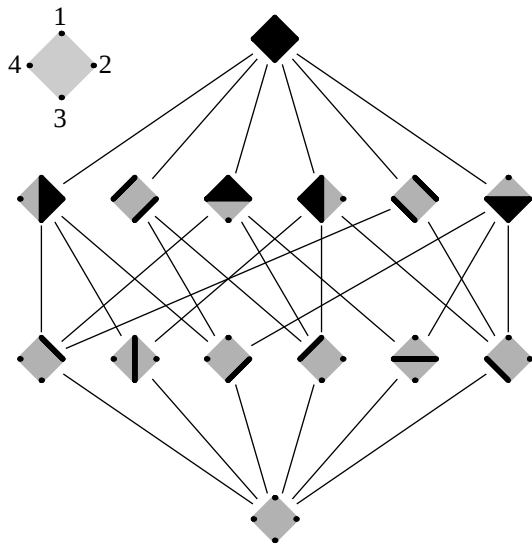
Largest circles:
hyperplanes for
 $s_1 = (1\ 2)$,
 $s_2 = (2\ 3)$, and
 $s_3 = (3\ 4)$.

$$m(s_1, s_2) = 3.$$

$$m(s_2, s_3) = 3.$$

$$m(s_1, s_3) = 2.$$

Noncrossing (NC) partitions (Kreweras, 1972)



Partitions of an n -cycle with noncrossing parts.

(Shown: $n = 4$, refinement order.)

NC partitions $\leftrightarrow$
certain elements of S_n .
Bijection: read parts
clockwise as cycles.

W -NC partitions (Athanasiadis, Bessis, Brady, Reiner, Watt, ~2000)

W : a finite Coxeter group with $S = \{s_1, \dots, s_k\}$ and reflections T .

Reduced T -word for $w \in W$: shortest possible word in alphabet T .

Absolute order: Prefix order on reduced T -words. Notation: $\leq_T$.

Coxeter element: $c = s_1 \cdots s_k$.

W -noncrossing partition lattice: elements of $[1, c]_T$.

Example ($W = S_n$, $c = (1\ 2)(2\ 3) \cdots (n-1\ n)$)

Reflections in S_n are (not-necessarily adjacent) transpositions.

S_n -noncrossing partitions map to classical noncrossing partitions.

(Interpret cycles as blocks.)

Why do this?

1. Eilenberg-MacLane spaces (and more) for Artin groups (e.g. the braid group).
2. Interesting algebraic combinatorics.

Motivation

NC(W) is a lattice.

First proved uniformly, Brady and Watt (2005).

Another proof (W crystallographic) Ingalls and Thomas (2006).

Initial motivation for the present work: A new proof that $\text{NC}(W)$ is a lattice, as follows: We construct a lattice $(W, \preceq)$ on the elements of W , and identify a sublattice of $(W, \preceq)$ isomorphic to $\text{NC}(W)$.

Beyond the initial motivation:

$(W, \preceq)$ turns out to have very interesting properties, very closely analogous to the properties of $\text{NC}(W)$.

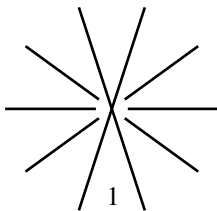
Proofs are simple and natural in the Coxeter context. (More broadly: in the context of simplicial hyperplane arrangements.)

This approach brings to light how $\text{NC}(W)$ arises naturally in the context of semi-invariants of quivers.

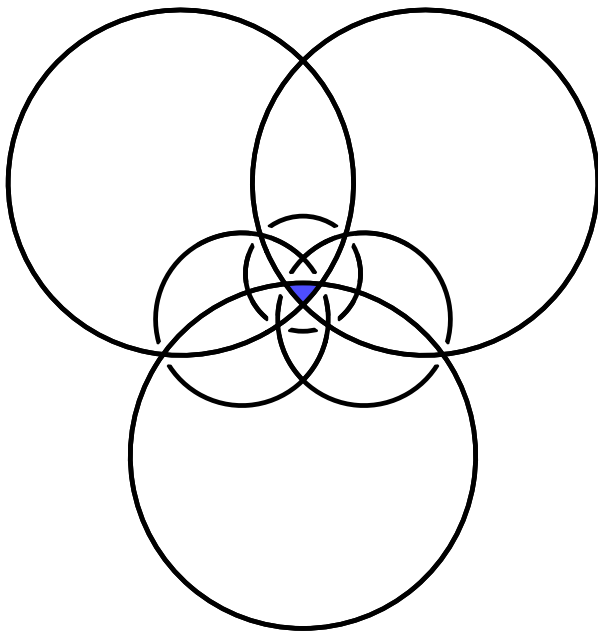
There are intriguing connections to certain “pulling” triangulations of associahedra and permutohedra.

Shards (an unmotivated definition)

Shards in a dihedral (or “rank 2”) Coxeter group: The two hyperplanes bounding the “identity region” are not cut. The remaining hyperplanes are cut in half.



Important technical point: all of the shards contain the origin. We “cut” along the intersection of the hyperplanes, then take closures of the pieces.



The shard intersection order

Let $\Psi(W)$ be the set of arbitrary intersections of shards. We partially order this set by reverse containment.

Immediate: $(\Psi(W), \supseteq)$ is a join semilattice. (Join is intersection.) It also has a unique minimal element (the empty intersection, i.e. the ambient vector space), so it is a lattice. Also immediate: $(\Psi(W), \supseteq)$ is atomic.

Less obvious: $(\Psi(W), \supseteq)$ is graded (ranked by codimension) and coatomic.

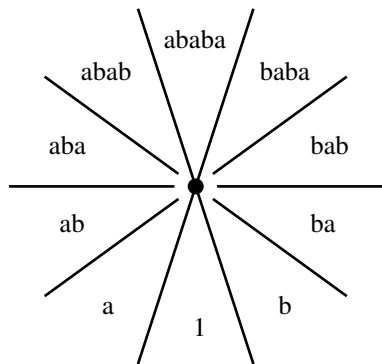
Surprising: The elements of $\Psi(W)$ are in bijection with the elements of W .

$$w \in W \longleftrightarrow \text{a region } R \longleftrightarrow \bigcap \{\text{shards below } R\}$$

In particular, $(\Psi(W), \supseteq)$ induces a partial order $\preceq$ on W .

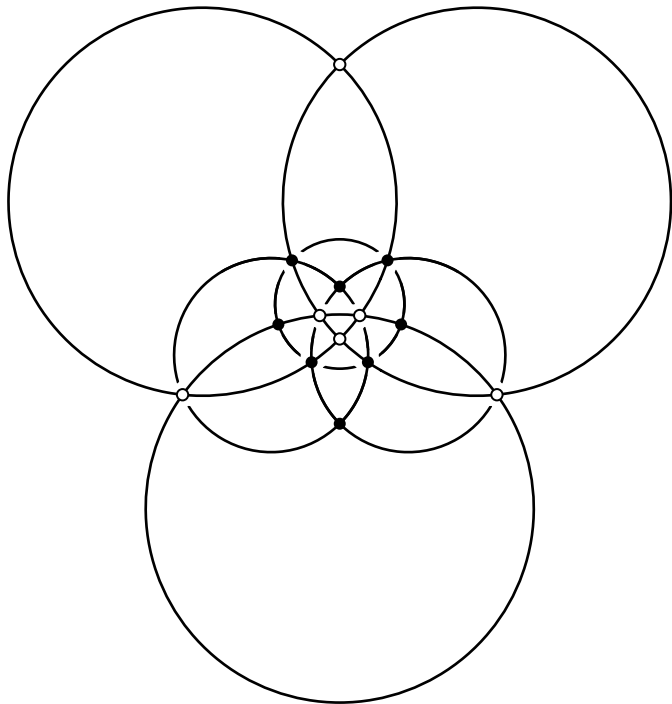
Also surprising: Every lower interval in $(\Psi(W), \supseteq)$ is isomorphic to $(\Psi(W_J), \supseteq)$ for some standard parabolic subgroup W_J .

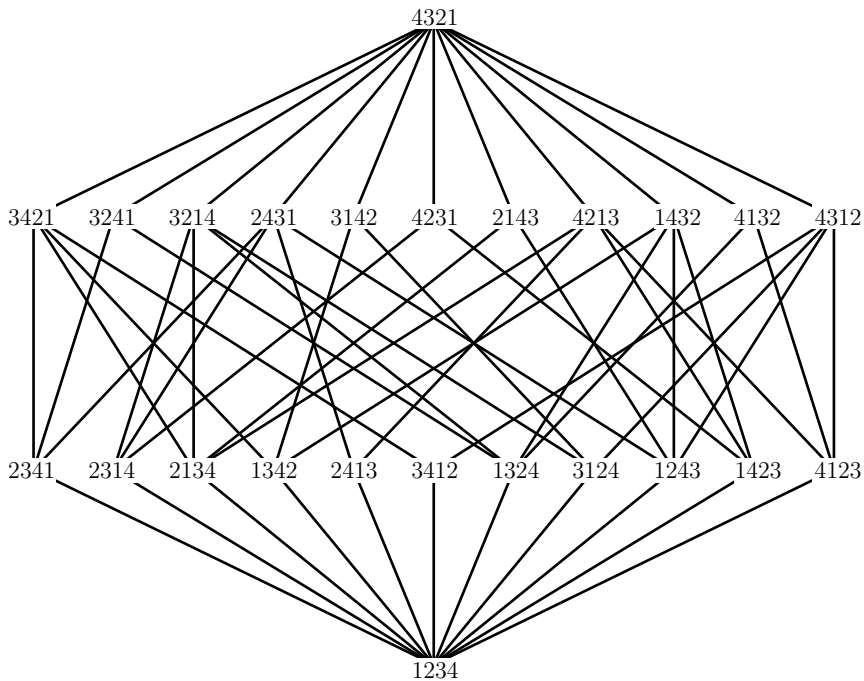
Shard intersections in $I_2(5)$



The poset $(\Psi(I_2(5)), \supseteq)$ has $\mathbb{R}^2$ as its unique minimal element and the origin as its unique maximal element. The 8 (1-dimensional) shards are pairwise incomparable under containment, and live at rank 1 (i.e. codimension 1).

The poset $(I_2(5), \preceq)$ has 1 as its unique minimal element and *ababa* as its unique maximal element. The other 8 elements of W are pairwise incomparable and live at rank 1.





Properties of $(W, \preceq)$ and $\text{NC}(W)$

(Except as noted: $(W, \preceq)$ results are new; $\text{NC}(W)$ results are not.)

$(W, \preceq)$	$\text{NC}(W)$
Lattice	Lattice (sublattice of $(W, \preceq)$)
Weaker than weak order	Weaker than Cambrian lattice (R., 2008 or modern folklore.)
Atomic and coatomic	Atomic and coatomic
Graded (W -Eulerian numbers)	Graded (W -Narayana)
Not self-dual	Self-dual
Lower intervals $\cong (W_J, \preceq)$	Lower intervals $\cong \text{NC}(W_J)$
Möbius number: $\pm$ number of “positive” elements of W .	Möbius number: $\pm$ number of “positive” elements of $\text{NC}(W)$.

Details on the Möbius number

Theorem

The Möbius function of $(W, \preceq)$ satisfies

$$\mu(1, w_0) = \sum_{J \subseteq S} (-1)^{|J|} |W_J|.$$

Proof.

Since lower intervals $[1, w]$ are isomorphic to $(W_{\text{Des}(w)}, \preceq)$, checking the defining recursion for μ becomes

$$\sum_{w \in W} \sum_{J \subseteq \text{Des}(w)} (-1)^{|J|} |W_J| = \sum_{J \subseteq S} (-1)^{|J|} |W_J| \sum_{\substack{w \in W \text{ s.t.} \\ J \subseteq \text{Des}(w)}} 1.$$

The inner sum is $|W|/|W_J|$, the number of maximal-length representatives of cosets of W_J in W . Thus the double sum reduces to zero.



Properties of $(W, \preceq)$ and $\text{NC}(W)$ (continued)

$(W, \preceq)$	$\text{NC}(W)$
Recursion counting maximal chains: sum over max'l proper standard parabolic subgroups. $\text{MC}(W) =$ $\sum_{s \in S} \left(\frac{ W }{ W_{\langle s \rangle} } - 1 \right) \text{MC}(W_{\langle s \rangle})$	Recursion counting maximal chains: sum over max'l proper standard parabolic subgroups. (R., 2007.) $\text{MC}(W) = \frac{h}{2} \sum_{s \in S} \text{MC}(W_{\langle s \rangle}).$

These types of recursions are very natural in the context of Coxeter groups/root systems. For example:

1. Recursions for the W -Catalan number (number of W -noncrossing partitions, clusters in the associated root system, W -nonnesting partitions, etc.)
2. Volume of W -permutohedron (weight polytope). This follows from Postnikov's formula in terms of Φ -trees.

Properties of $(W, \preceq)$ and $\text{NC}(W)$ (concluded)

$(W, \preceq)$	$\text{NC}(W)$
Maximal chains $\longleftrightarrow$ maximal simplices in a pulling triangulation of the W -permutohedron. (S_n case: Loday described the triangulation, 2005.)	Maximal chains $\longleftrightarrow$ maximal simplices in a pulling triangulation of the W -associahedron. (S_n case: Loday, 2005.) (General case: R., 2008.)
k -Chains $\longleftrightarrow$ k -simplices in the same triangulation of the W -permutohedron.	k -Chains $\longleftrightarrow$ k -simplices in the same triangulation of the W -associahedron. (R., 2008.)

Loday: Noticed that maximal simplices in a certain pulling triangulation of the S_n -associahedron biject with parking functions. Constructed the analogous triangulation of the S_n -permutohedron and asked what played the role of parking functions.

Details on the triangulations

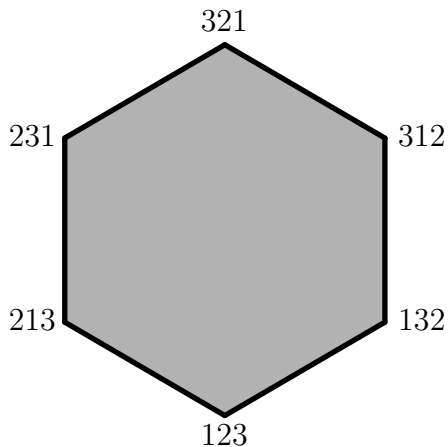
The bijection between intersections of shards and elements of W extends to a bijection between k -chains in $(W, \preceq)$ and k -simplices in a pulling triangulation of the W -permutohedron.

In particular: The order complex of $(W, \preceq)$ has f -vector equal to the f -vector of a pulling triangulation of the W -permutohedron.

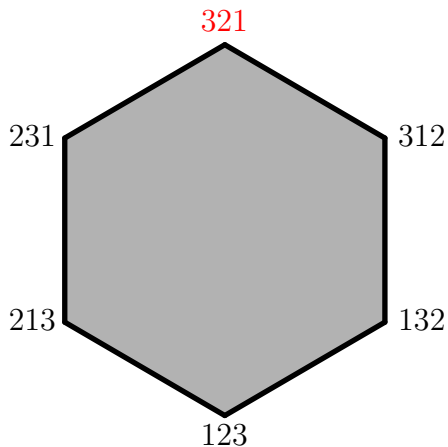
Key point: For any $w \in W$, the lower interval $[1, w]$ in $(W, \preceq)$ is isomorphic to $(W_J, \preceq)$ for some W_J . The elements of W_J are in bijection with vertices of the face below w in the permutohedron.

All of this works for $\text{NC}(W)$ and the W -associahedron as well. Maximal chains in $\text{NC}(S_n)$ are in bijection with parking functions, so we recover the Loday result as a special case.

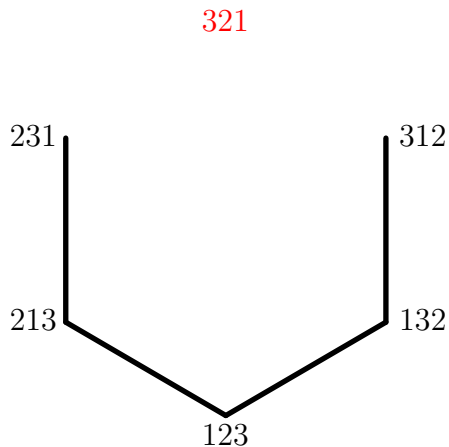
S_3 Permutohedron example



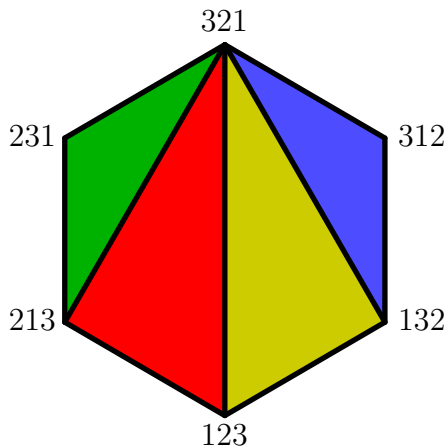
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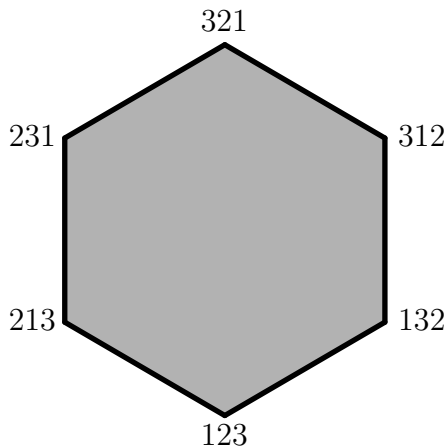
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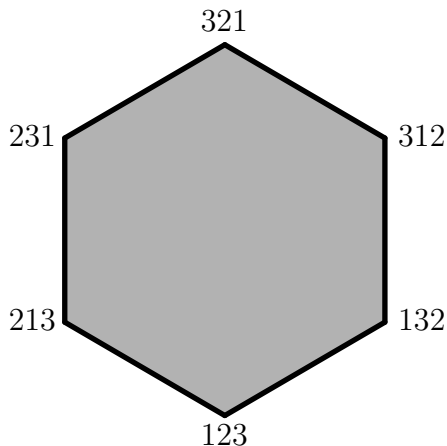
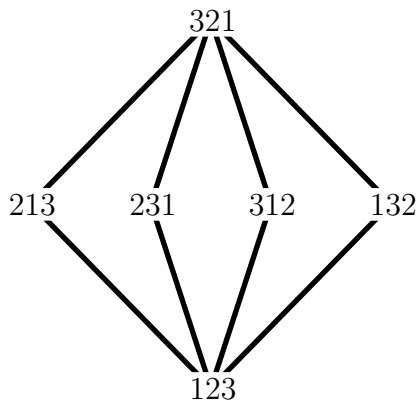
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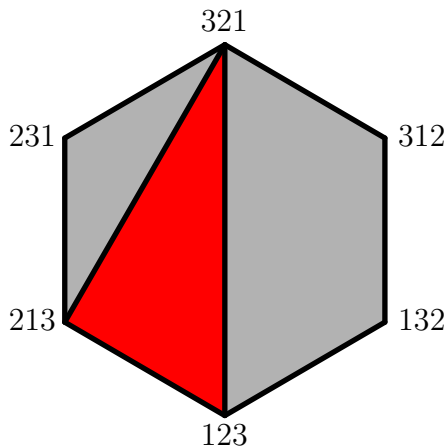
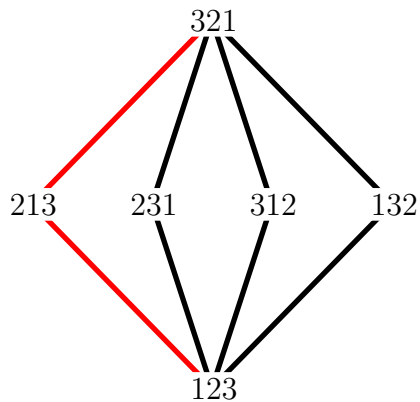
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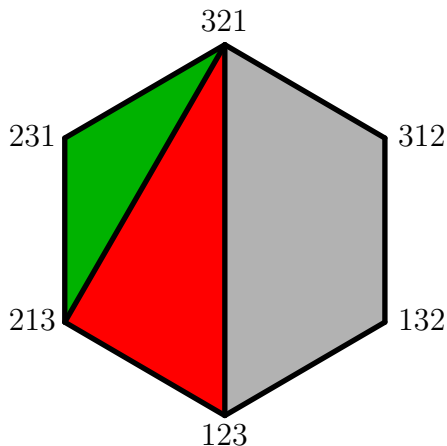
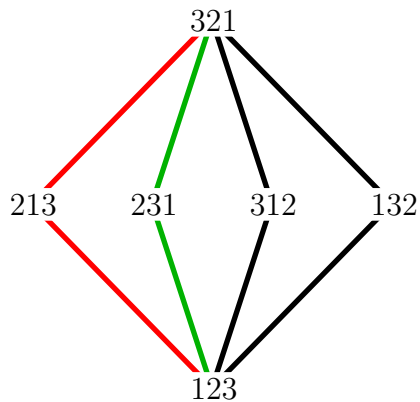
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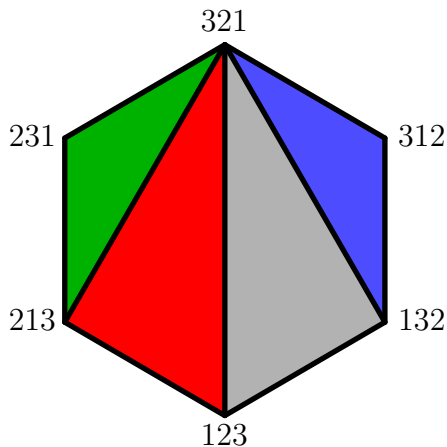
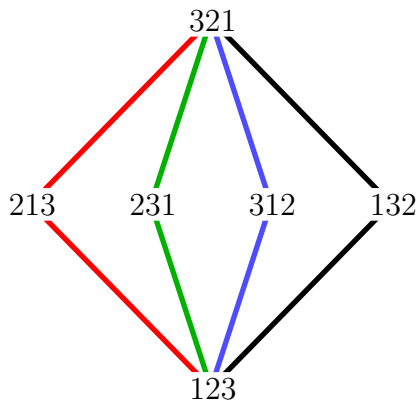
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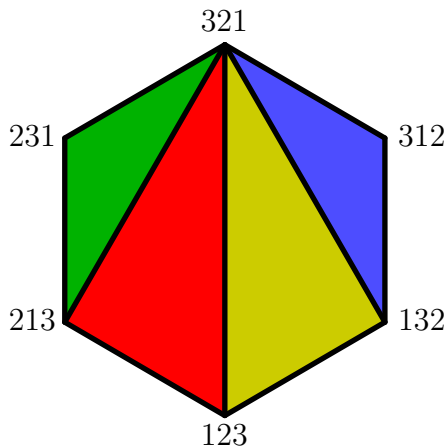
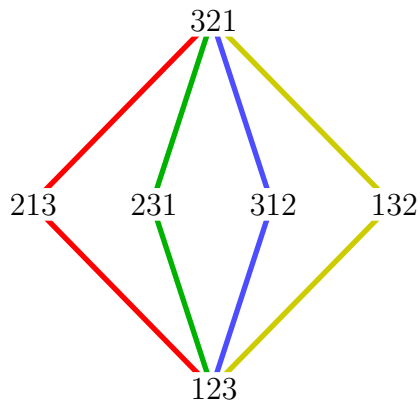
S_3 Permutohedron example



S_3 Permutohedron example



S_3 Permutohedron example



Proofs of properties of $(W, \preceq)$

The key point in the proofs is the bijection between shard intersections $\Psi(W)$ and elements of W .

All of the arguments use fairly simple tools, including:

- A characterization of “canonical join representations” in the weak order (R., Speyer, 2008).

- Lemmas on shards proved in my earlier papers on the lattice theory of the weak order. (These involve simple geometric and lattice-theoretic arguments.)

- New lemmas in the same spirit.

Why shards?

Before the shard intersection order, the original purpose for shards:

Shards encode lattice congruences of the weak order on W .

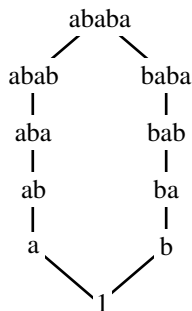
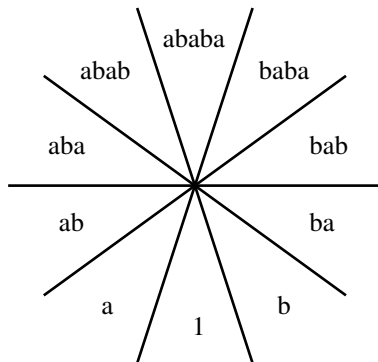
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The weak order on $I_2(5)$: Bottom element: the identity element.

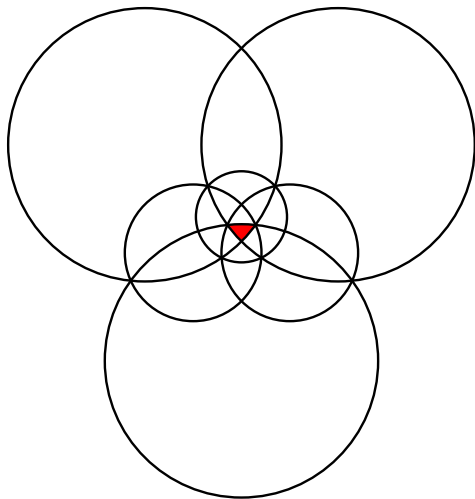
Going up: Crossing reflecting hyperplanes away from the identity.



The weak order on S_4

Bottom element: the identity element.

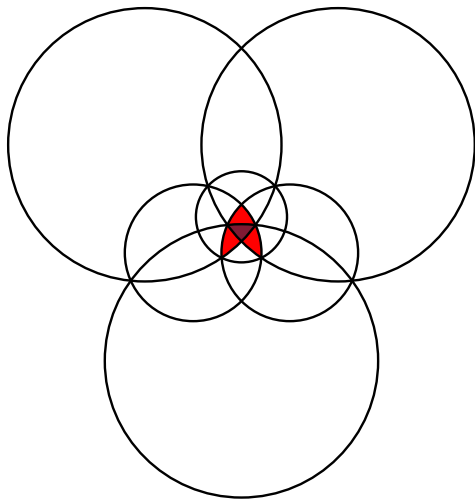
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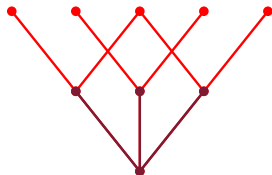
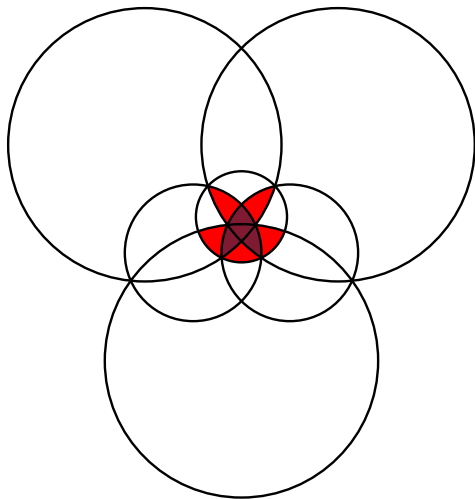
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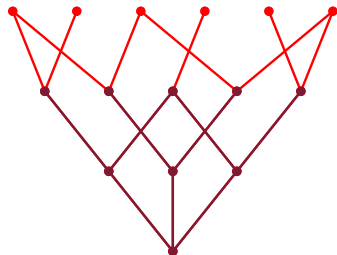
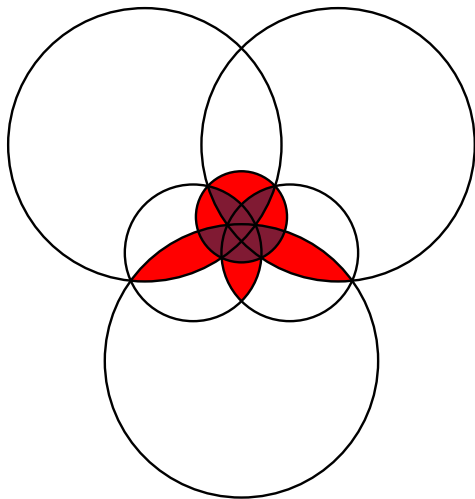
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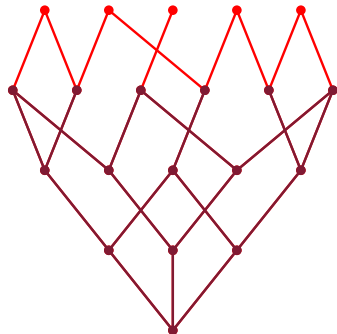
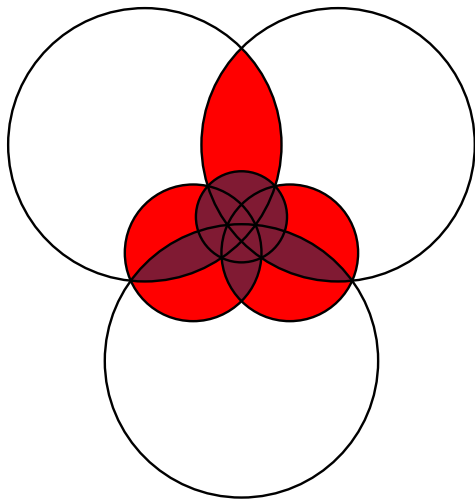
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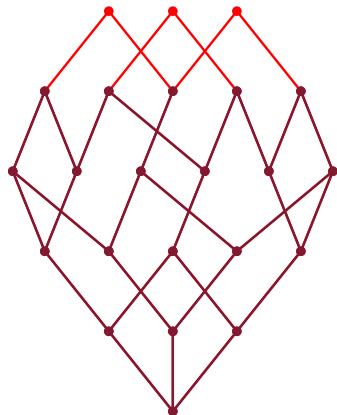
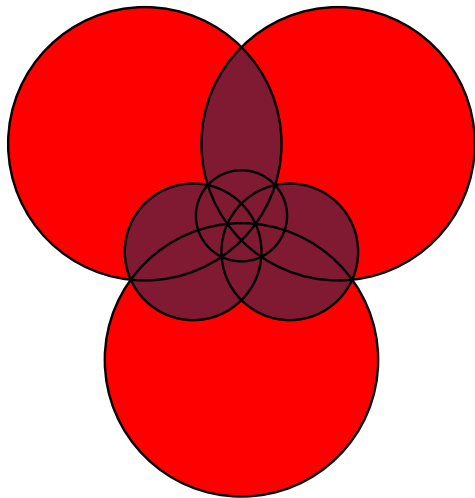
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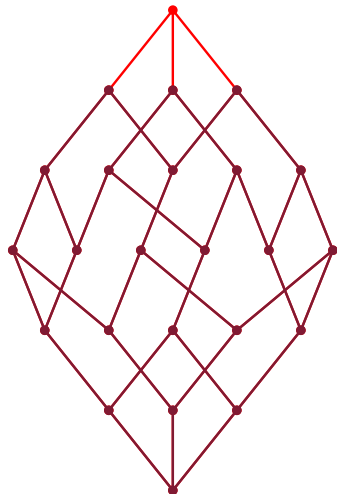
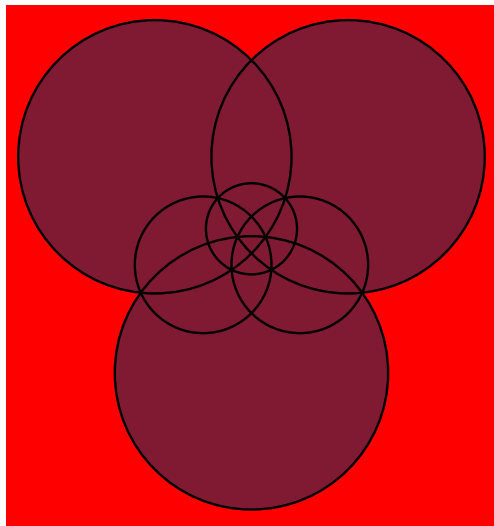
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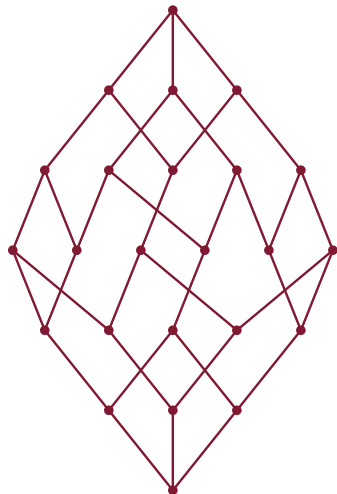
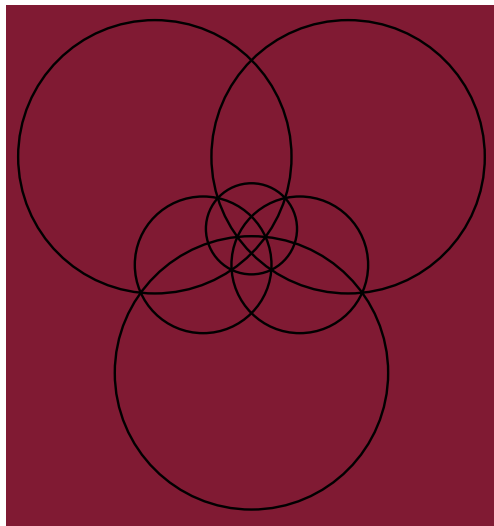
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The weak order on S_4

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Going up: Crossing reflecting hyperplanes away from the identity.



Lattice congruences 101

A congruence on a finite lattice L is an equivalence relation $\equiv$ that respects the operations $\vee$ (least upper bound) and $\wedge$ (greatest lower bound).

Easy: congruence classes are intervals in L .

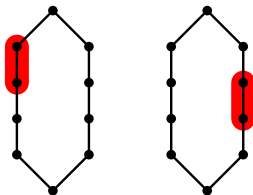
Therefore: The relation $\equiv$ is determined by transitivity, once one knows all equivalences of the form $x \equiv y$ for $x \leq y$.

We say that $\equiv$ **contracts** the edge $x \leq y$ if $x \equiv y$.

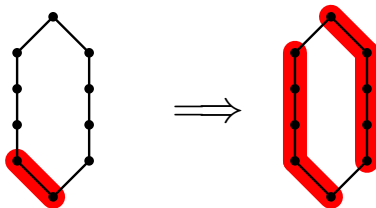
Edges cannot be contracted independently. There are some **forcing relations**.

Edge-forcing example: “Dihedral” lattices

A “side” edge can be contracted independently. E.g.:



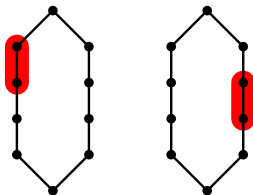
A “bottom” edge forces all side edges and the opposite “top” edge.



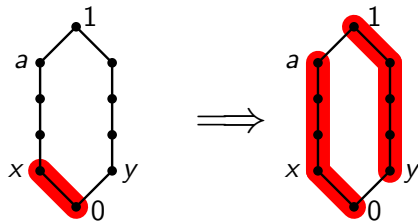
Dually, a “top” edge forces all side edges and the opposite “bottom” edge.

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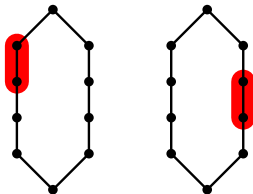
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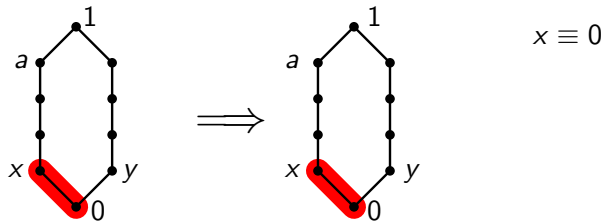
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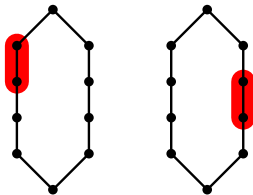
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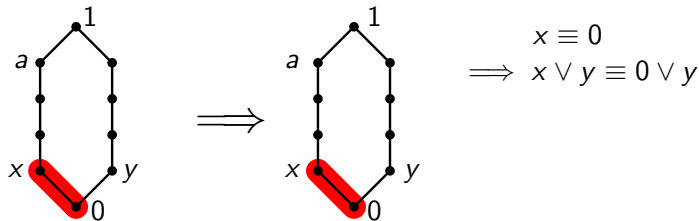
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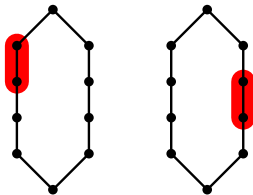
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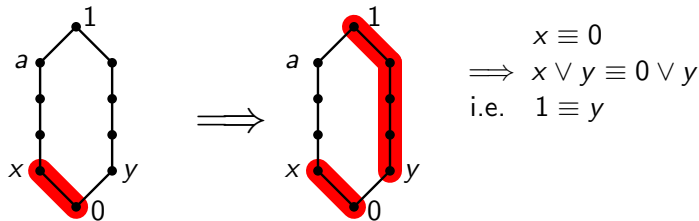
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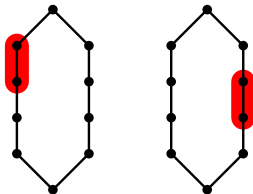
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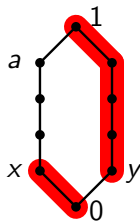
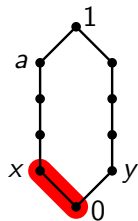
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A “bottom” edge forces all side edges and the opposite “top” edge.

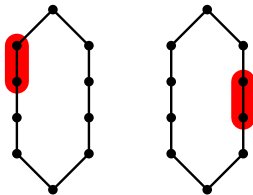


$$\begin{aligned} & x \equiv 0 \\ \Rightarrow & x \vee y \equiv 0 \vee y \\ \text{i.e.} & 1 \equiv y \\ \Rightarrow & a \wedge 1 \equiv a \wedge y \end{aligned}$$

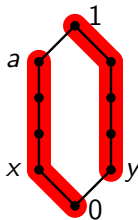
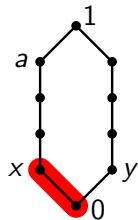
Dually, a “top” edge forces all side edges and the opposite “bottom” edge.

Edge-forcing example: “Dihedral” lattices

A “side” edge can be contracted independently. E.g.:



A “bottom” edge forces all side edges and the opposite “top” edge.



$$\begin{aligned} & x \equiv 0 \\ \implies & x \vee y \equiv 0 \vee y \\ \text{i.e.} & 1 \equiv y \\ \implies & a \wedge 1 \equiv a \wedge y \\ \text{i.e.} & a \equiv 0 \end{aligned}$$

Dually, a “top” edge forces all side edges and the opposite “bottom” edge.

Lattice congruences of the weak order

The weak order on W is a lattice.

Congruences on the weak order have nice geometric properties. Interpret a lattice congruence “ $\equiv$ ” as an equivalence relation on the regions cut out by the reflecting hyperplanes. For each congruence class C , let $\cup C$ denote the union of the regions in C .

Theorem (R., 2004)

The cones $\cup C$ are the maximal cones of a complete fan.

The quotient of the weak order modulo $\equiv$ arises geometrically from the coarser fan just as the weak order arises from the original fan.

The weak order has many intervals that are dihedral lattices.

Theorem (R., 2002)

All edge forcings for lattice congruences of the weak order are determined locally within dihedral intervals.

What shards are

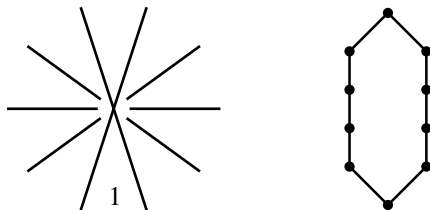
We will be gluing regions together according to congruence classes.

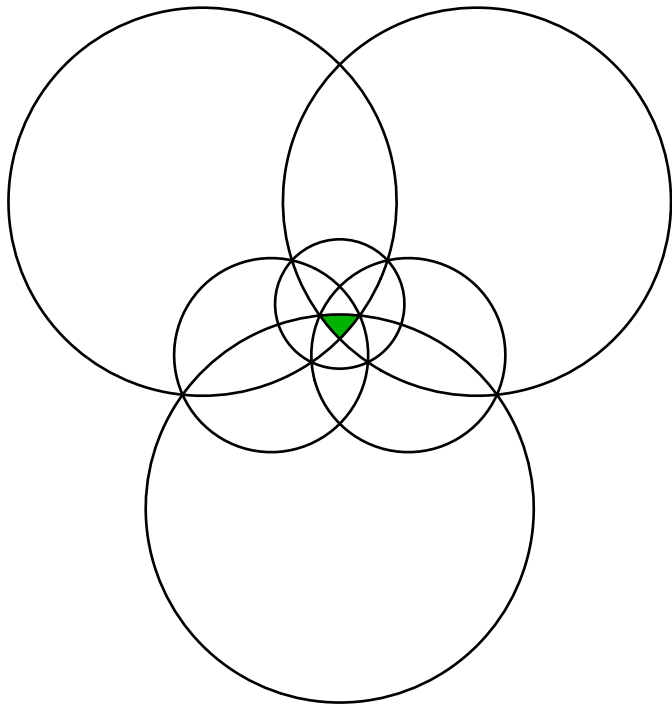
So: contracting an edge means removing the wall between two adjacent cones.

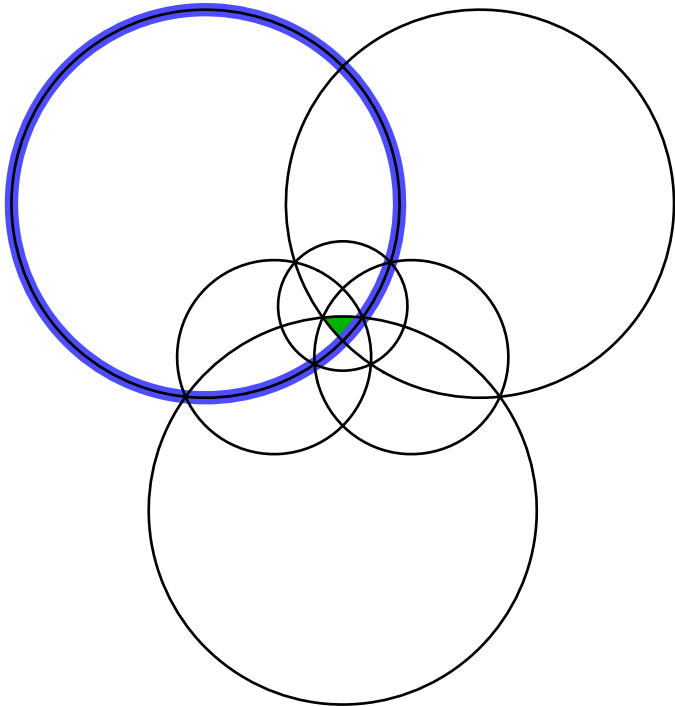
A shard is a maximal collections of walls which must always be removed together in a lattice congruence (because of edge-forcing). Each shard turns out to consist of walls all in the same hyperplane.

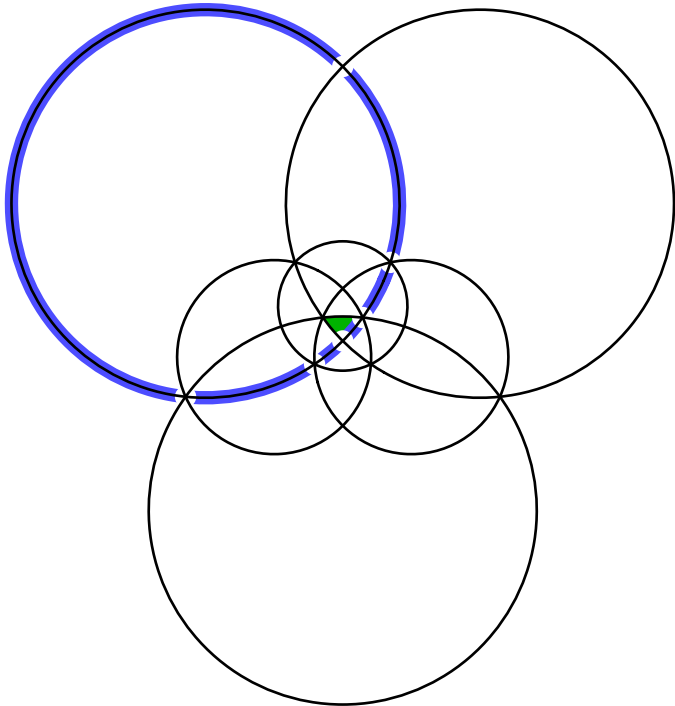
Edge-forcing also implies some forcing relations among shards.

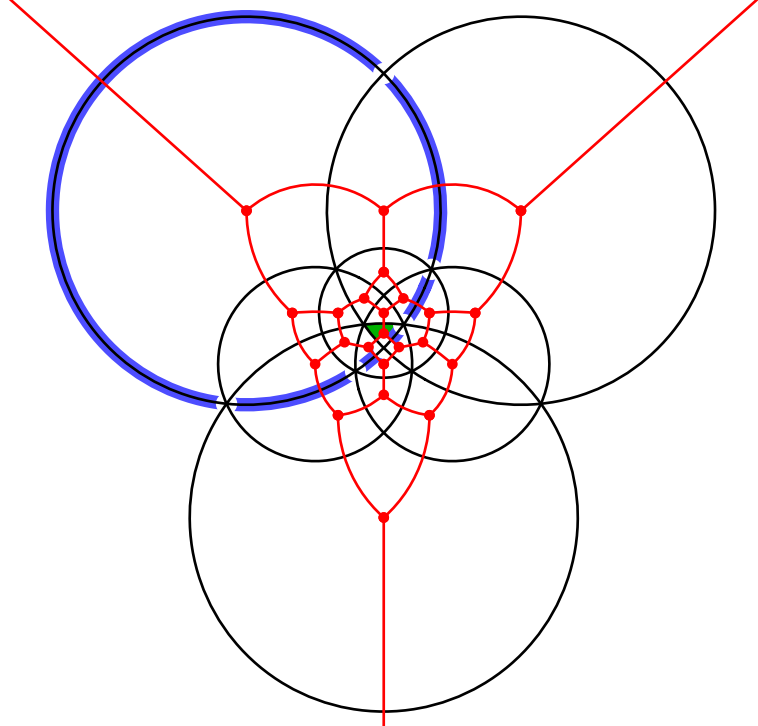
Example:

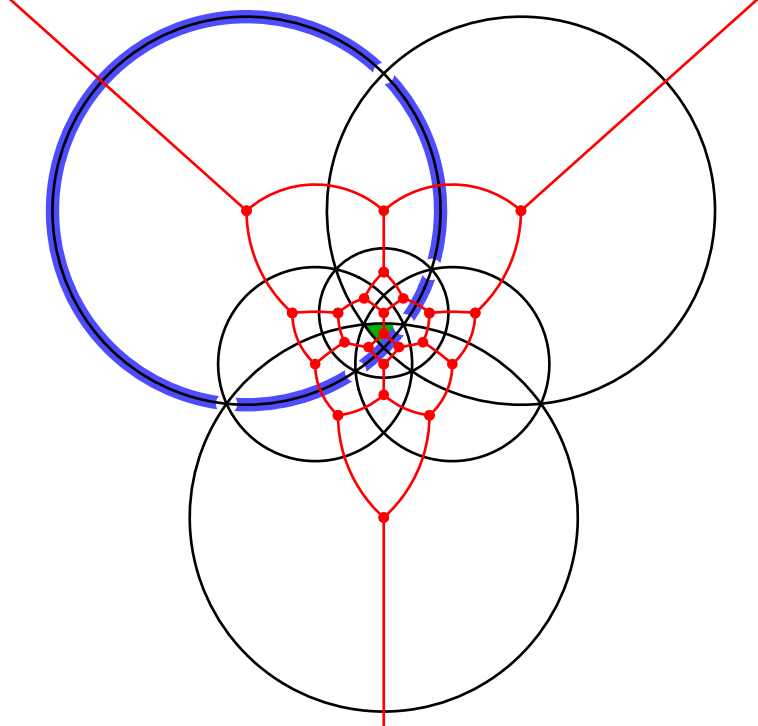


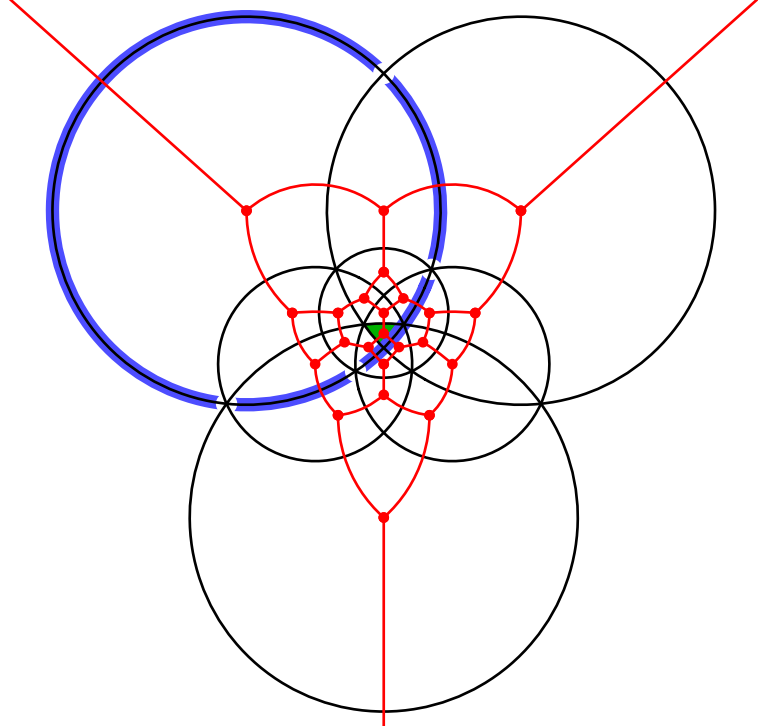


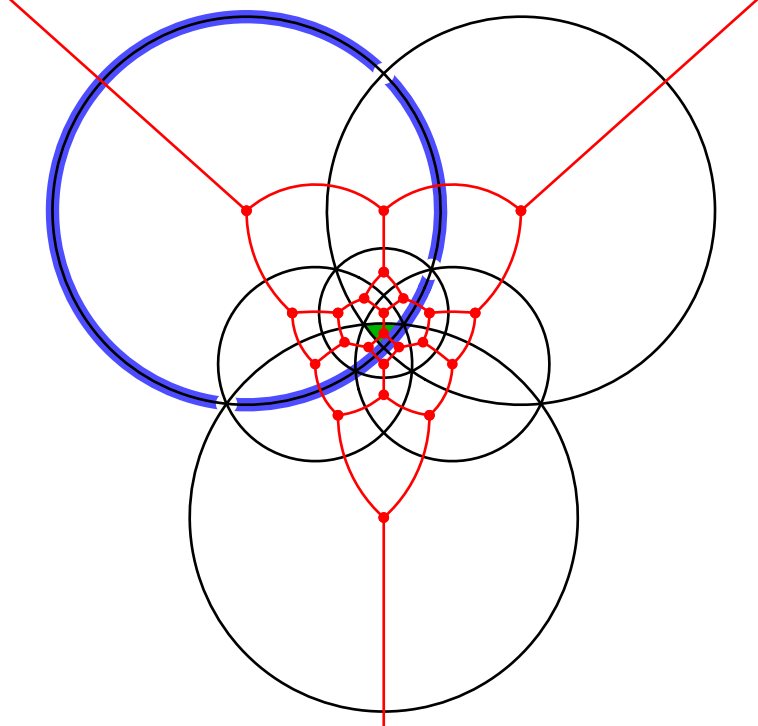


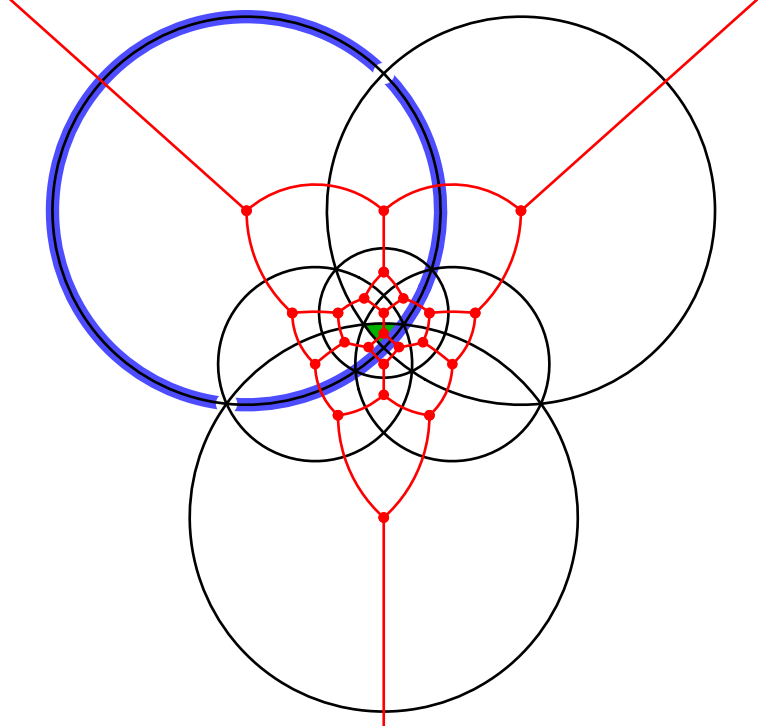


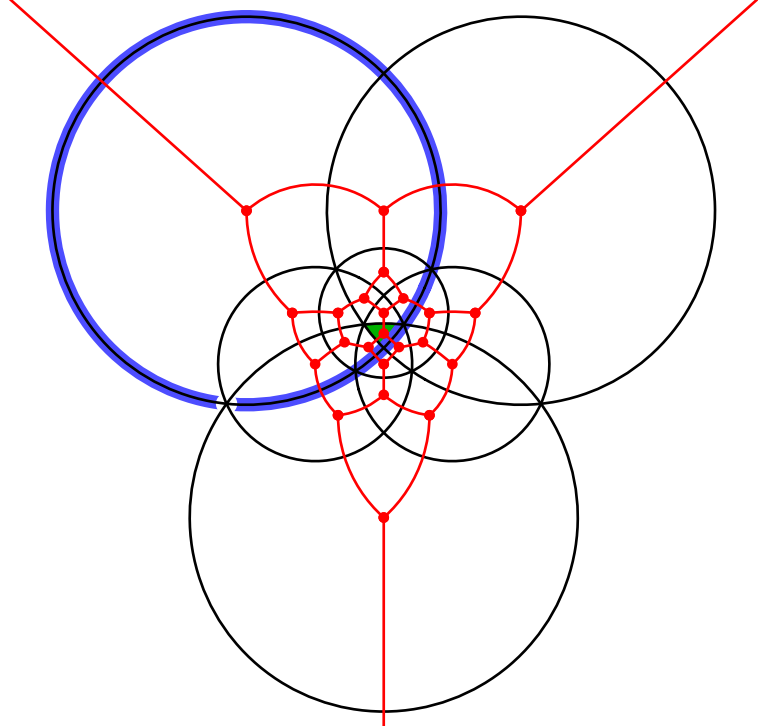


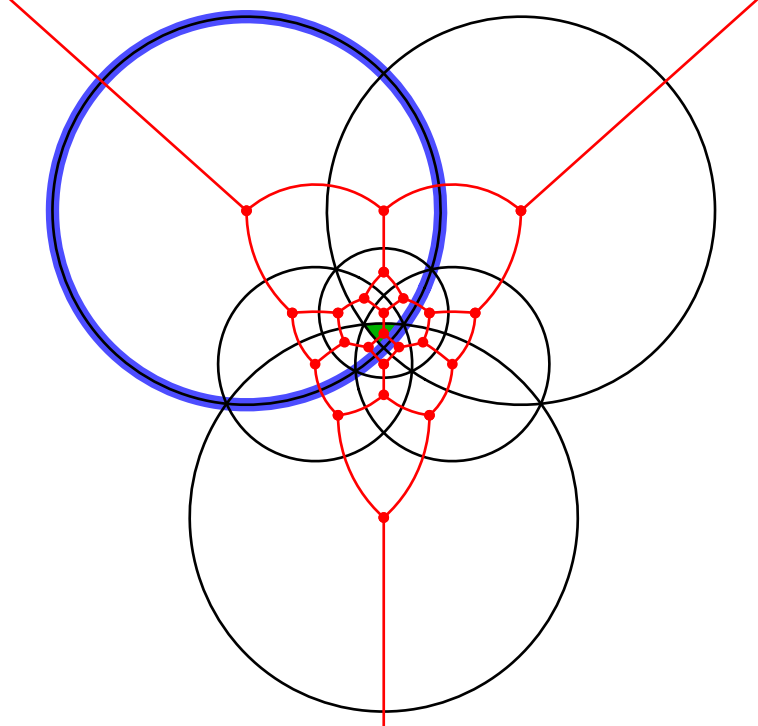


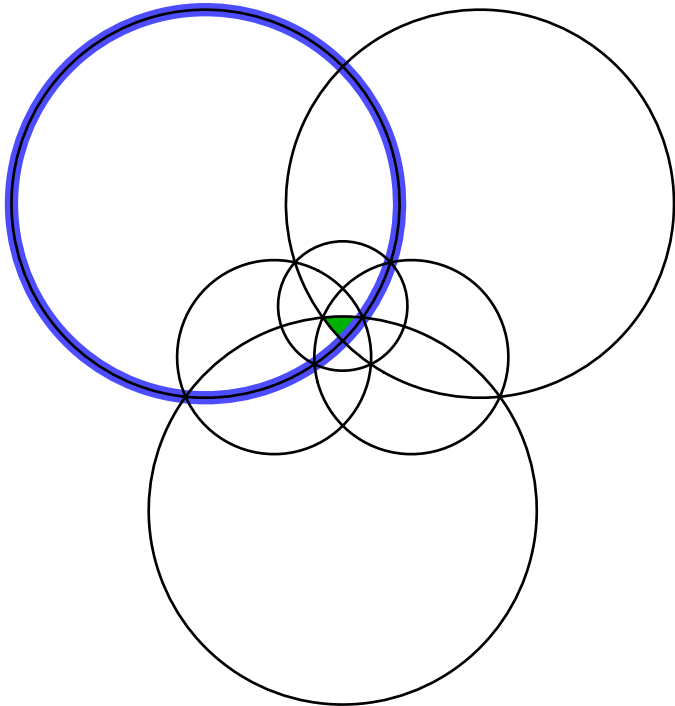


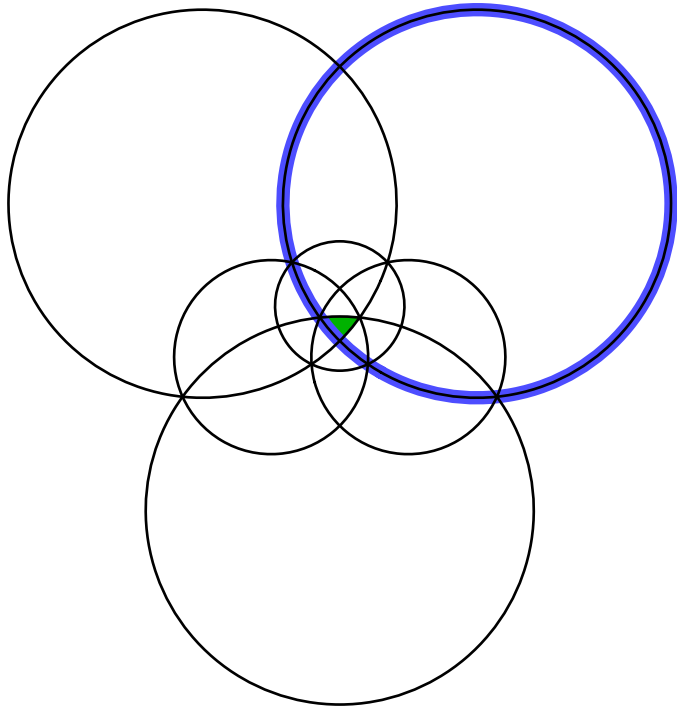


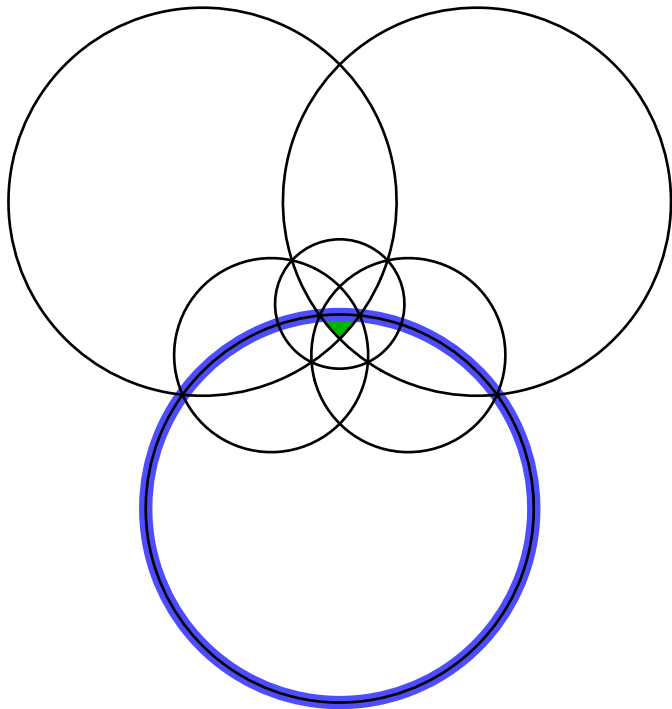


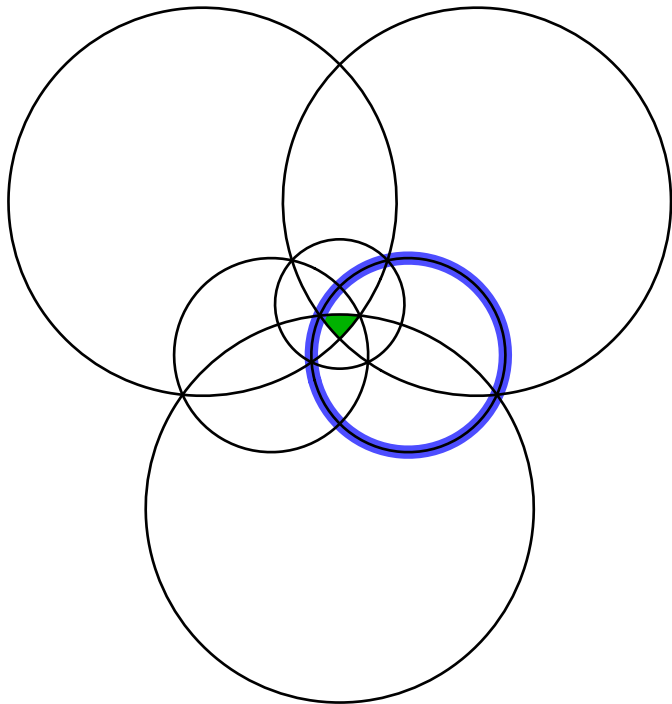


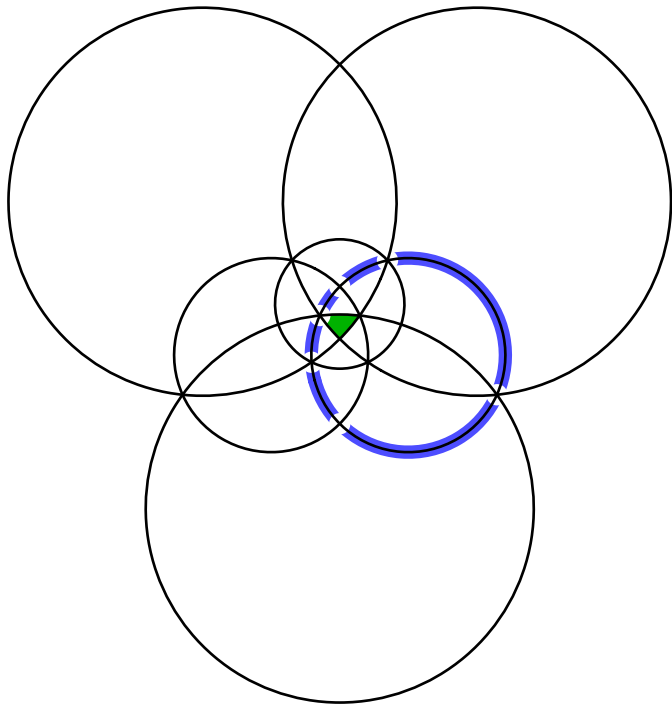


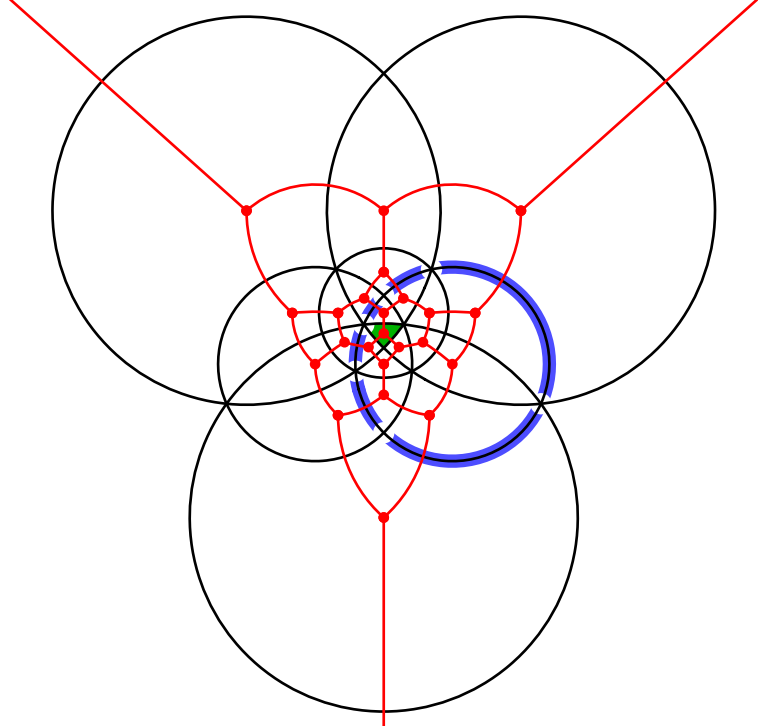


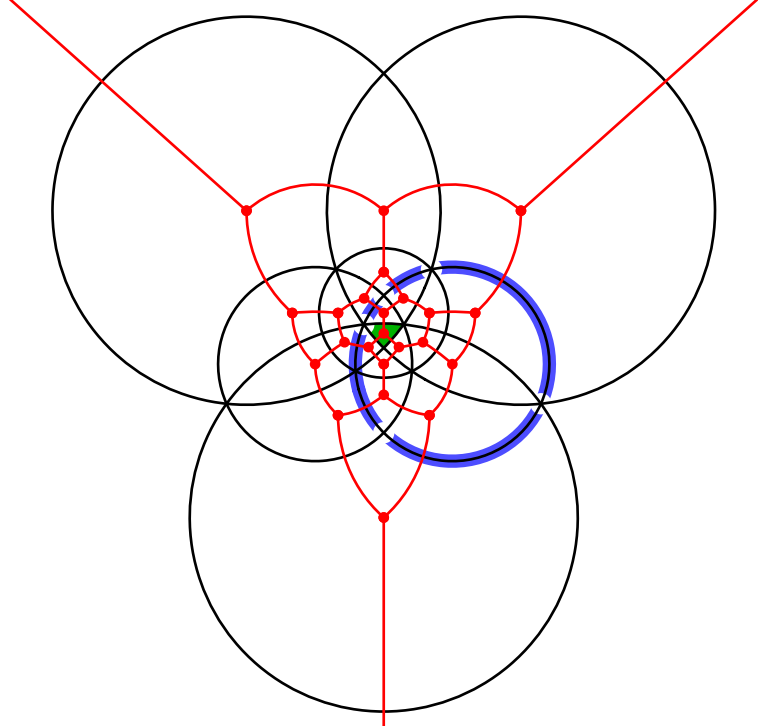


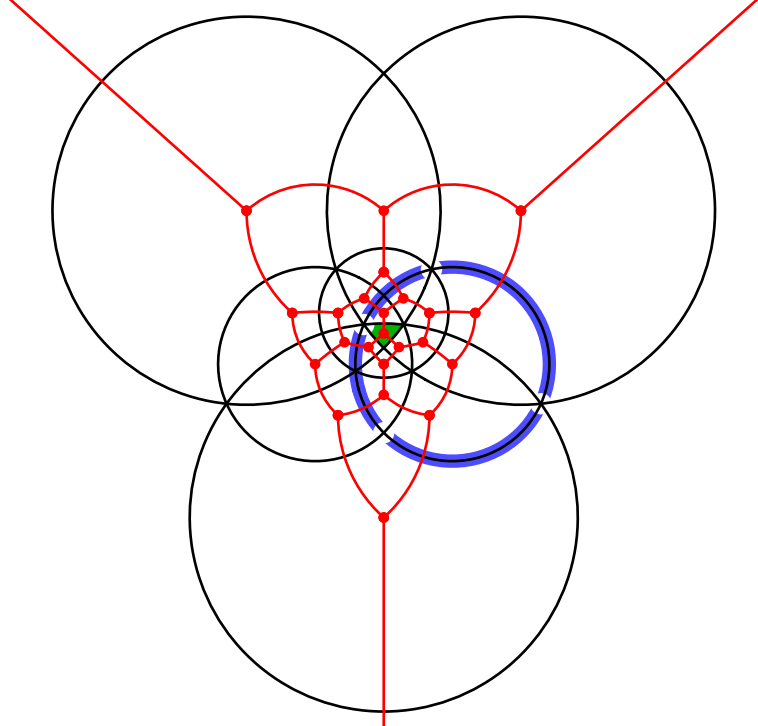


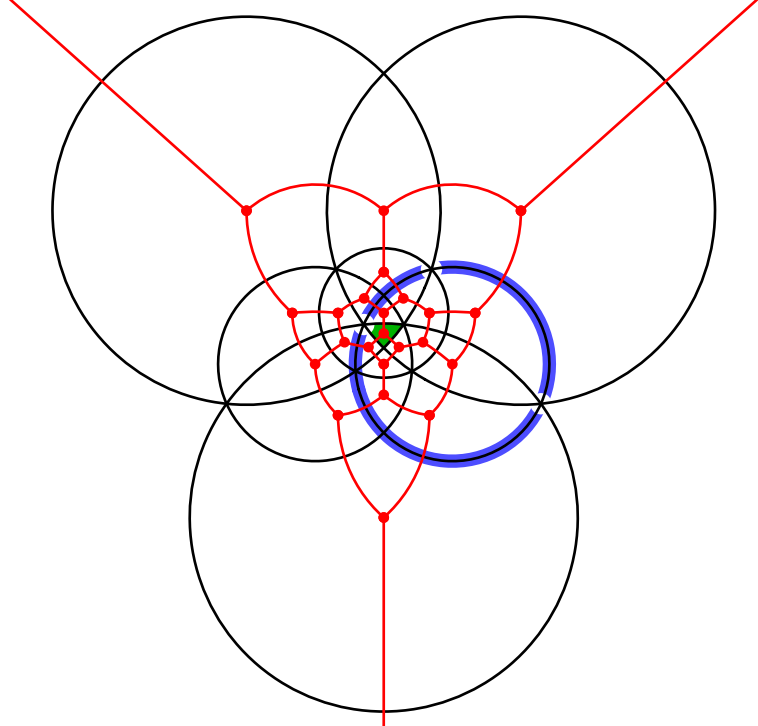


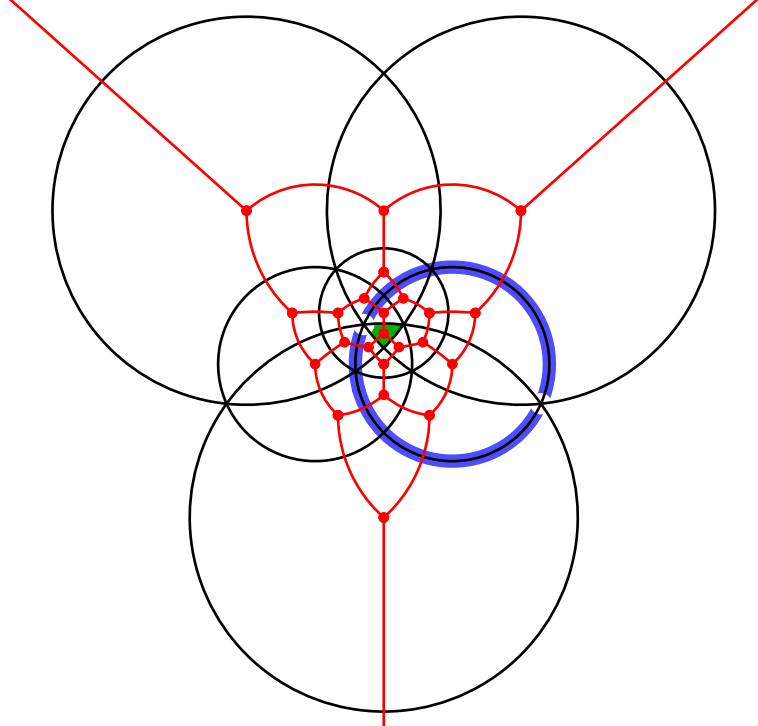


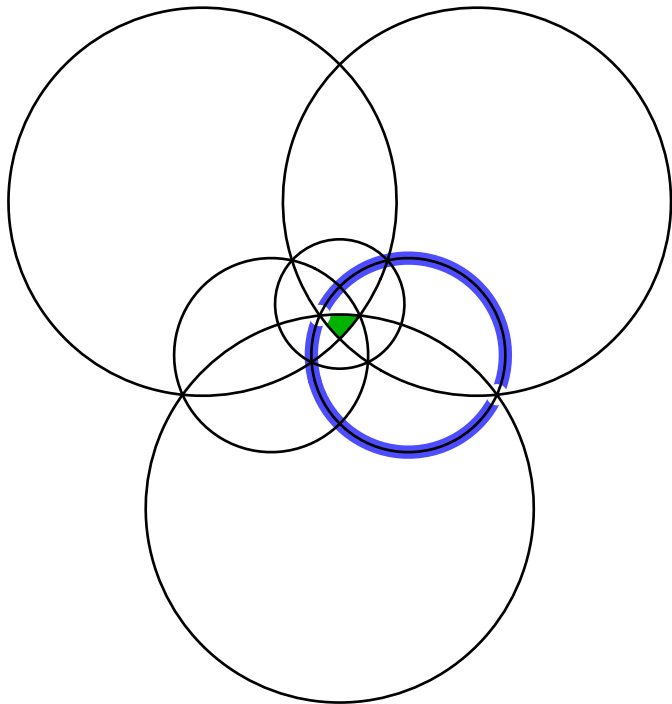


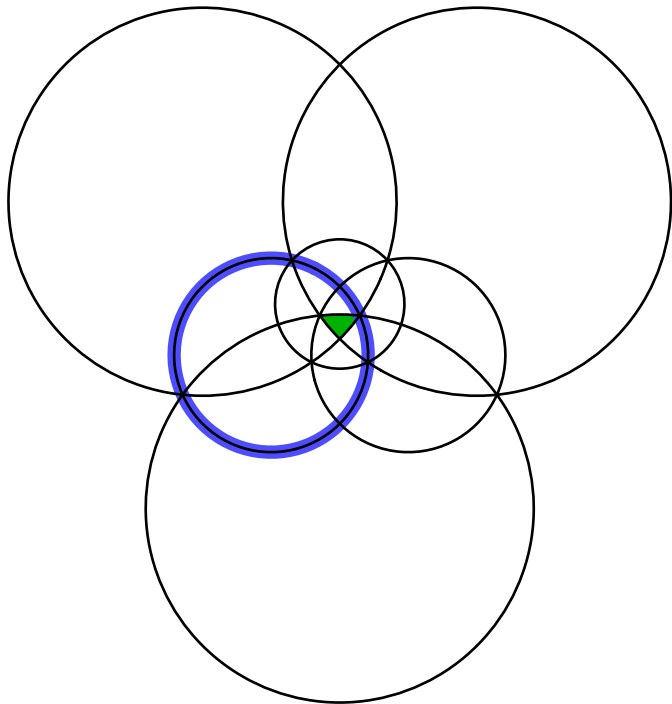


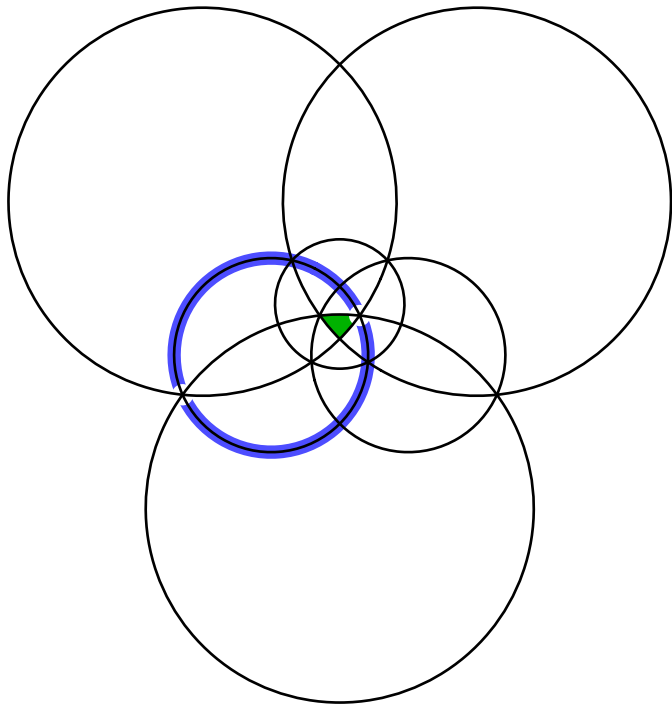


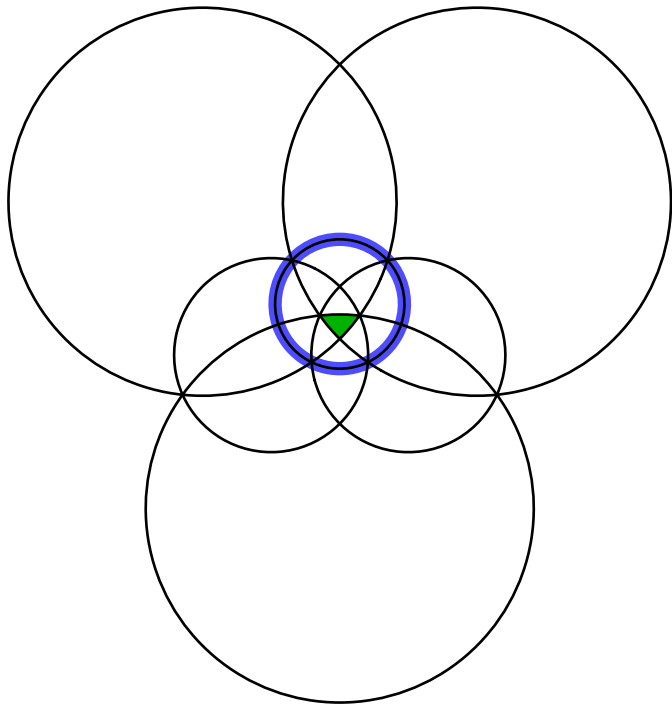


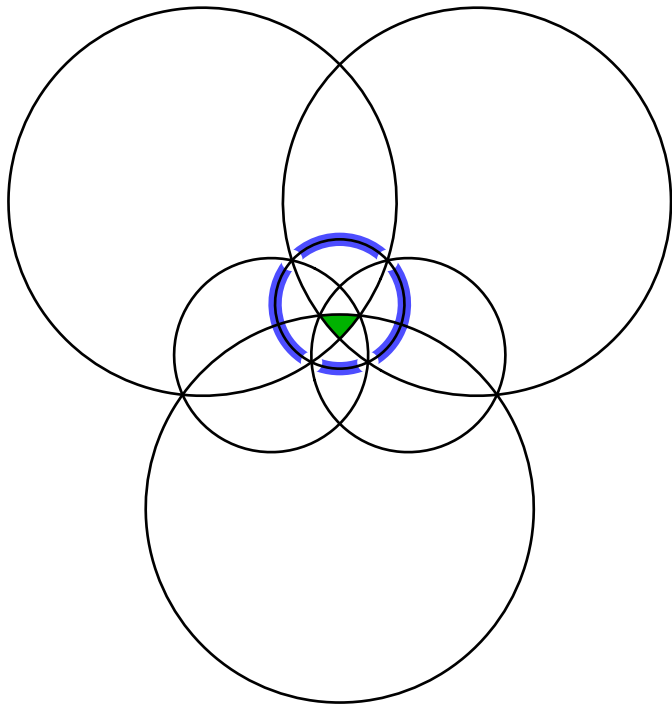




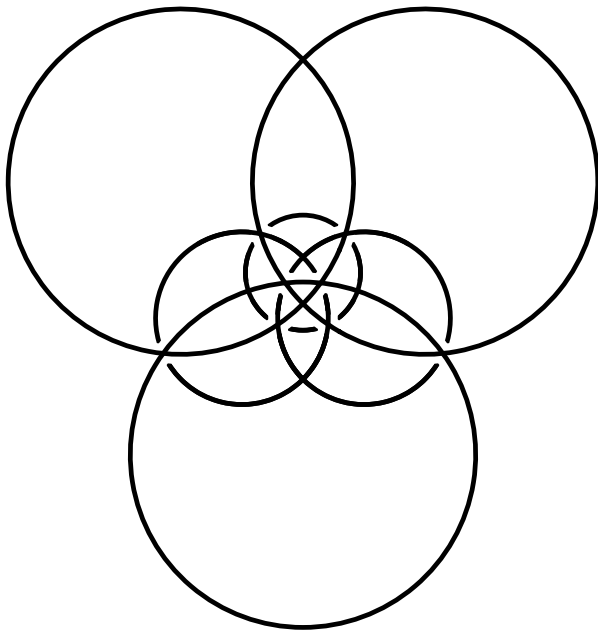




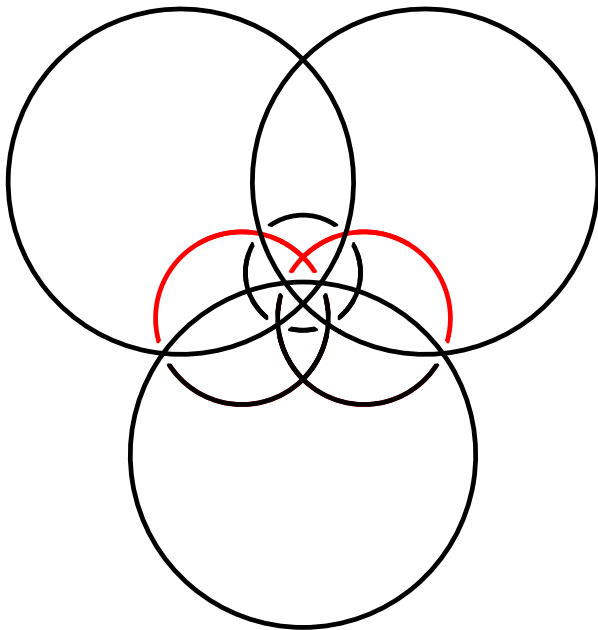




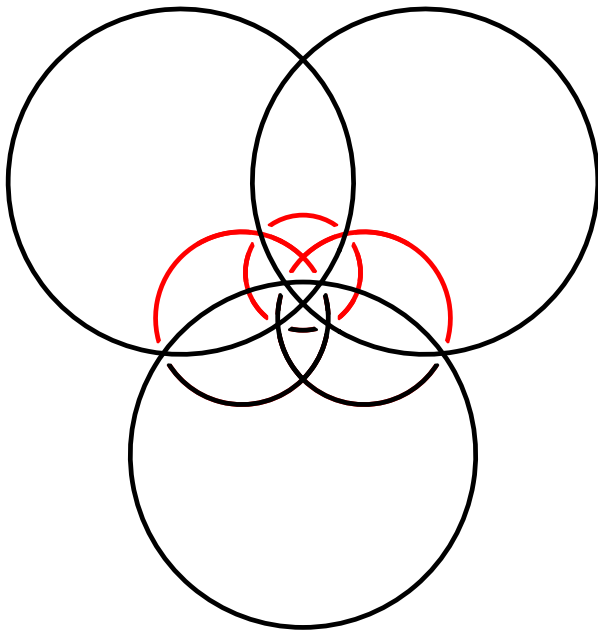
Shard removal, forcing and fans in S_4



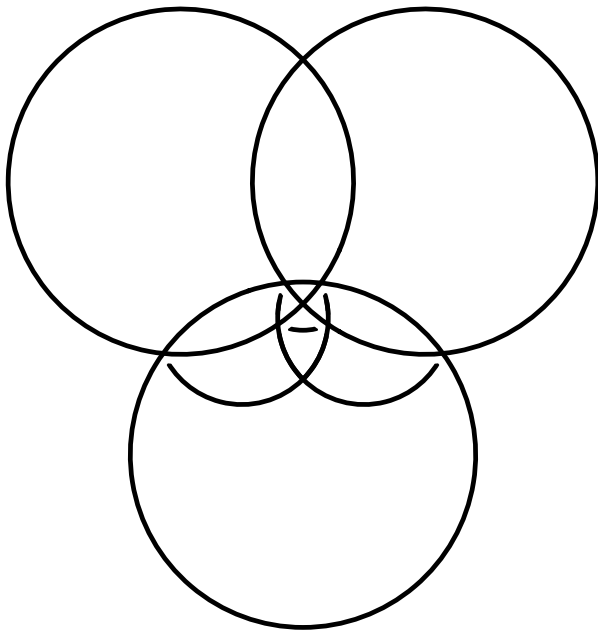
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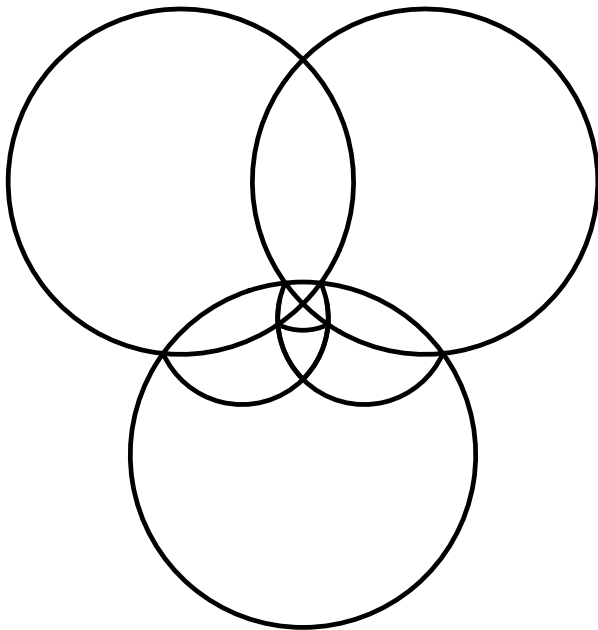
Shard removal, forcing and fans in S_4



Shard removal, forcing and fans in S_4



Shard removal, forcing and fans in S_4



Shard intersections and lattice congruences

Since shards are so central to lattice congruences on the weak order, it is perhaps not surprising that lattice congruences “play nicely” with the shard intersection order.

Specifically, let $\pi_{\downarrow}^{\Theta}(W)$ be the collection of “bottom elements” of congruence classes of a congruence Θ . Then the restriction $(\pi_{\downarrow}^{\Theta}(W), \preceq)$:

- is a lattice (a join-sublattice of $(W, \preceq)$);

- is graded, atomic and coatomic;

- has lower intervals $(\pi_{\downarrow}^{\Theta'}(W_J), \preceq)$

- has Möbius number analogous to that of $(W, \preceq)$;

- has order-complex whose simplices biject with the simplices of a pulling triangulation of a certain CW-ball.

Shard intersections and noncrossing partitions

There is a special lattice congruence Θ_c called the c -Cambrian congruence, with very special properties:

The Cambrian fan, (obtained by removing shards according to Θ_c) is combinatorially isomorphic with the generalized associahedron for W . (R., Speyer, 2007.)

$\pi_{\downarrow}^{\Theta_c}(W)$ is the set of c -sortable elements. These are in bijection with both noncrossing partitions and clusters in the corresponding cluster algebra of finite type. (R., 2006.)

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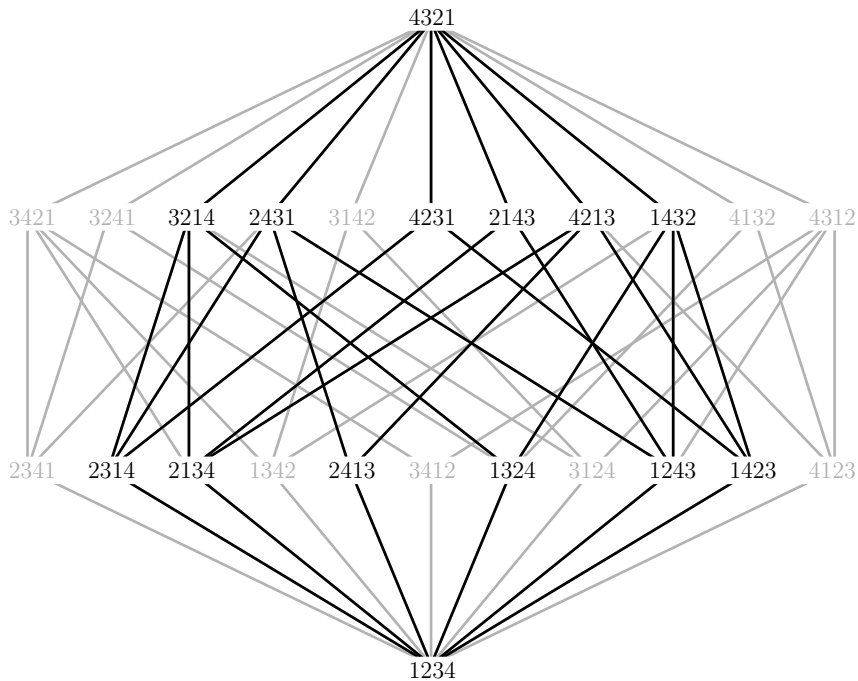
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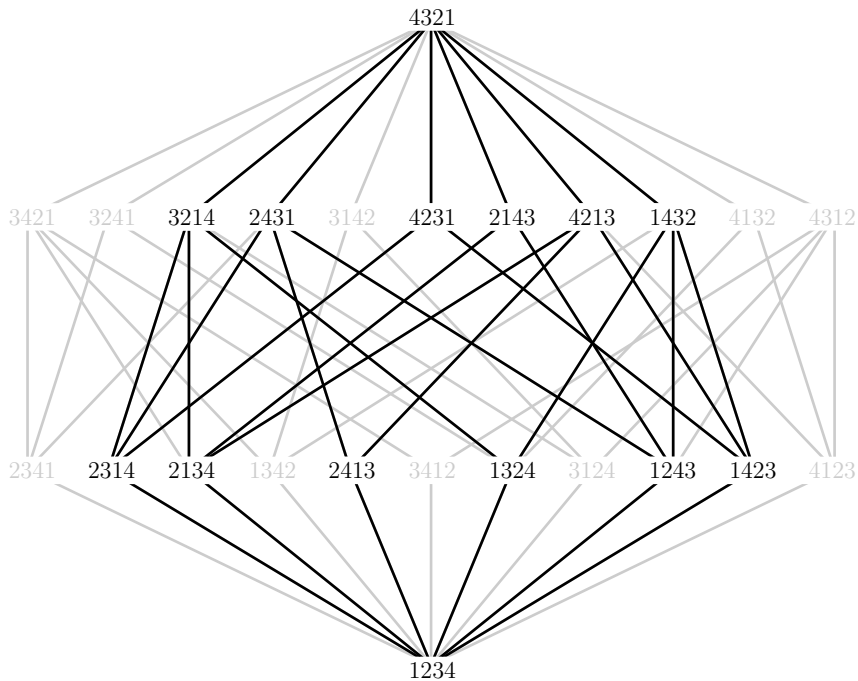
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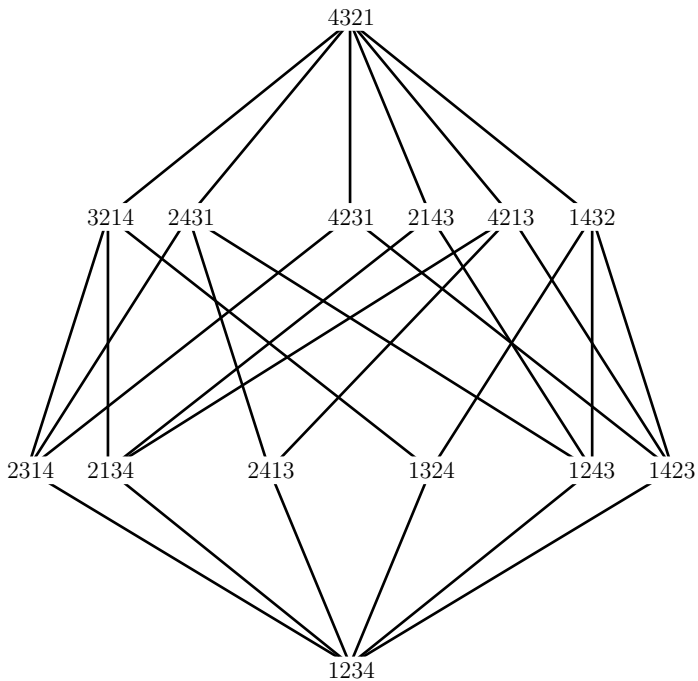
NEW! The lattice $(\pi_{\downarrow}^{\Theta_c}, \preceq)$ is isomorphic to $\text{NC}_c(W)$.

As a consequence, **$\text{NC}_c(W)$ is a lattice**. (In fact, $\text{NC}_c(W)$ is a sublattice of $(W, \preceq)$.)

The earlier proof (by Brady and Watt) that $\text{NC}(W)$ is a lattice also used the polyhedral geometry of cones. Their proof is “dual” to the new proof (in the broadest outlines but not in any of the details).



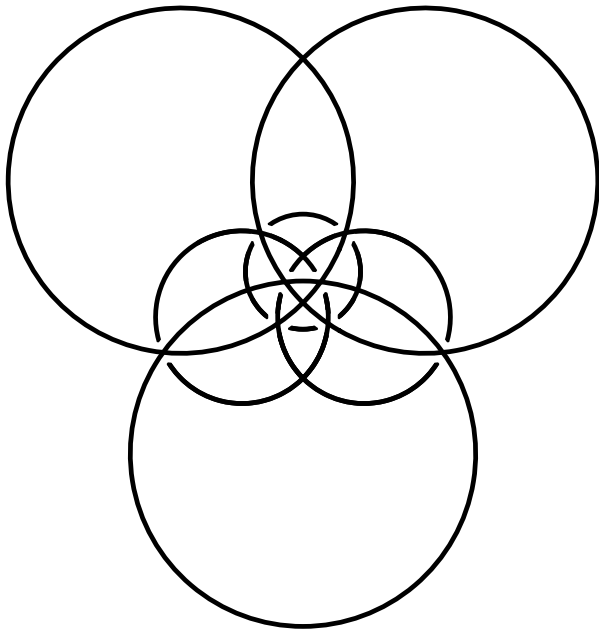




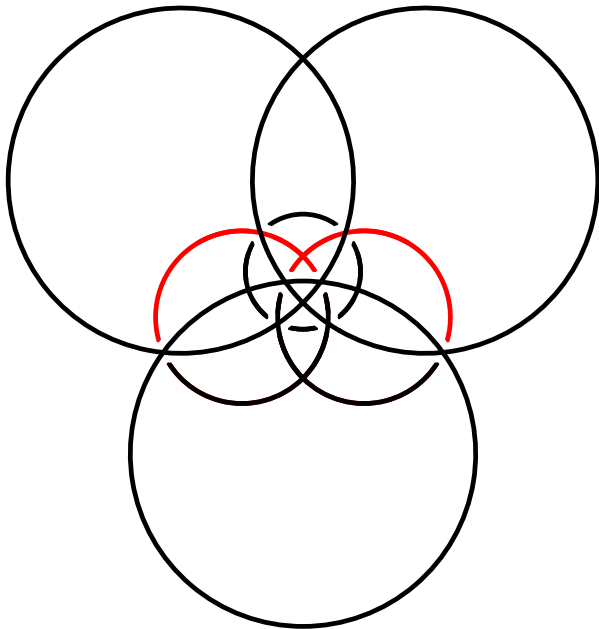
What are the Cambrian congruences?

There is a small set $\Sigma_{W,c}$ of shards such that the c -Cambrian congruence corresponds to removing the shards in $\Sigma_{W,c}$ and all other shards whose removal is then forced.

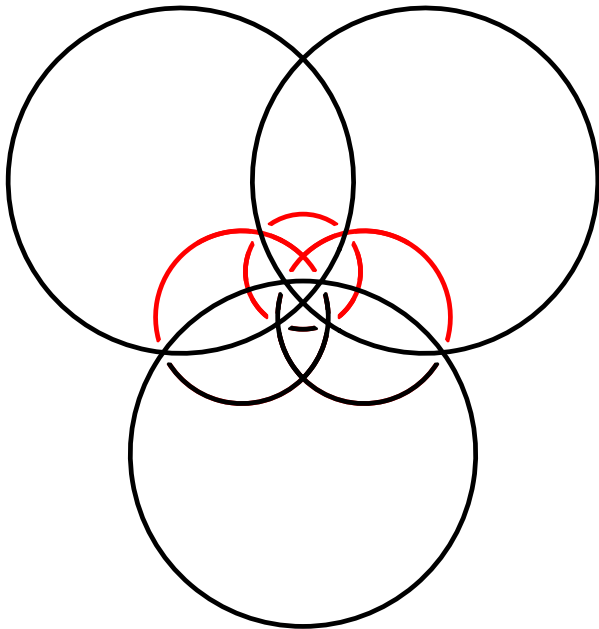
The Cambrian fan



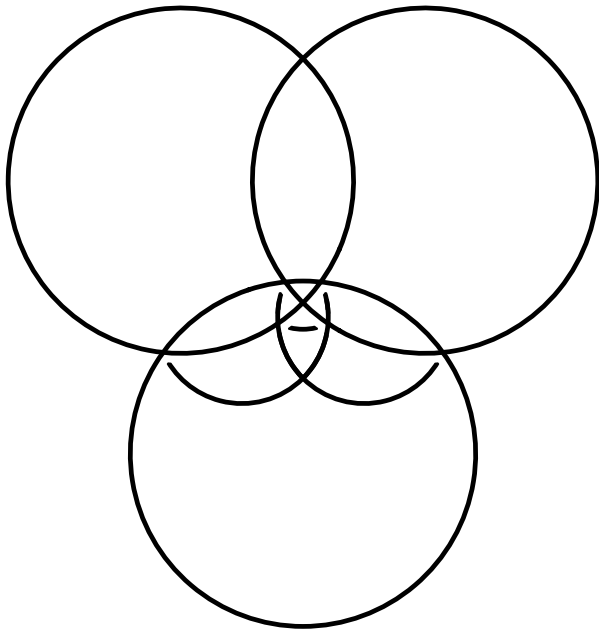
The Cambrian fan



The Cambrian fan



The Cambrian fan



The Cambrian fan (Normal fan to W -associahedron)

