The Hopf algebra, lattice, and polytope of rectangulations

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(joint with Shirley Law)

NC State University

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Combinatorial Hopf algebras

A combinatorial Hopf algebra is a:

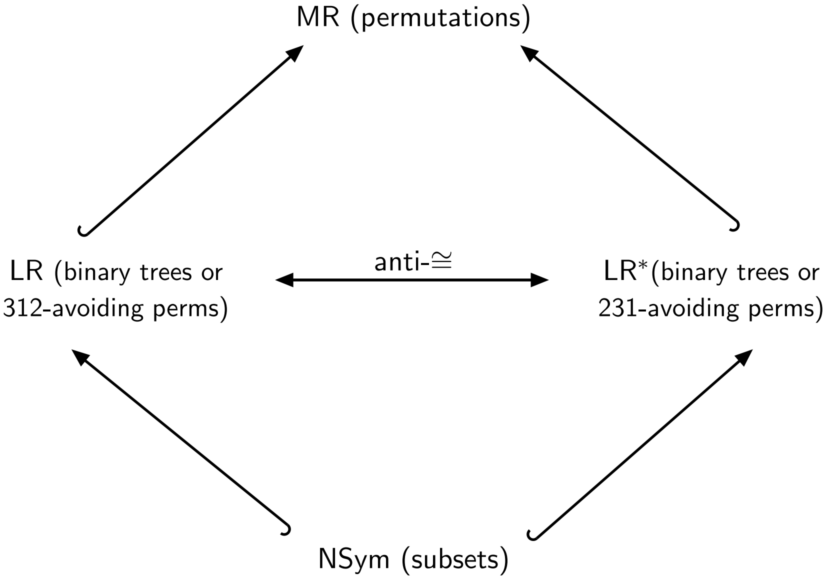
- graded algebra $\mathcal{H}$ over $\mathbb{Q}$
- basis B indexed by "combinatorial objects"
- equipped with a graded coproduct
- axiom on compatibility of product and coproduct

$$\begin{array}{llll} \text{Product:} & m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} & & (+\text{ axioms}). \\ \text{Coproduct:} & \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} & & (+\text{ dual axioms}). \end{array}$$

$$m(b_1 \otimes b_2) = \sum_{b \in B} c_b b$$

$$\Delta(b) = \sum_{b_1, b_2 \in B} d_{b_1 b_2} (b_1 \otimes b_2)$$

Context for this talk



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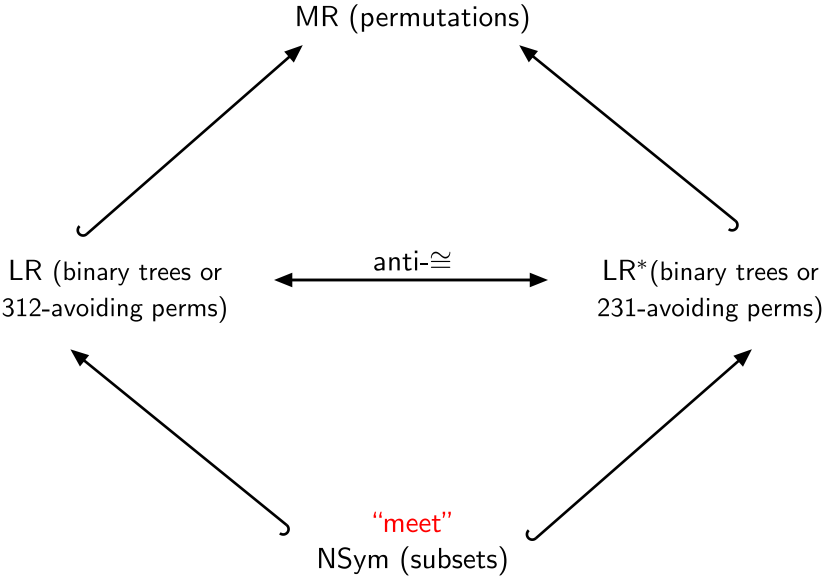

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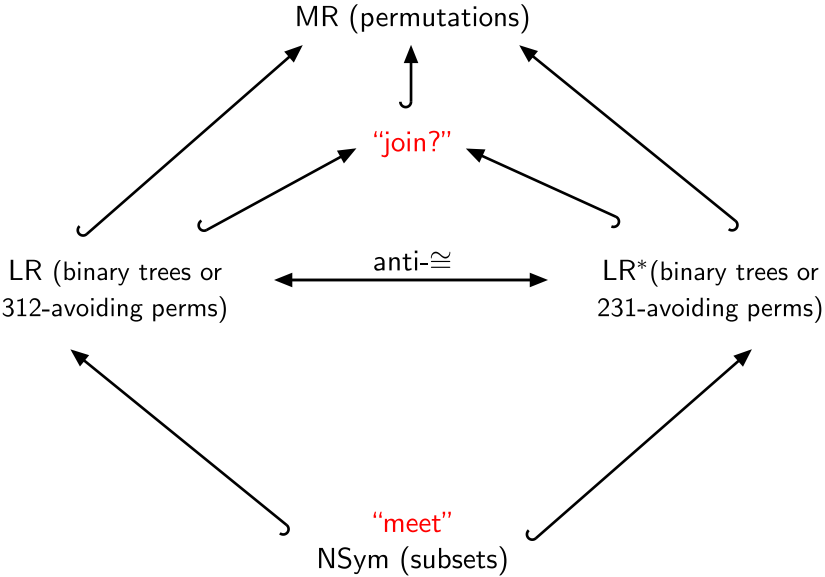

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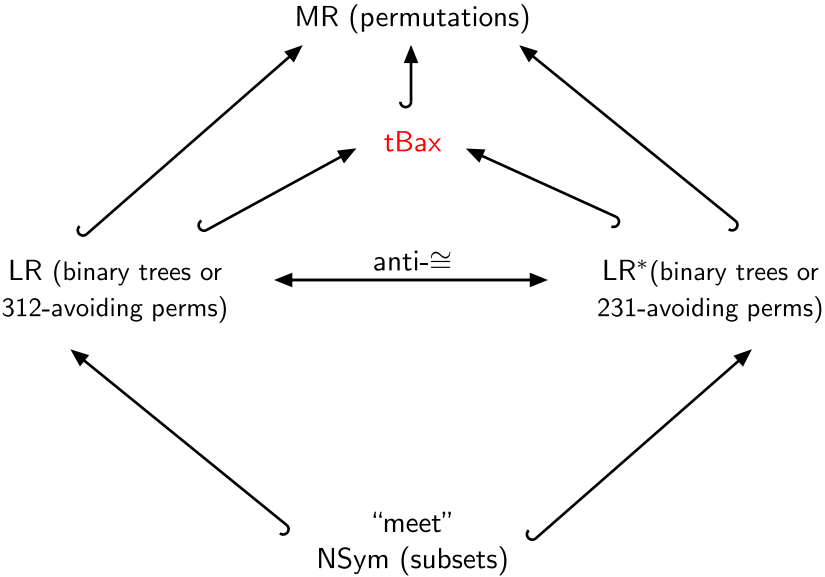


Baxter permutations and twisted Baxter permutations

Baxter permutations (Baxter, '64):

A pattern-avoidance condition.

Enumerated by Chung, Graham, Hoggatt, Kleiman, '78.

$$\binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

Twisted Baxter permutations (R., '04):

A very similar pattern-avoidance condition.

Enumeration:

Same as Baxter permutations. (West, '06, Law-R., '09.)

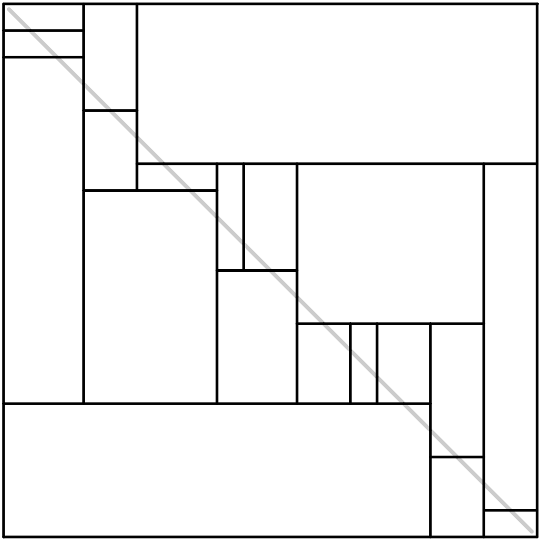
The Hopf algebra tBax

tBax is a sub Hopf algebra of MR. It arises as a special case of a lattice-theoretic machine (R., '04) that produces sub-Hopf algebras of MR.

Product and coproduct in tBax: defined extrinsically in terms of the ebbedding into MR.

Motivation for this work: Find an intrinsic description of tBax in terms of some set of combinatorial objects.

Rectangulations (Ackerman, Barequet, Pinter, '06, Felsner, Fusy, Noy, Orden, '08, Dulucq and Guibert, '96)



(Diagonal) rectangulations are decompositions of a square into rectangles such that each rectangle's interior intersects the diagonal.

Enumeration: same as (twisted) Baxter permutations.

Permutations to planary binary trees

$\rho_t : S_n \to \{\text{planar binary trees}\}$.

Example: $\rho_t(467198352)$

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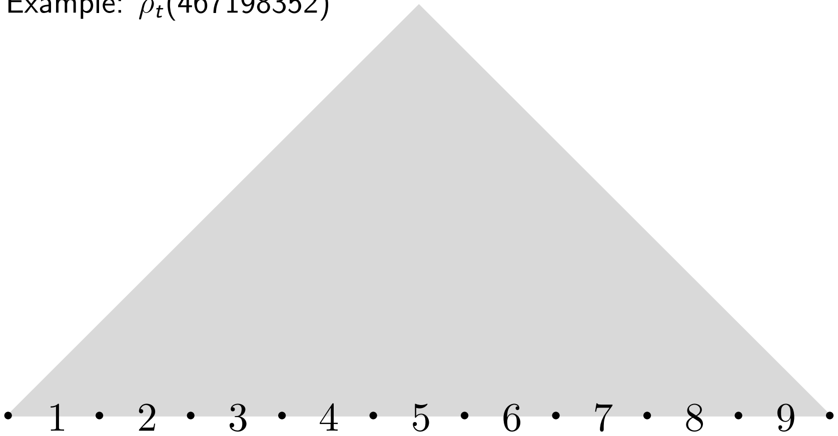

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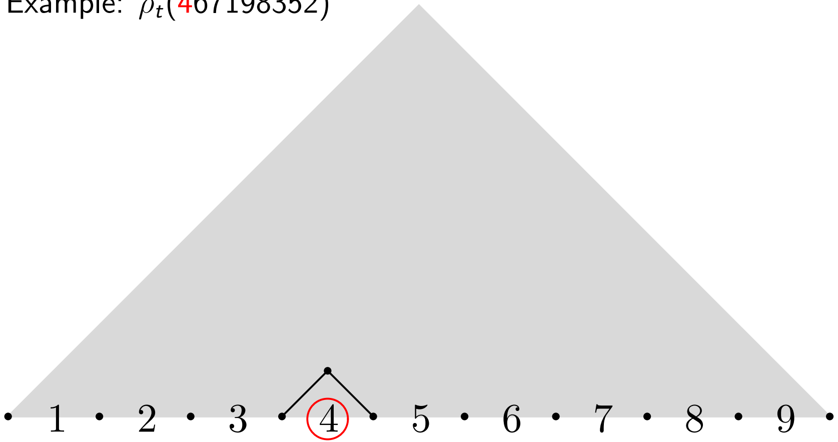

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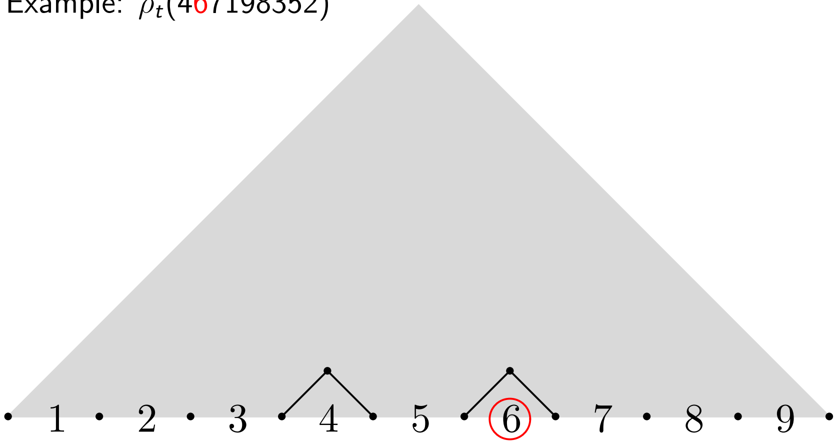

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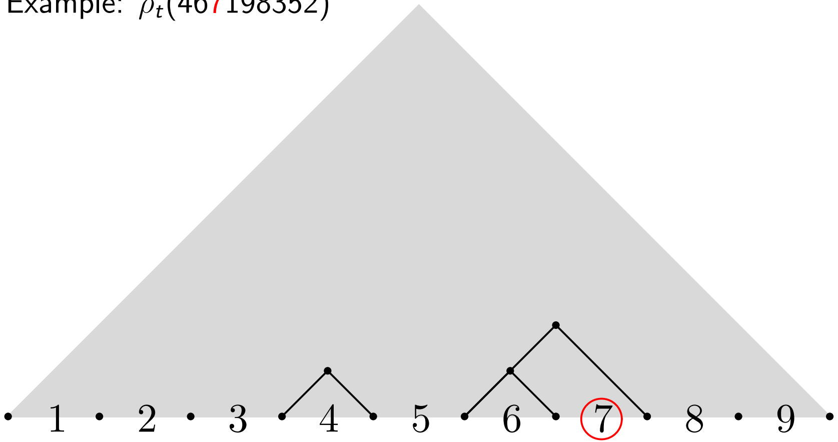
Example: $\rho_t(46\textcolor{red}{7}198352)$



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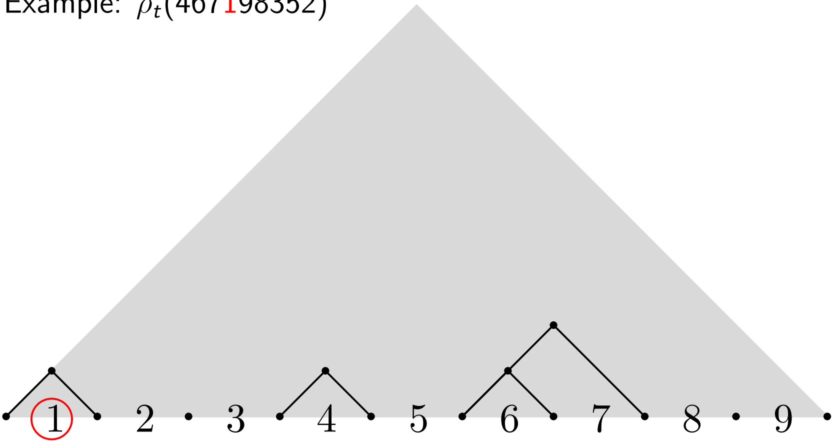

Permutations to planary binary trees

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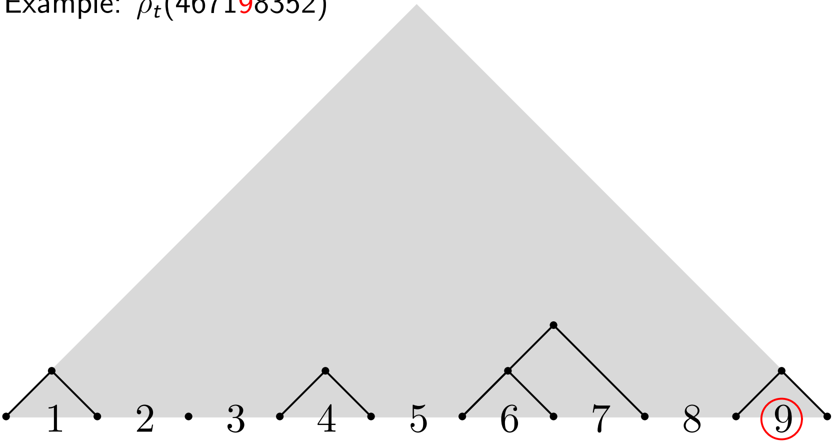
Example: $\rho_t(4671\color{red}{9}8352)$



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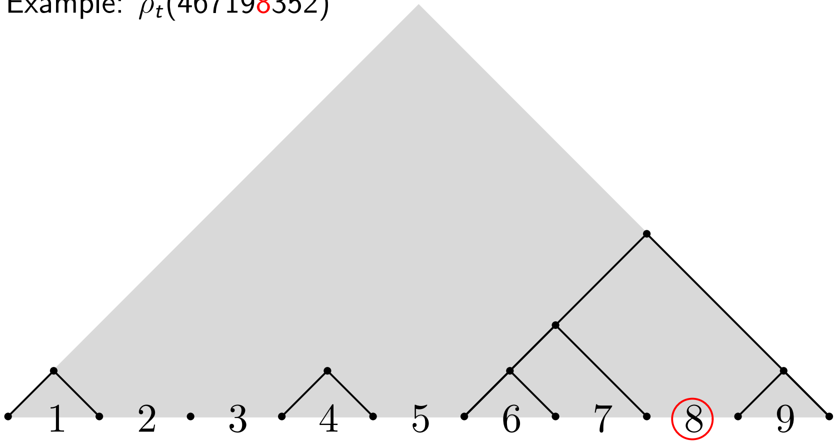

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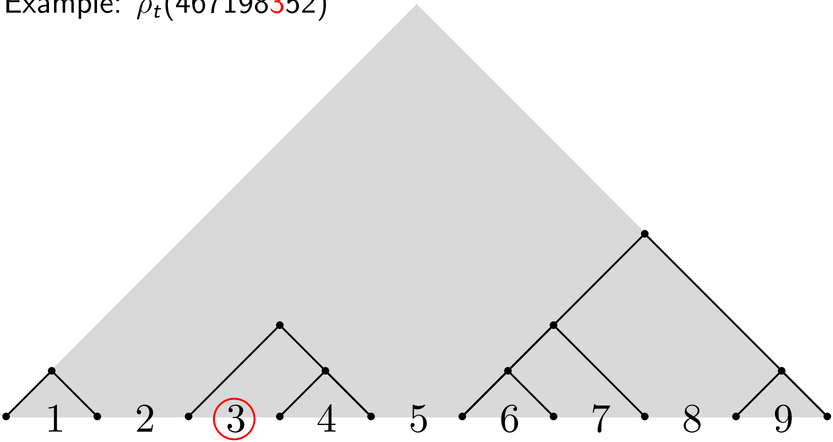

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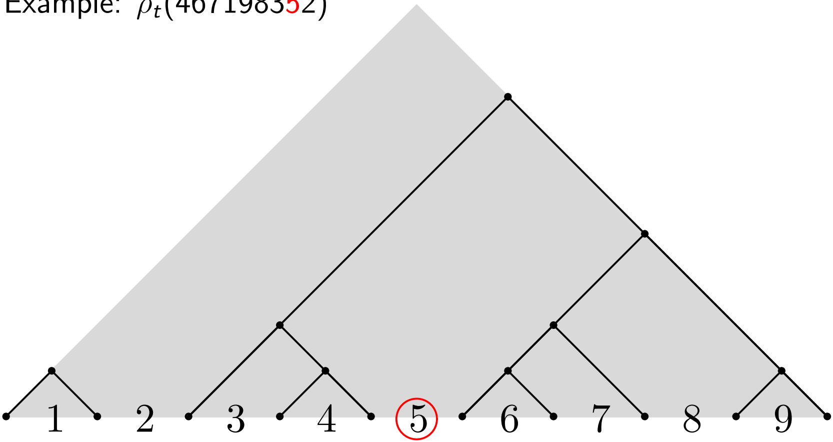

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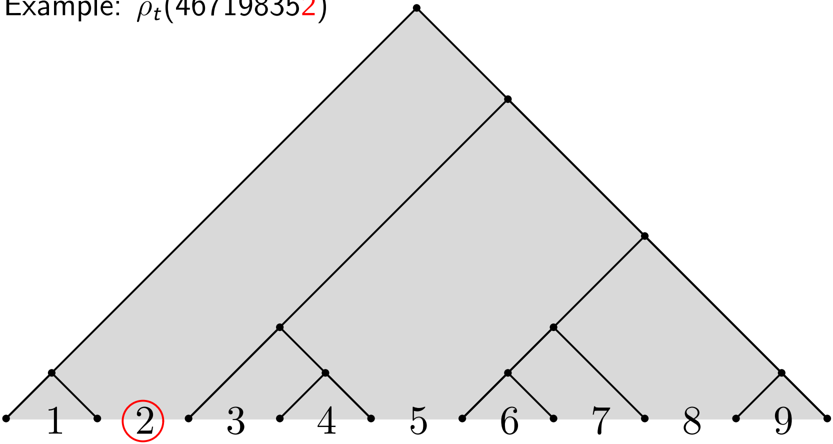

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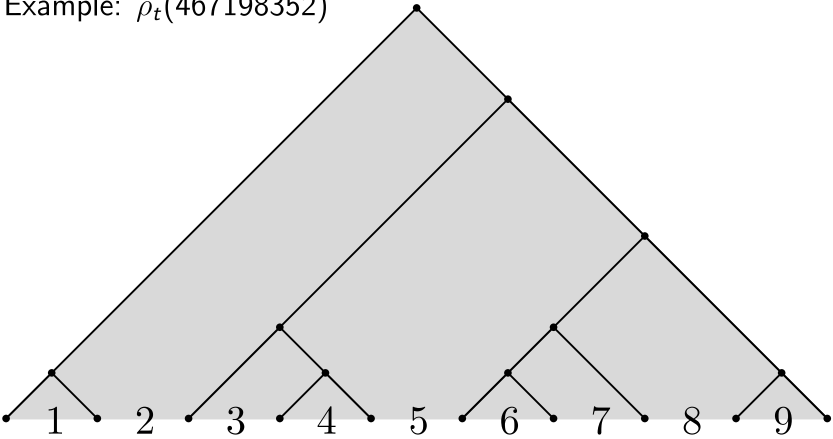

Permutations to planary binary trees, alternate map

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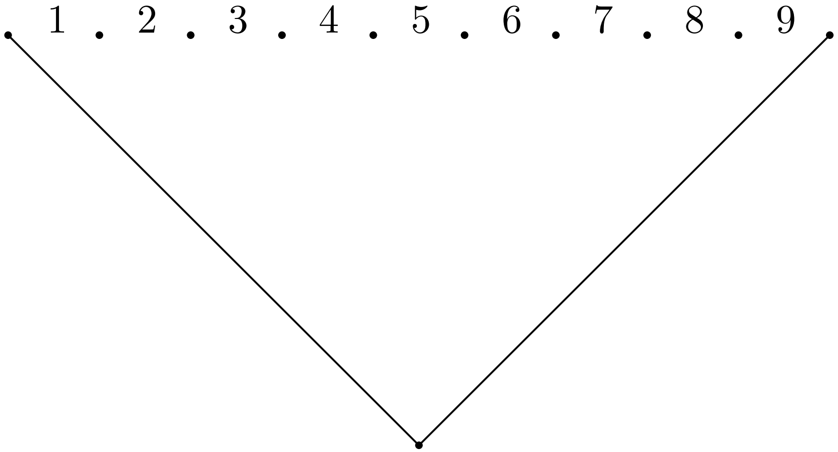

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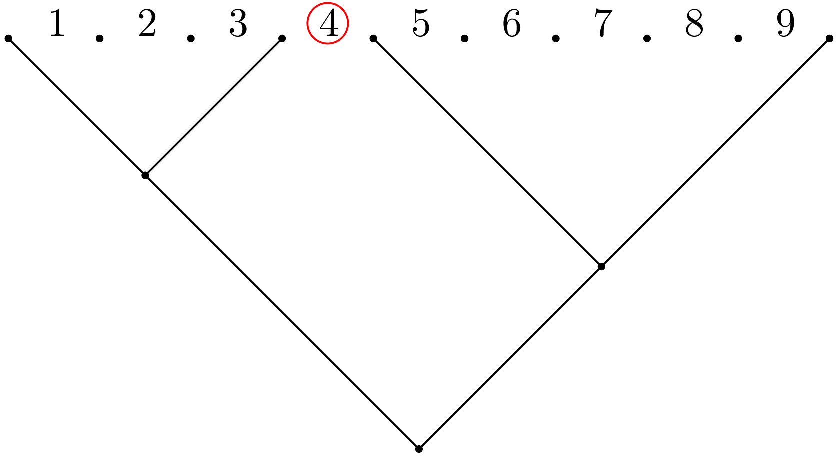

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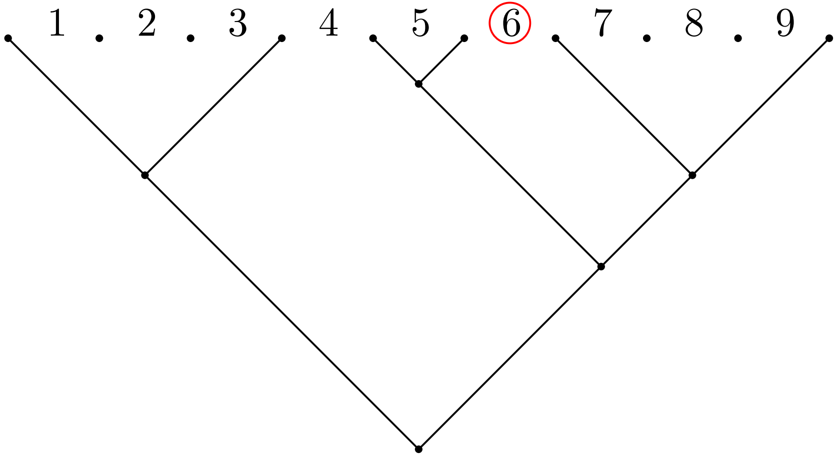

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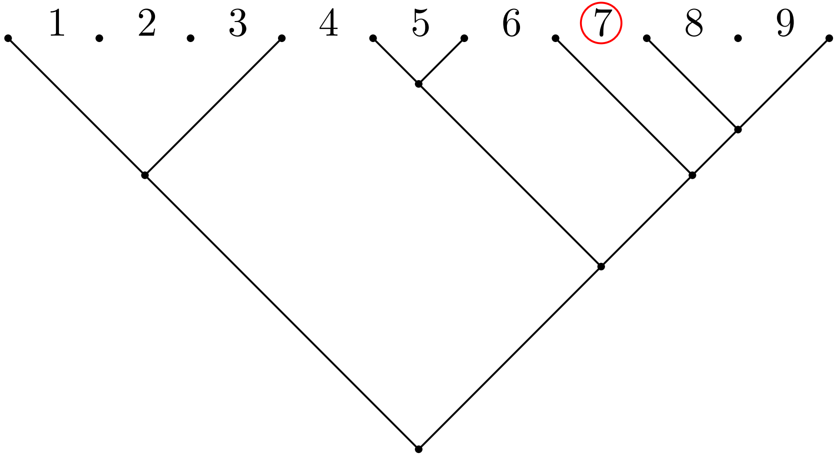

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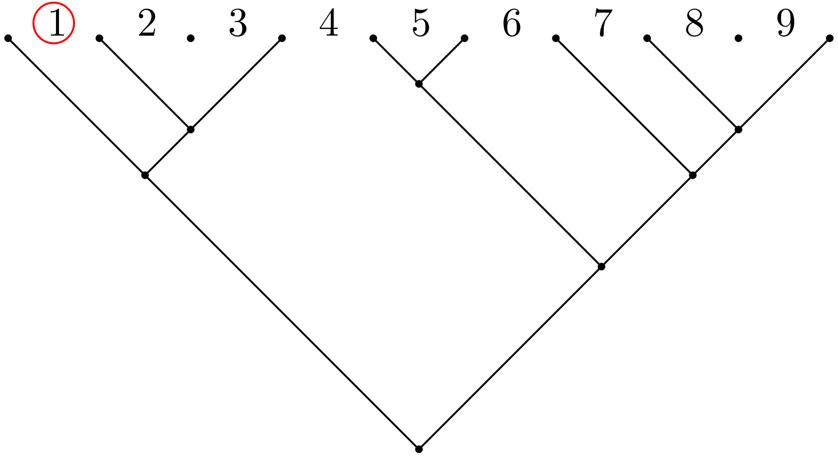

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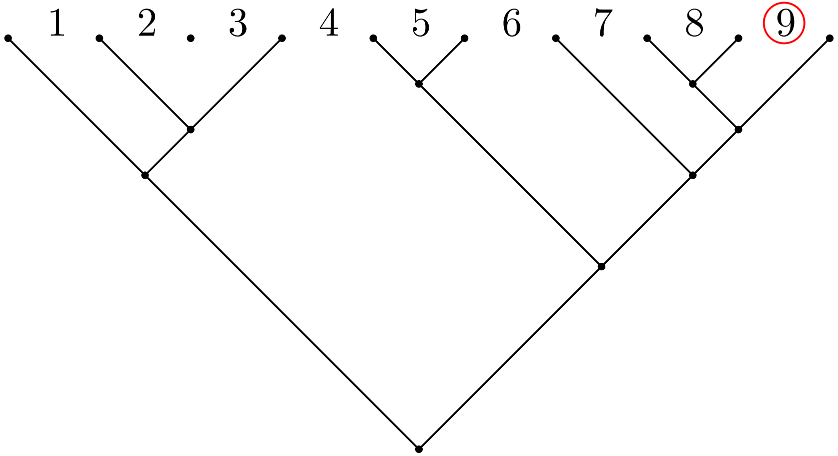

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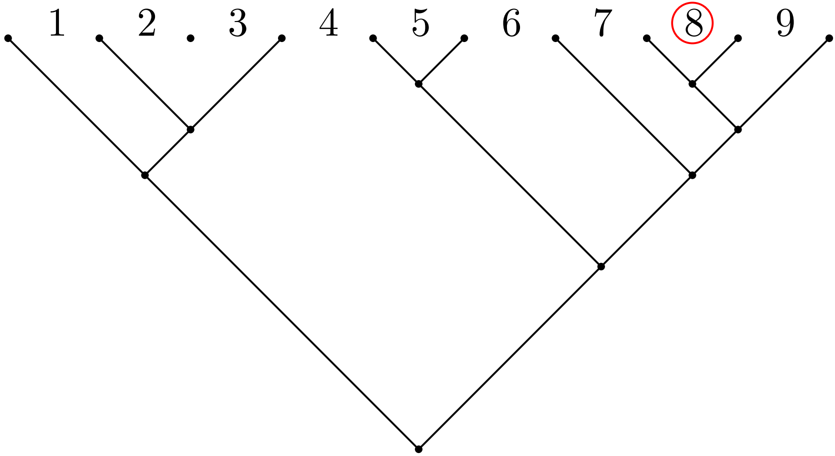

Permutations to planary binary trees, alternate map

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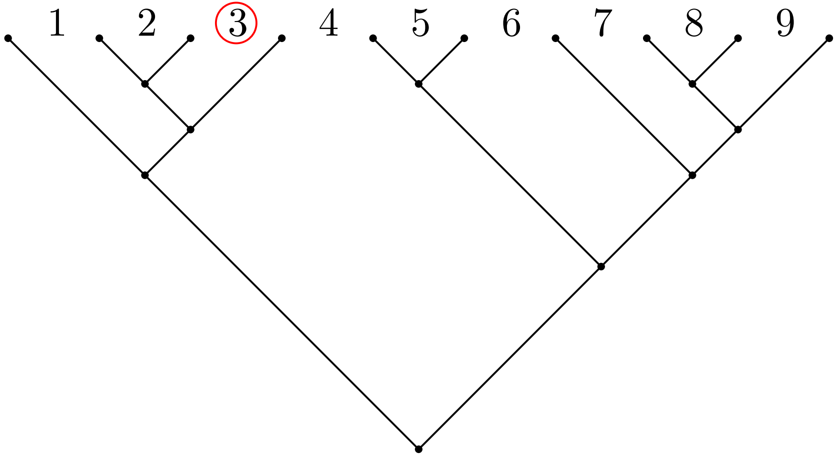
Example: $\rho_b(467198\textcolor{red}{3}52)$



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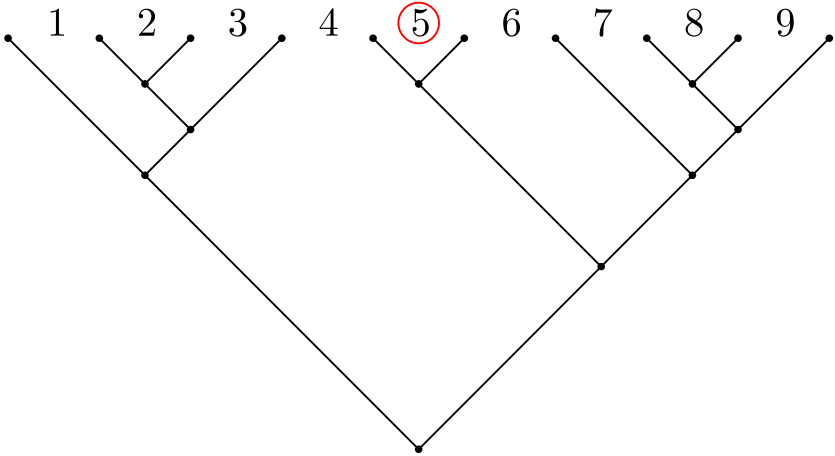

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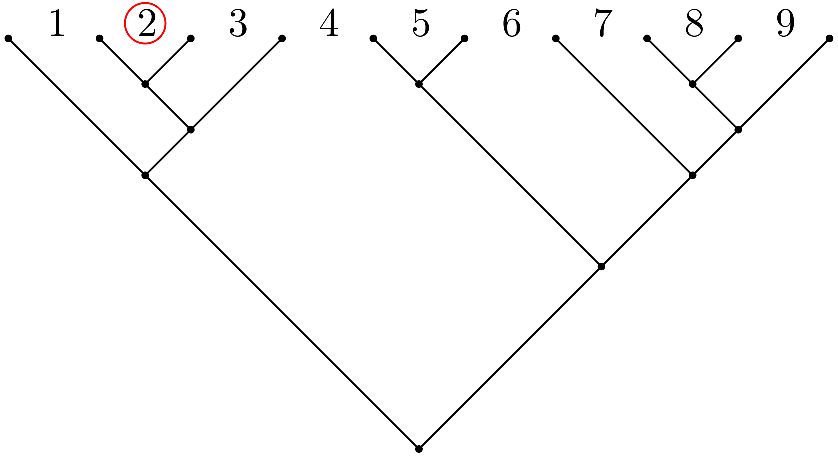

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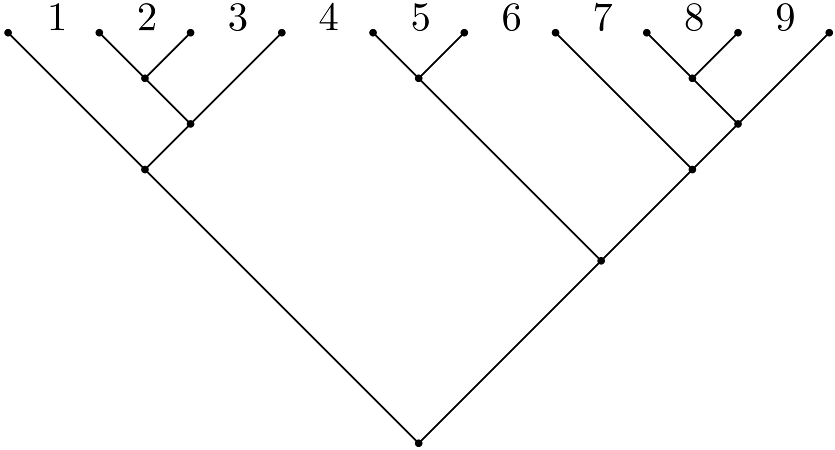

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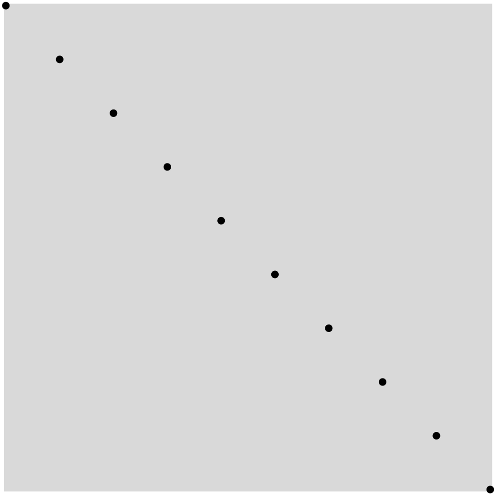

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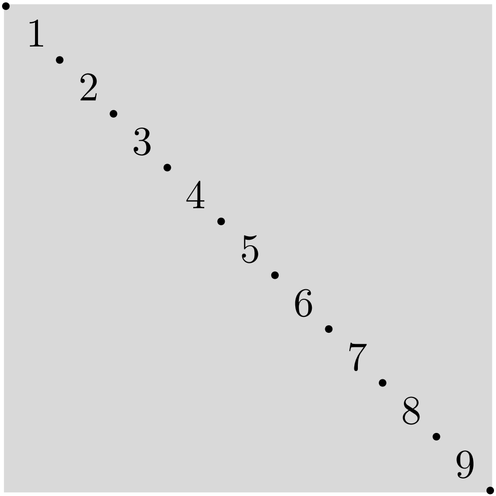

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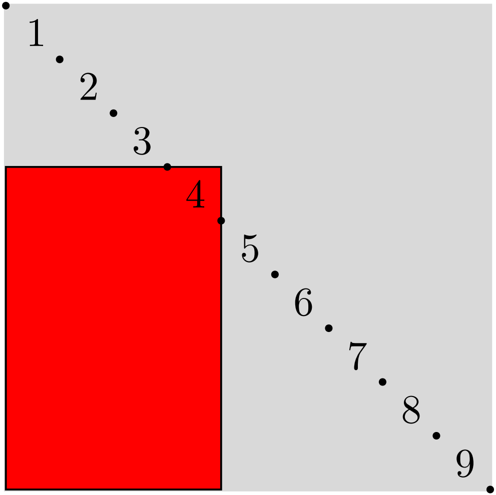

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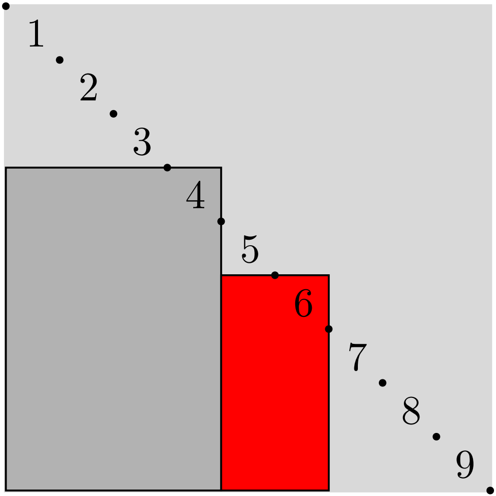

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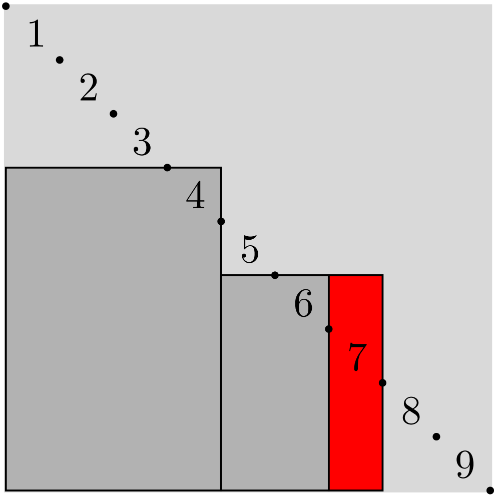

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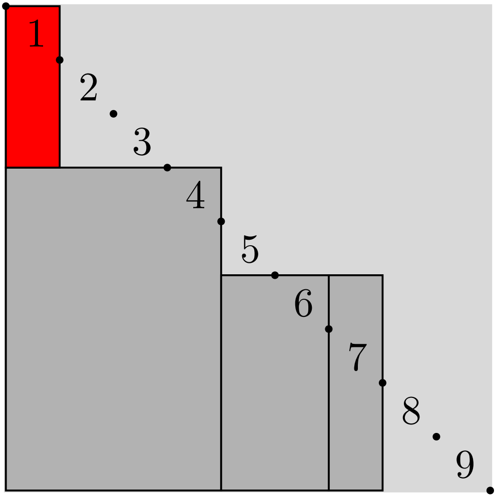

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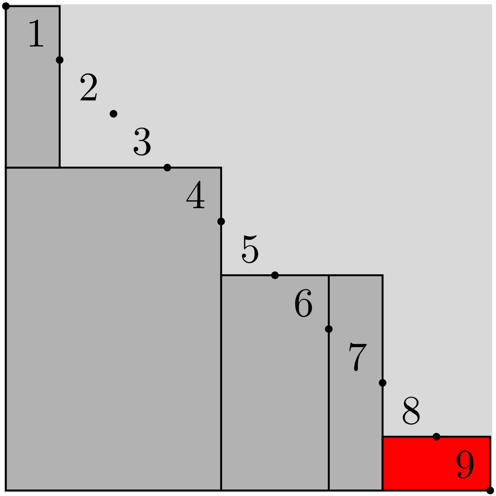

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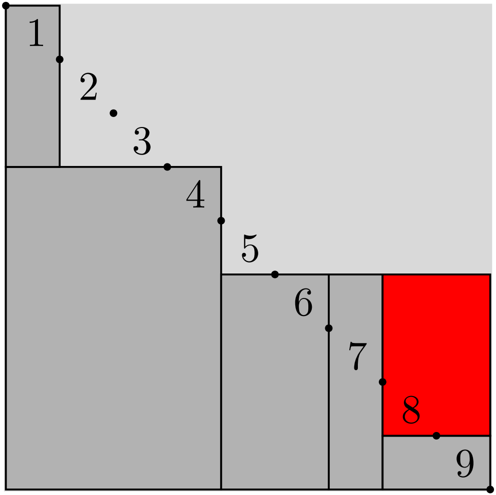

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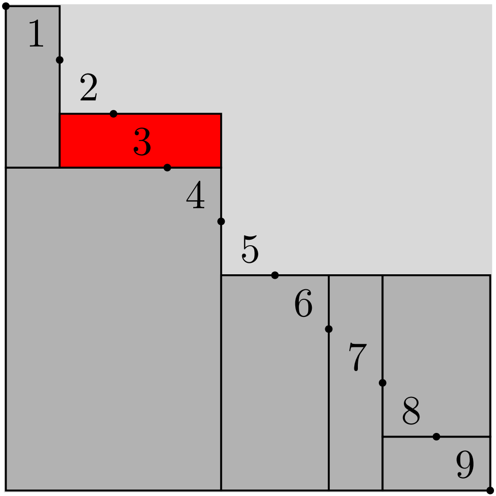

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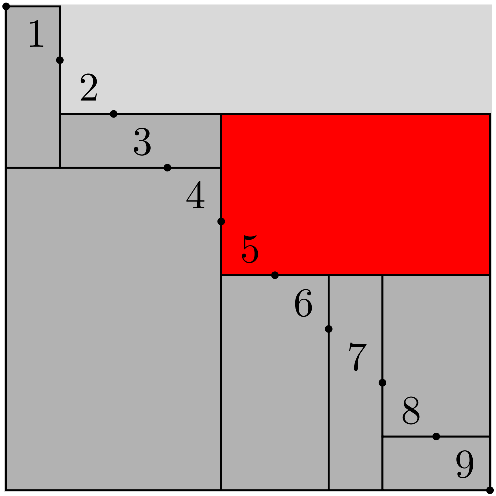

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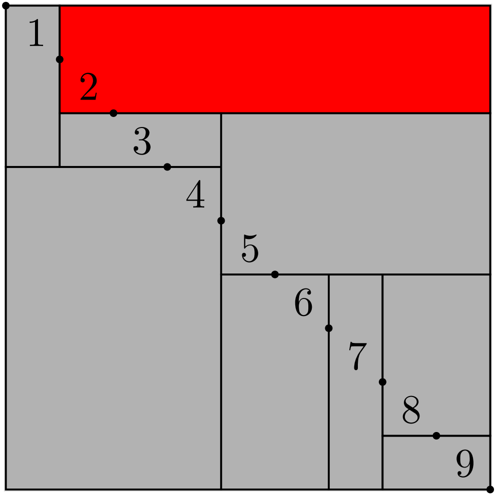

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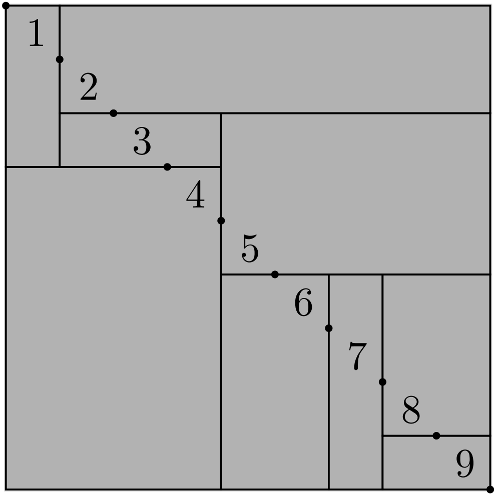


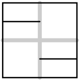
Results

Hopf algebra: There is a combinatorially natural product and coproduct on rectangulations, making a Hopf algebra Rec. The dual map to ρ embeds Rec as a sub Hopf algebra of MR, isomorphic (equal) to tBax.

Lattice: ρ induces a partial order on rectangulations. This is a lattice and ρ is a lattice homomorphism. Cover relations are given by "pivots" on rectangulations. (Cf. the Tamari lattice.)

Polytope: There is a polytope whose vertices are labeled by rectangulations and whose edges correspond to pivots on rectangulations. (Cf. the associahedron.) This polytope is the Minkowski sum of two realizations of the associahedron. (We use results of Hohlweg and Lange, '06 about these realizations.)

Product on Rec

 ·  = sum of completions of 

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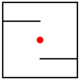
 $\cdot$  $=$ sum of completions of 

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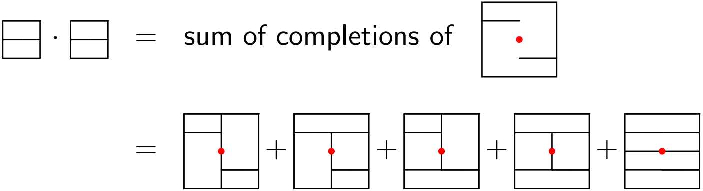

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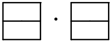
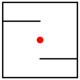
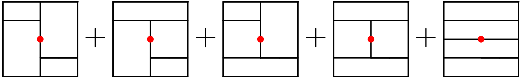
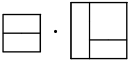
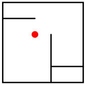
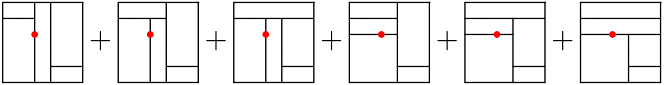
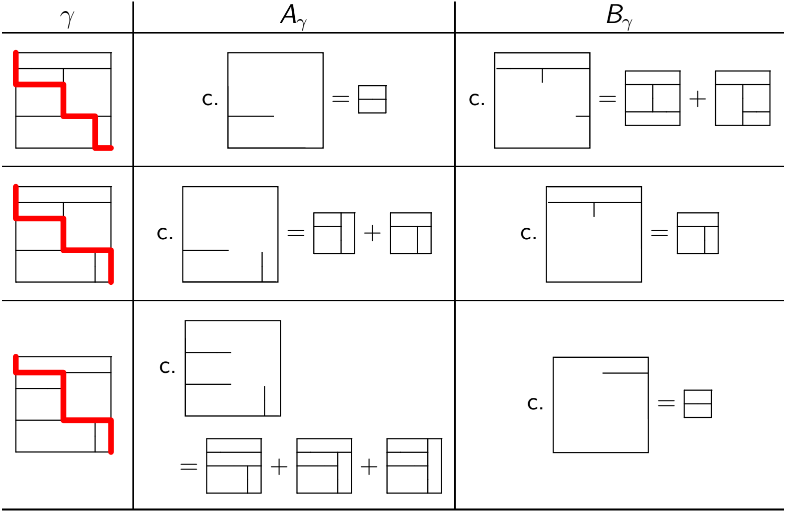
 = sum of completions of 

= 

 = sum of completions of 

= 

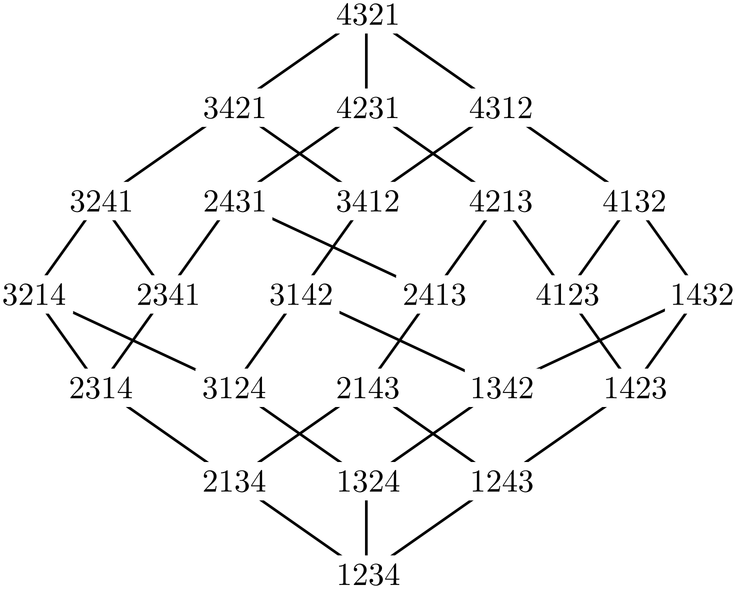
Coproduct on Rec: Sum $A_\gamma \otimes B_\gamma$ over paths γ



(+ terms for 6 other paths).

"c." is "sum of completions obtained by extending line segments."

Lattice of rectangulations



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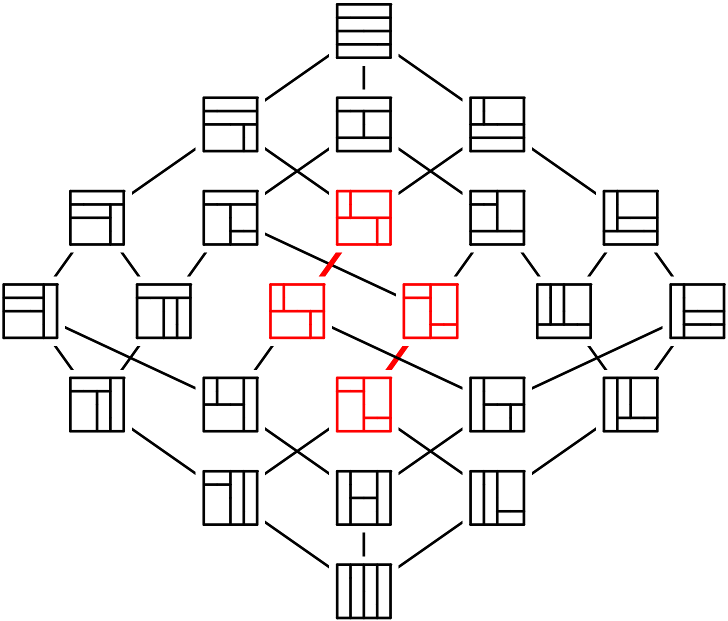
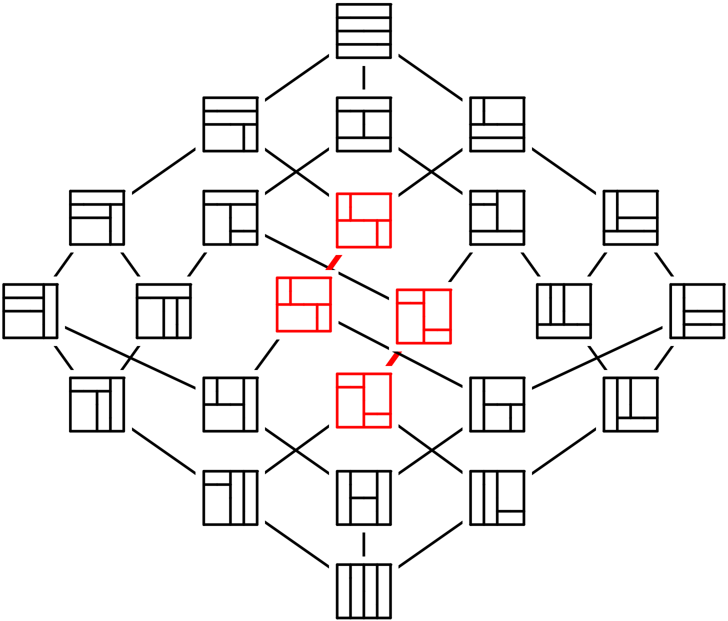


Lattice of rectangulations



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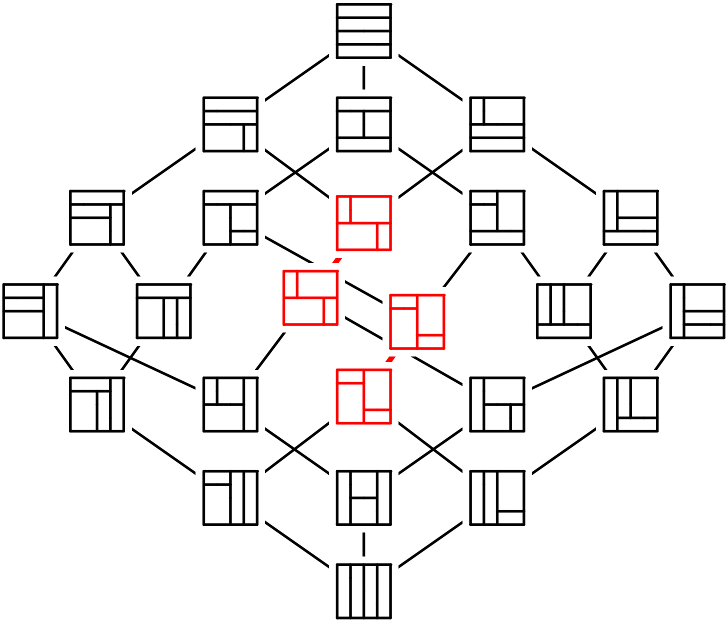

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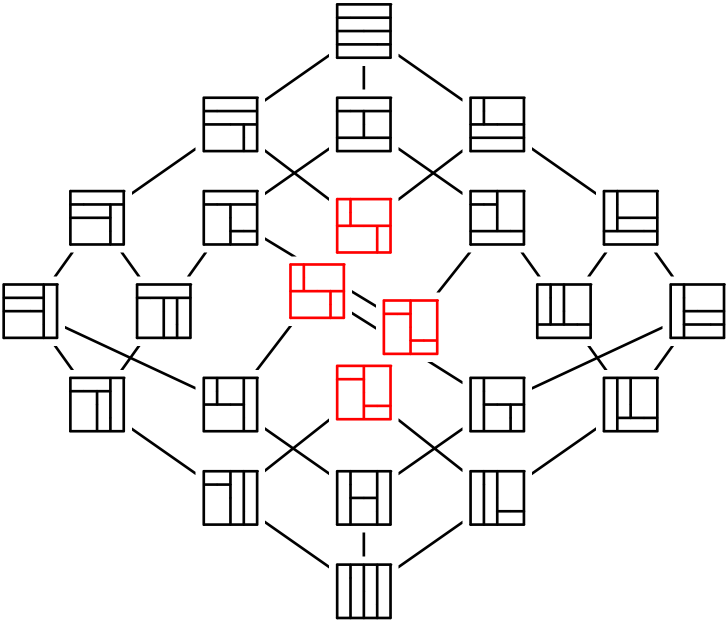

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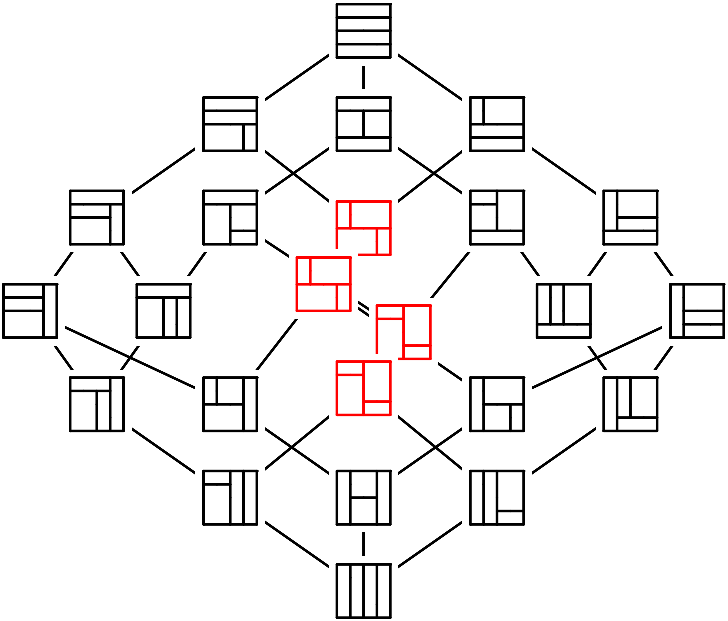

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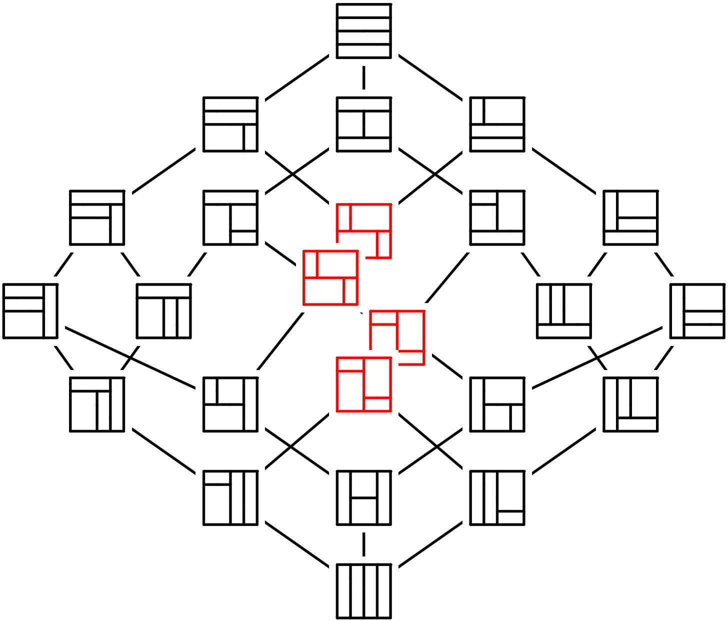

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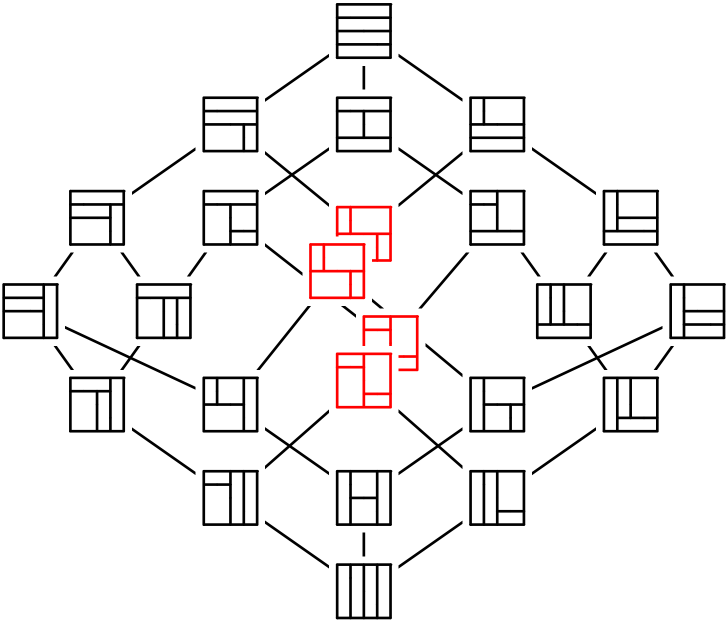

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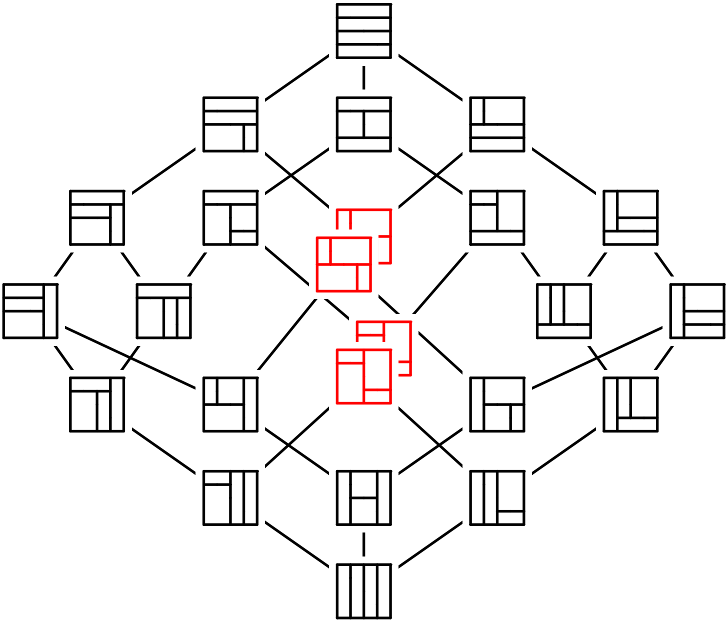

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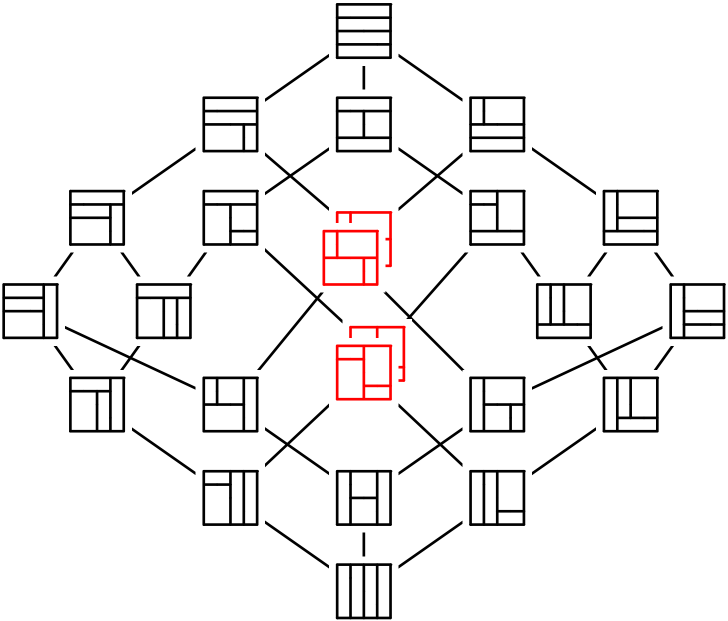

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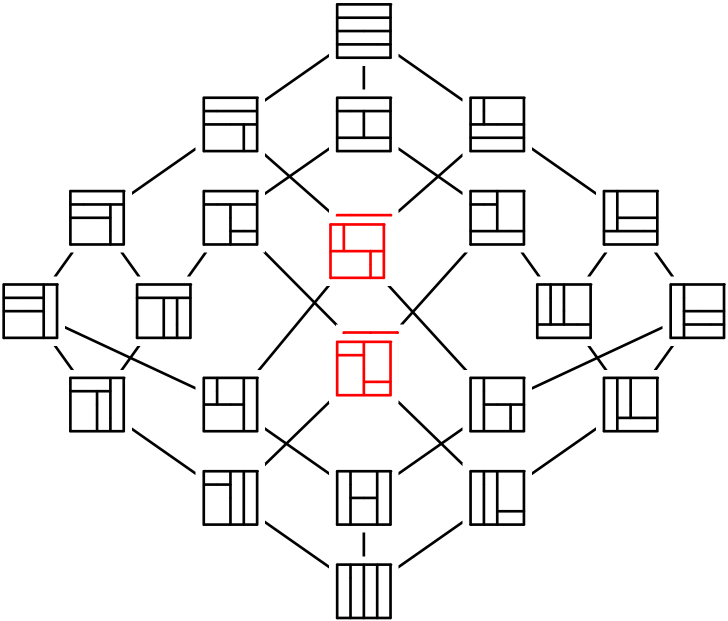

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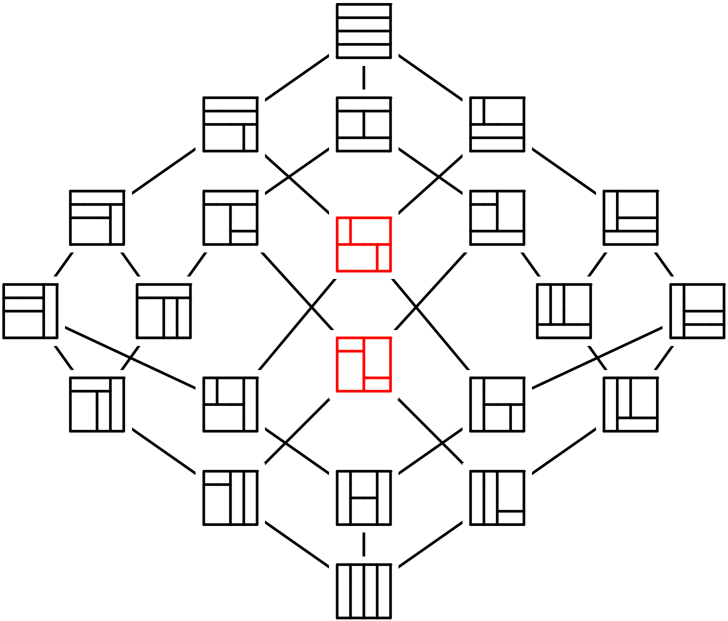

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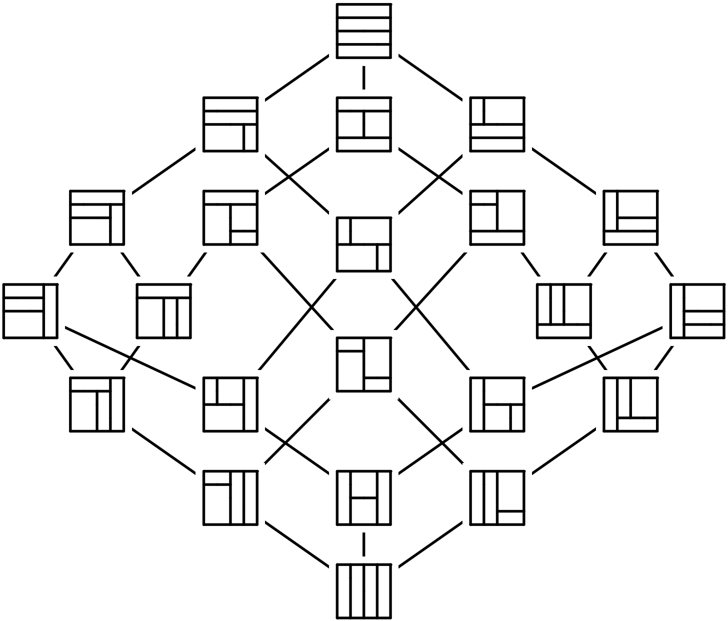
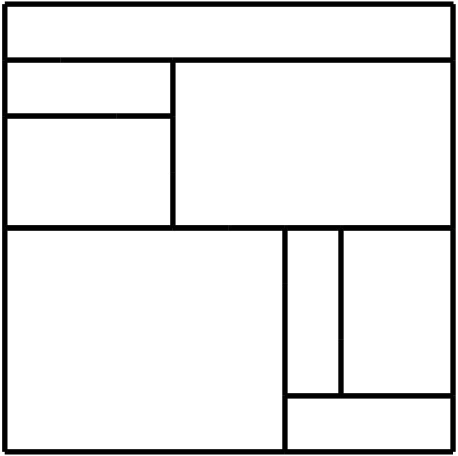


Pivots



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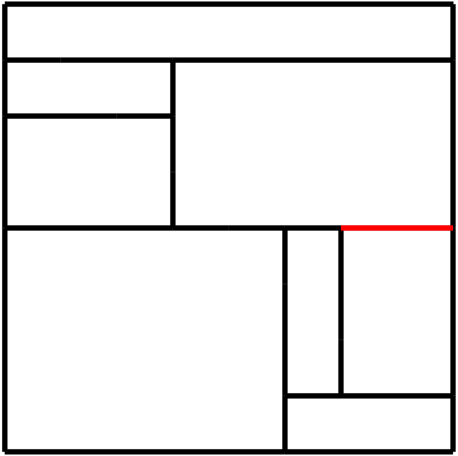

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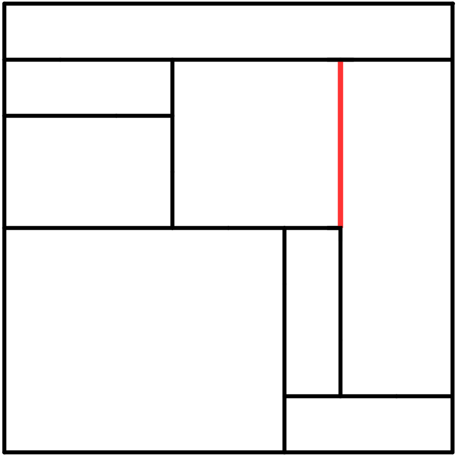

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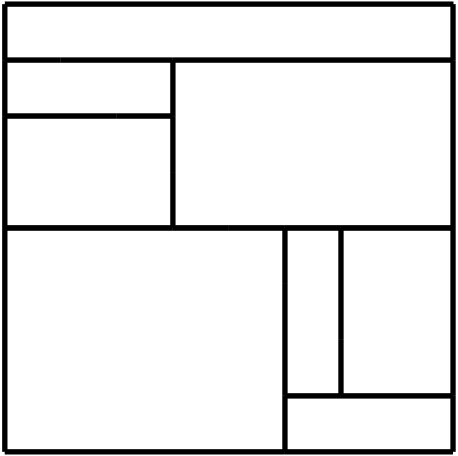

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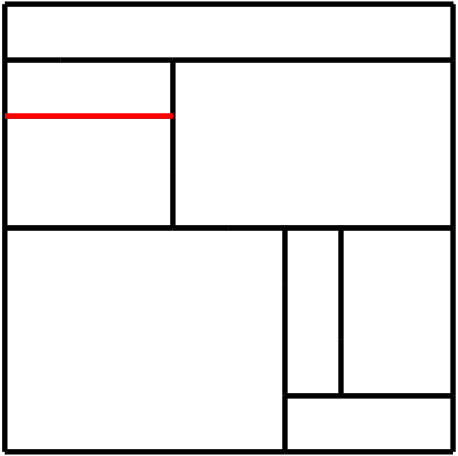

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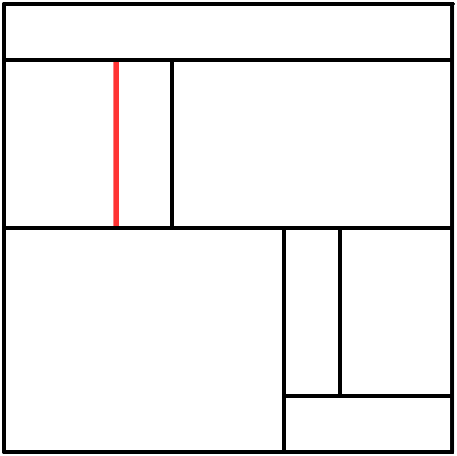

Thank you.