Lattice theory of the poset of regions, with applications to W-Catalan combinatorics.

Nathan Reading

NC State University

AMS Special Session on Arrangements and Related Topics
New Orleans, LA, January 5–6, 2007

Main point

$\mathcal{A}$: a simplicial hyperplane arrangement.
Given a lattice congruence Θ on the poset of regions of $\mathcal{A}$, we define a coarsening $\mathcal{F}_\Theta$ of the fan defined by $\mathcal{A}$.

In the special case where $\mathcal{A}$ is a Coxeter arrangement, the poset of regions is the weak order on W. For a particular choice of Θ, $\mathcal{F}_\Theta$ is the Cambrian fan:
Maximal cones are counted by the W-Catalan number.
Cambrian fan is combinatorially dual to the W-associahedron.

In particular this constructs the combinatorial backbone of cluster algebras of finite type directly from the lattice theory and geometry of the weak order.

The poset of regions (Edelman, 1985)

$\mathcal{A}$: a (central) hyperplane arrangement in a real vector space.
Regions: connected components of the complement of $\mathcal{A}$.
B: a distinguished "base" region.
Separating set of a region R: The set of hyperplanes in $\mathcal{A}$ separating R from B.
Poset of regions $\mathcal{P}(\mathcal{A}, B)$: The partial order on regions given by containment of separating sets. Alternately, take the zonotope dual to $\mathcal{A}$ and direct its 1-skeleton by a linear functional.

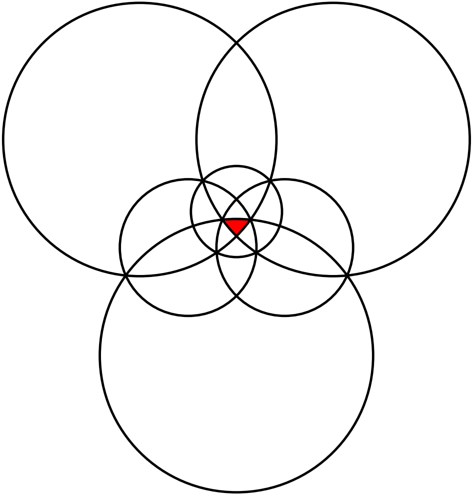
Examples

Finite Boolean lattices: Take $\mathcal{A}$ to be the coordinate hyperplanes.
Weak order on a finite Coxeter group W: Take $\mathcal{A}$ to be the set of all reflecting hyperplanes of W.

The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

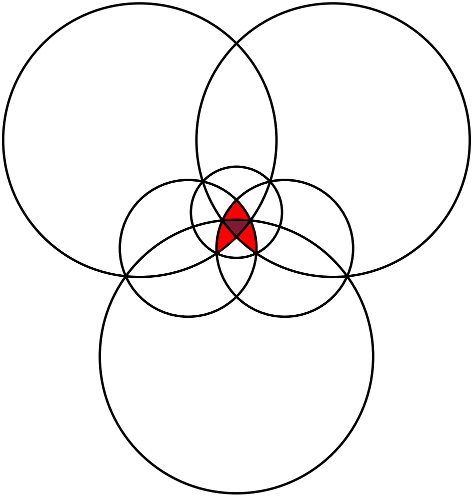
Zonotope: permutohedron. Poset of regions: weak order on S_4.

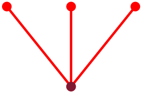


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.

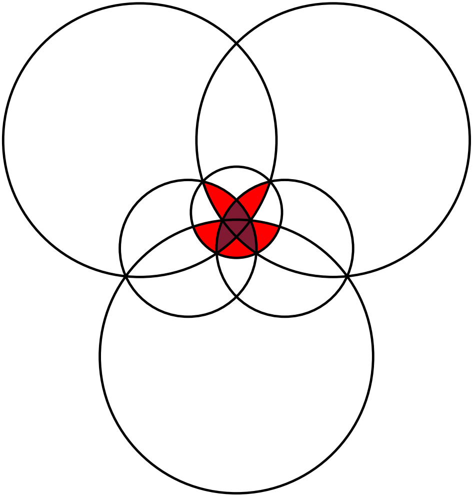


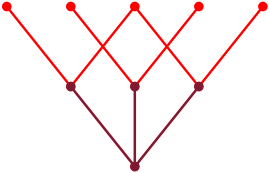


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.

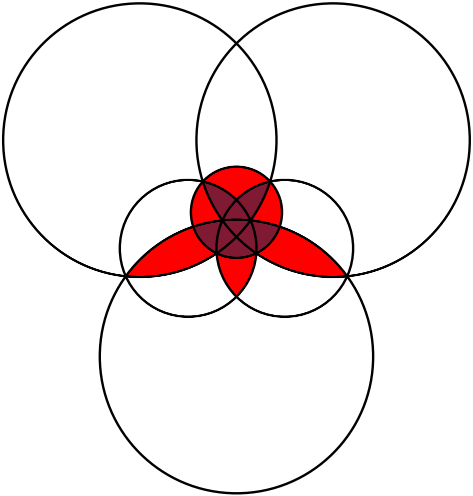


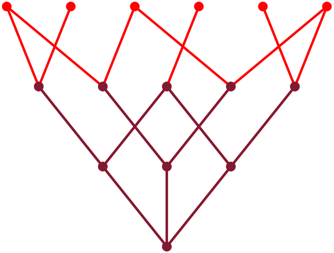


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.

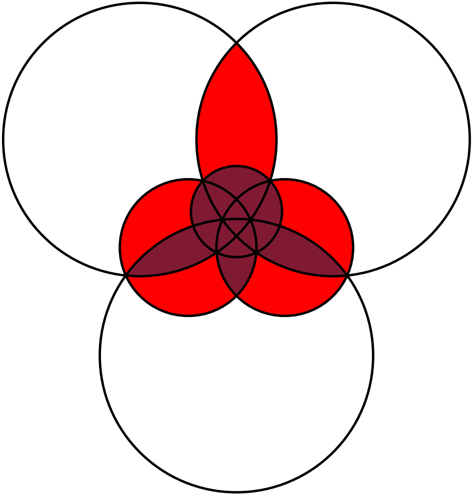


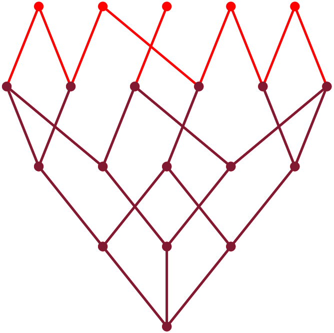


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.

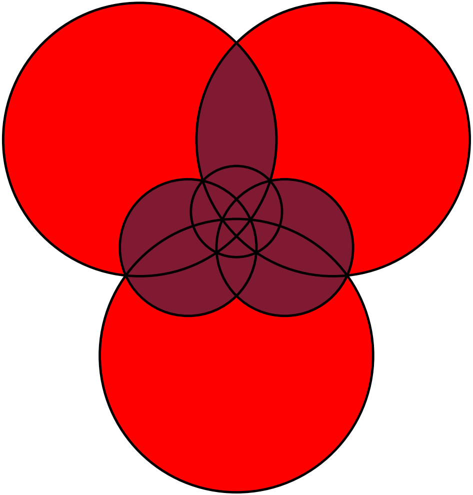


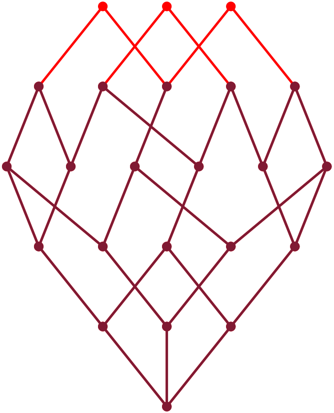


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.

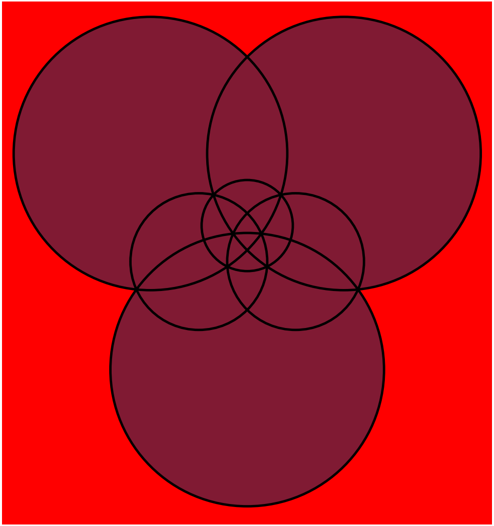


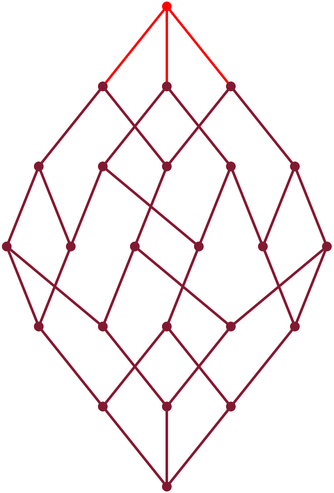


The poset of regions (continued)

Example (Planes of reflective symmetry of regular tetrahedron)

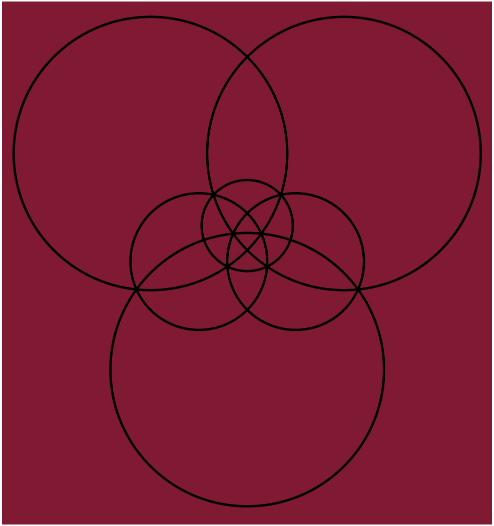
Zonotope: permutohedron. Poset of regions: weak order on S_4.

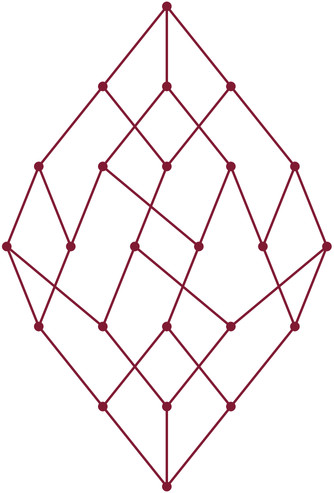




Example (Planes of reflective symmetry of regular tetrahedron)

Zonotope: permutohedron. Poset of regions: weak order on S_4.





Congruences of a finite lattice

L: a lattice.

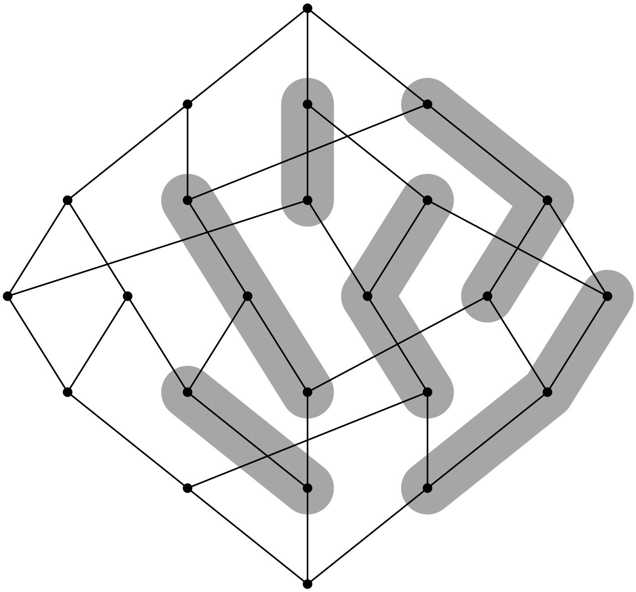
Congruence on L: an equivalence relation on L given by the fibers of some lattice homomorphism $L \to L'$.

Key facts about congruences of finite lattices

1. Each congruence class is an interval.
2. Projection to bottom element of class is order-preserving.
3. Projection to top element of class is order-preserving.

In fact 1, 2 and 3 characterize congruences on finite lattices.

Example (A lattice congruence on the weak order on S_4)



Fans

V: a real vector space.

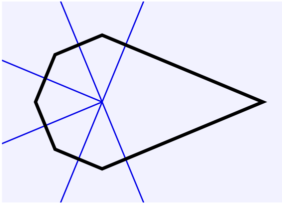
A (complete) fan is a decomposition of V into convex cones with "nice" intersections. (Cf. polyhedral complex).

Example (The normal fan of a polytope P in V)

Define an equivence relation on functionals in the dual space to V:

$$f \equiv f' \text{ if and only if } f, f' \text{ maximized on the same face of } P.$$

For example, a polygon and its normal fan:



Example (Fan defined by a central hyperplane arrangement)

Cones in this fan are the regions, together with all their faces.
This is the normal fan of the corresponding zonotope.

Coarsening fans by lattice congruences (R., 2004)

Simplicial fan: all cones are simplicial.
Simplicial hyperplane arrangement: cuts space into a simplicial fan.

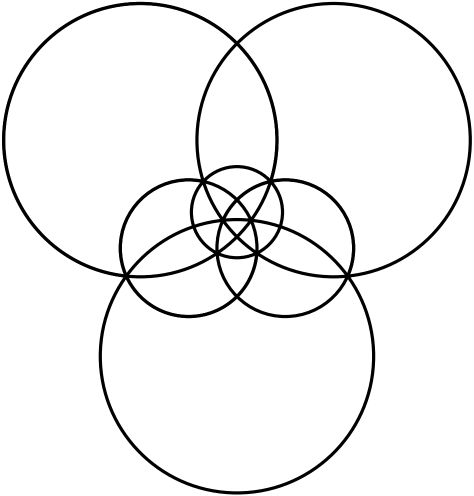
Theorem (Bjorner, Edelman, Ziegler, 1987)

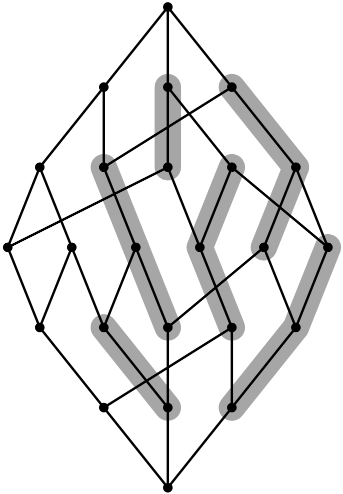
If $\mathcal{A}$ is simplicial then $\mathcal{P}(\mathcal{A}, B)$ is a lattice for any base region B.

Θ: any lattice congruence on $\mathcal{P}(\mathcal{A}, B)$.
$\mathcal{F}_\Theta$: a collection of cones:
Maximal cones of $\mathcal{F}_\Theta$: unions, over congruence classes of Θ, of maximal cones of the fan defined by $\mathcal{A}$.

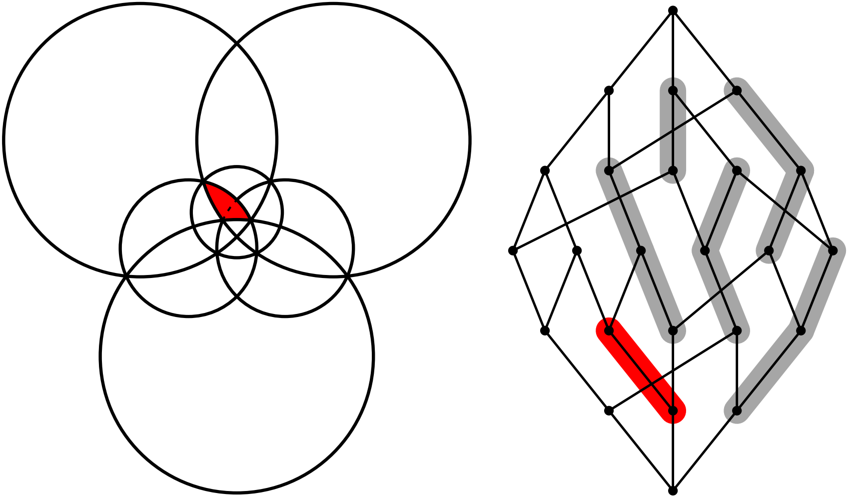
These maximal cones are convex (congruence classes are intervals in $\mathcal{P}(\mathcal{A}, B)$). Using the order-preserving projections, one checks that they intersect "nicely." Thus $\mathcal{F}_\Theta$ is a fan.

Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)

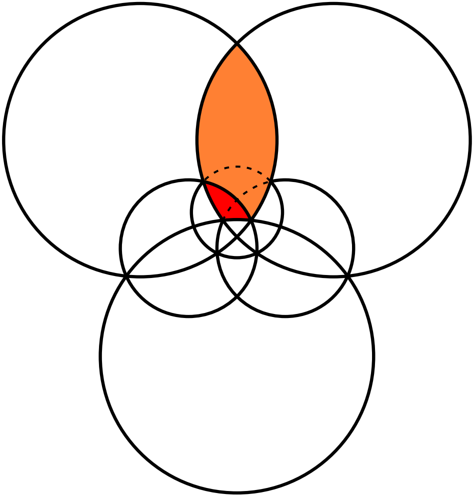


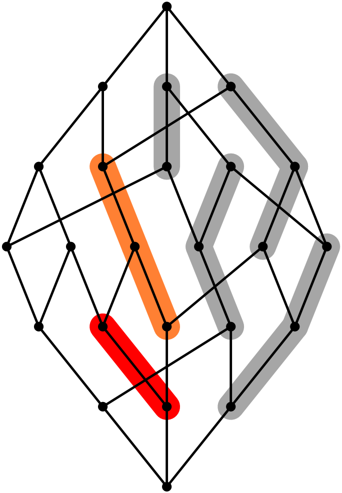


Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)

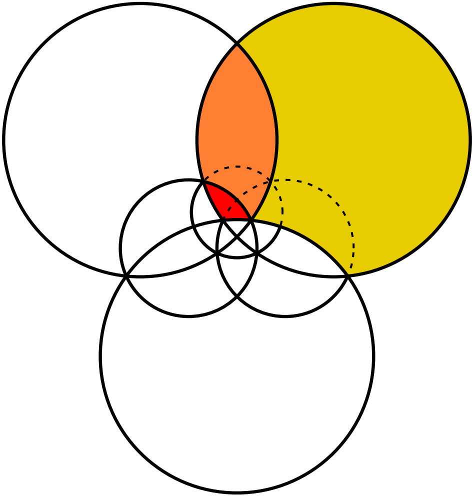


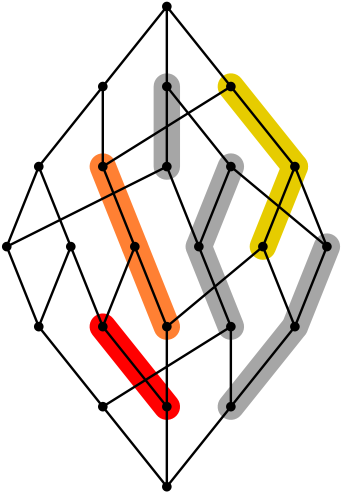
Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)



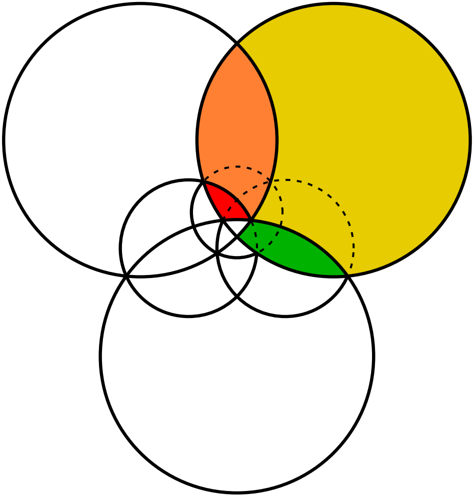


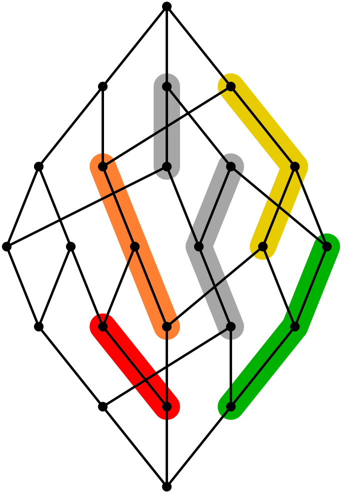
Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)



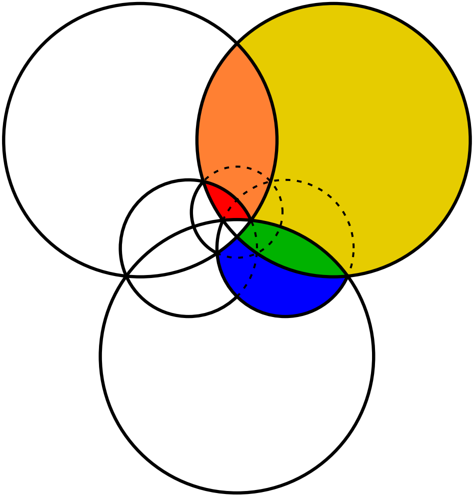


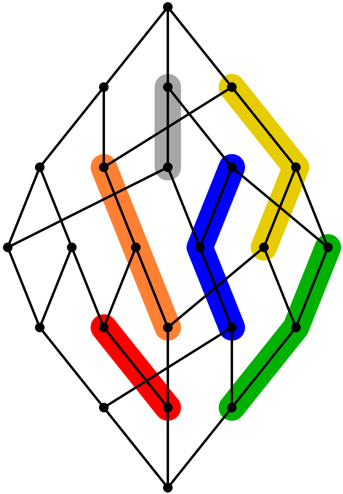
Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)



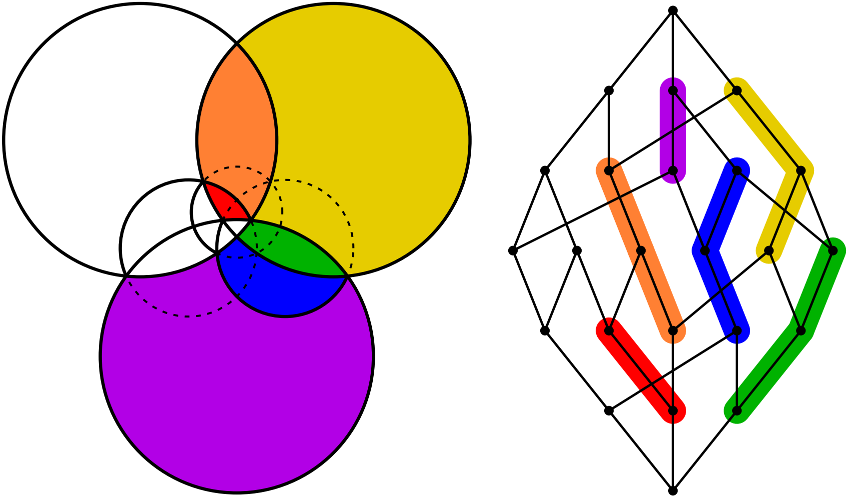


Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)

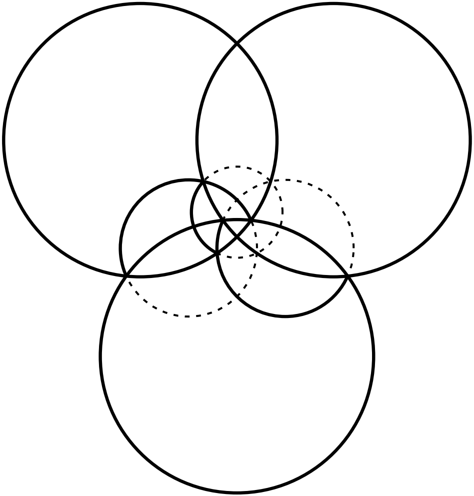


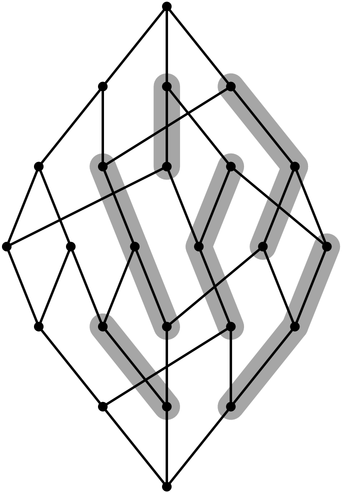


Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)



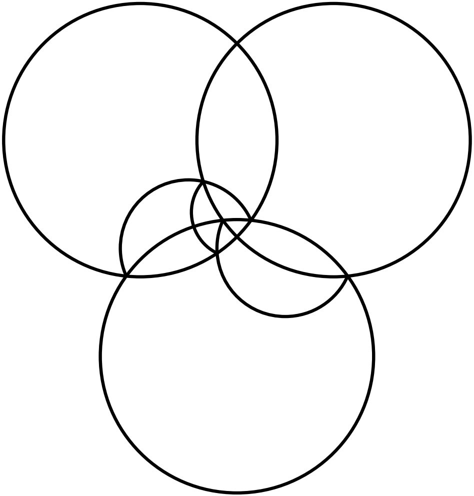
Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)

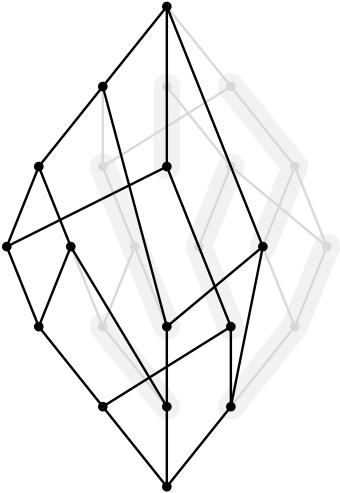




Example ($\mathcal{F}_\Theta$ for a congruence on the weak order on S_4)

$\mathcal{F}_\Theta =$ normal fan of associahedron. $\mathcal{P}(\mathcal{A}, B)/\Theta =$ Tamari lattice.





Generating a lattice congruence

Θ: a congruence relation on a finite lattice.
Since Θ-classes are intervals, Θ is completely determined by edge-equivalences $x \equiv y$ where x covers y.

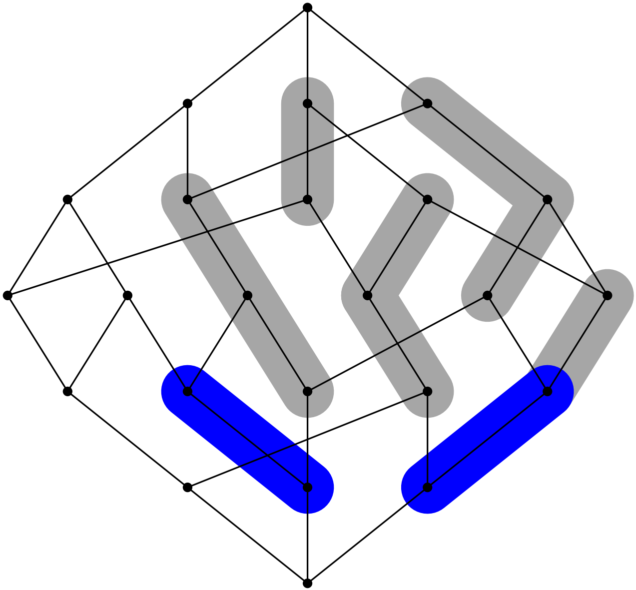
Furthermore, for any set of edge-equivalences, there is a unique coarsest congruence containing those equivalences. In other words, a congruence can be generated by specifying a small number of edge-equivalences.

Generalized associahedra (R., 2004–R., Speyer, 2006)

W: a finite Coxeter group W.
By a simple rule, specify a small number of edge-equivalences, generating a congruence Θ. The fan $\mathcal{F}_\Theta$ is combinatorially isomorphic to the normal fan of the W-associahedron.
Recently, Hohlweg, Lange and Thomas showed that $\mathcal{F}_\Theta$ is the normal fan of a polytope.

Example

This congruence is generated by the **blue edge-equivalences**.



Forcing

Edge-equivalences are not independent; one edge-equivalence in general forces many others.

Example

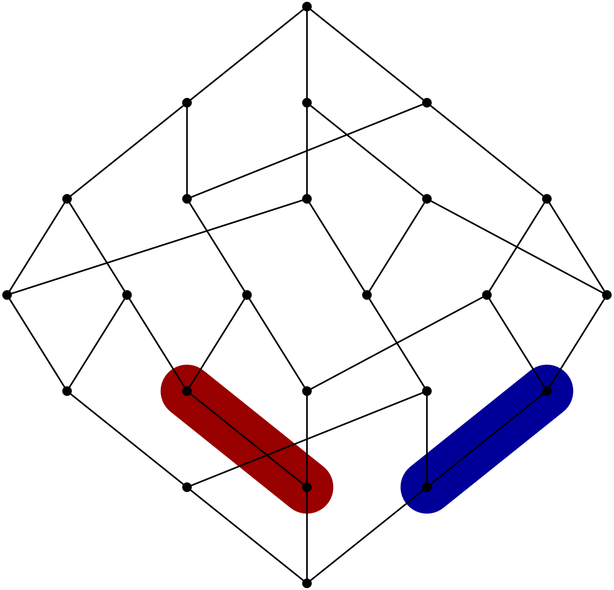
Forcing in $\mathcal{P}(\mathcal{A}, B)$ (R., 2004)

For many simplicial arrangements $\mathcal{A}$, forcing is completely determined by such local moves on intervals in $\mathcal{P}(\mathcal{A}, B)$.
(In particular, when $\mathcal{A}$ is the arrangement for a reflection group.)

Furthermore, this forcing can be described entirely in terms of the geometry of the arrangement.

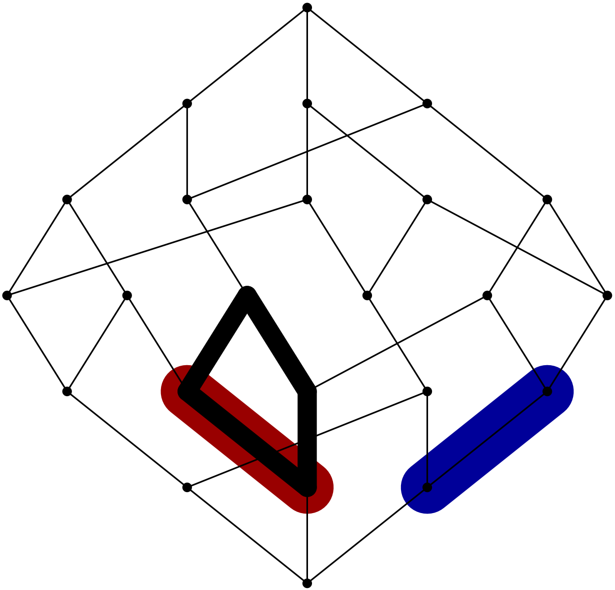
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



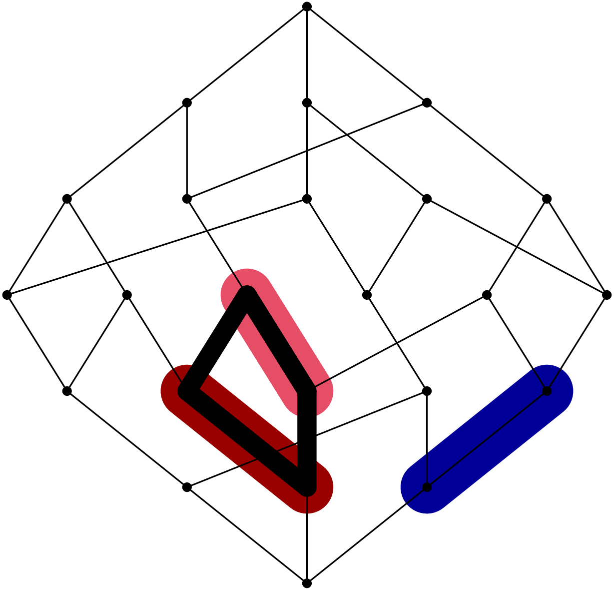
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



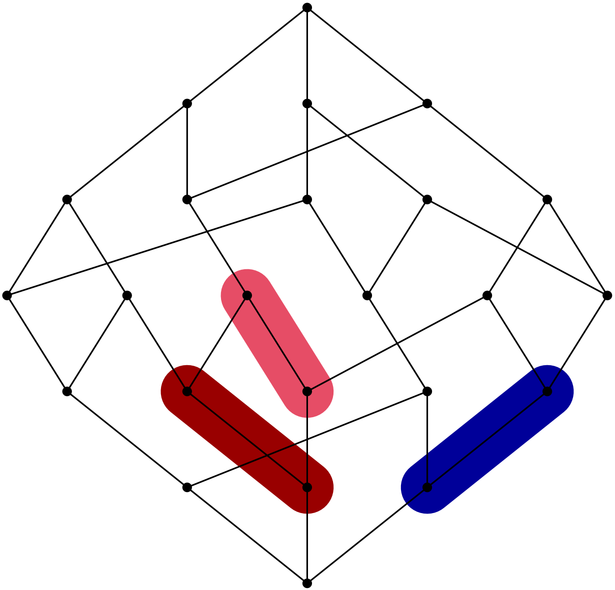
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



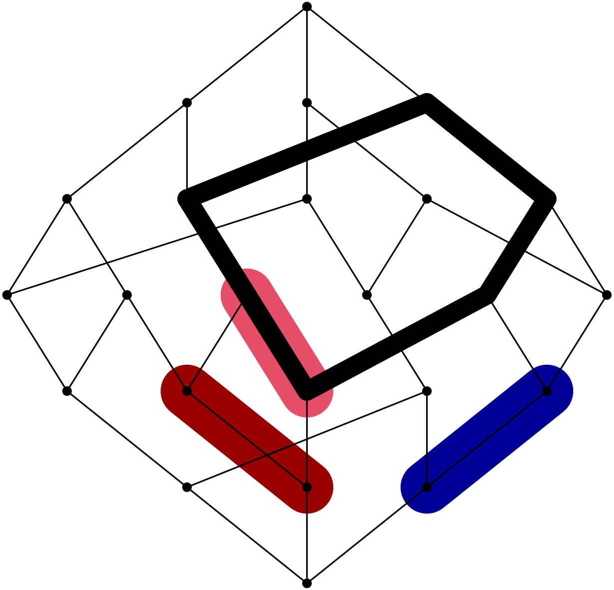
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



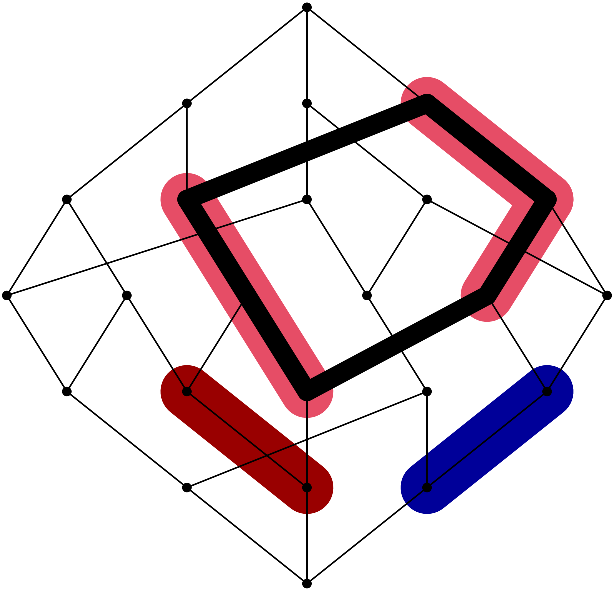
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



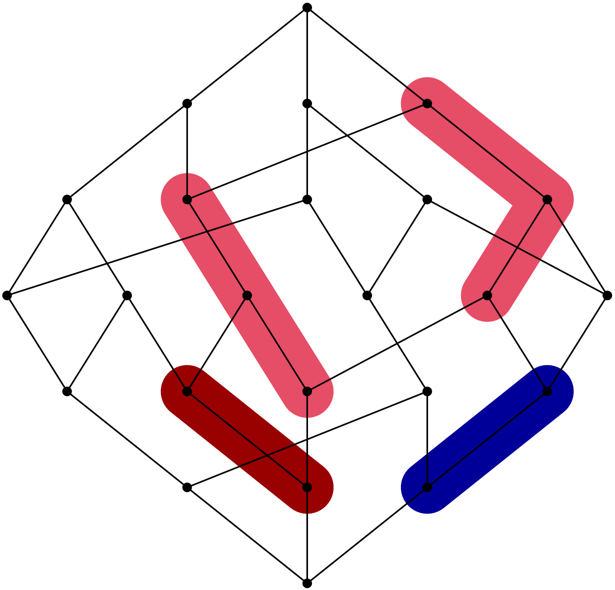
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



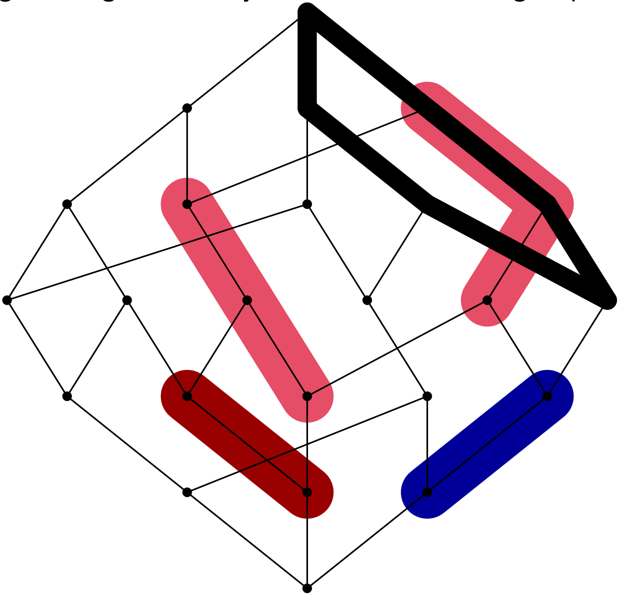
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



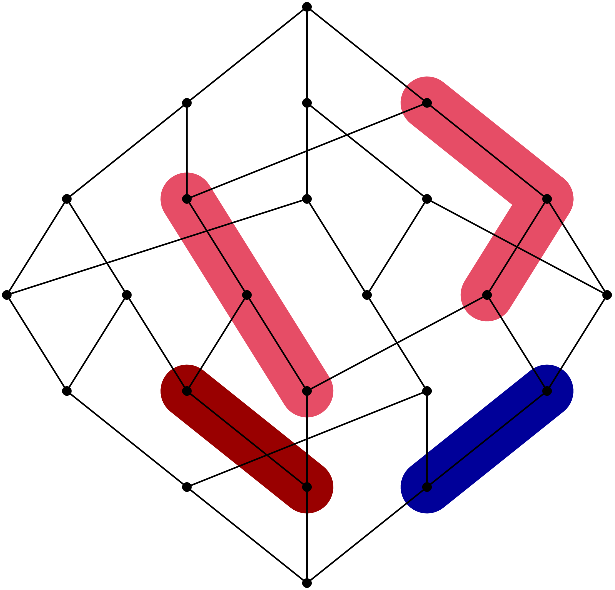
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



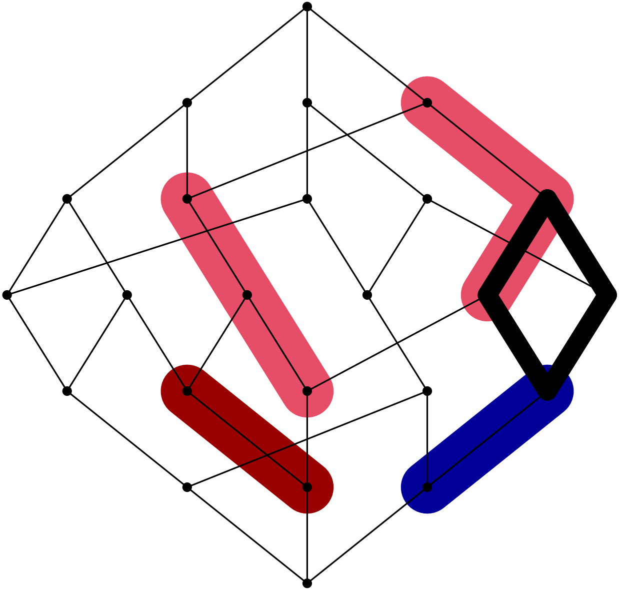
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



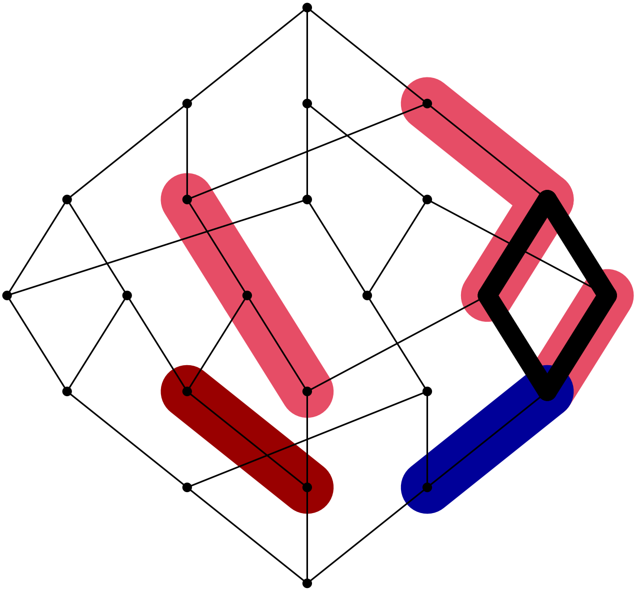
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



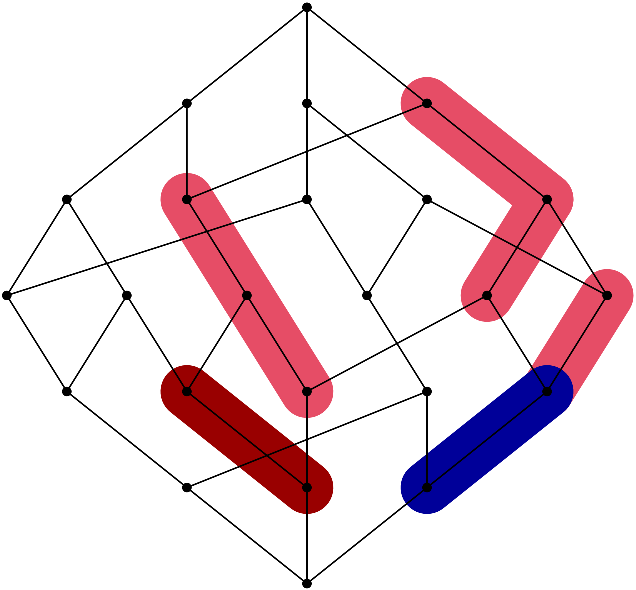
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



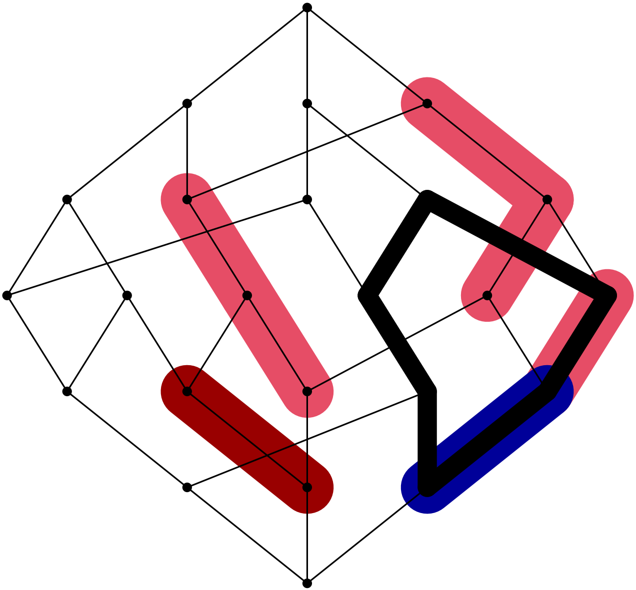
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



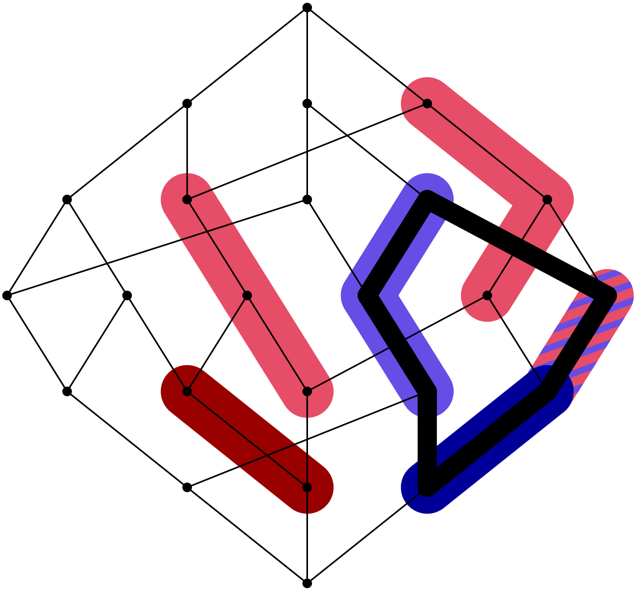
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



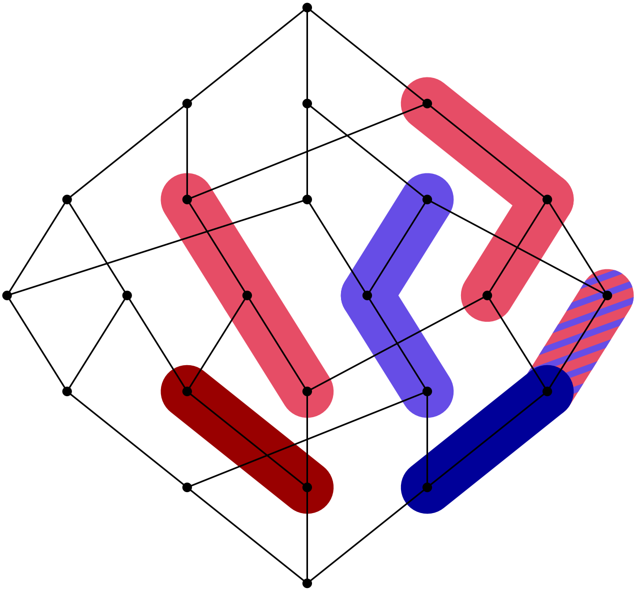
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



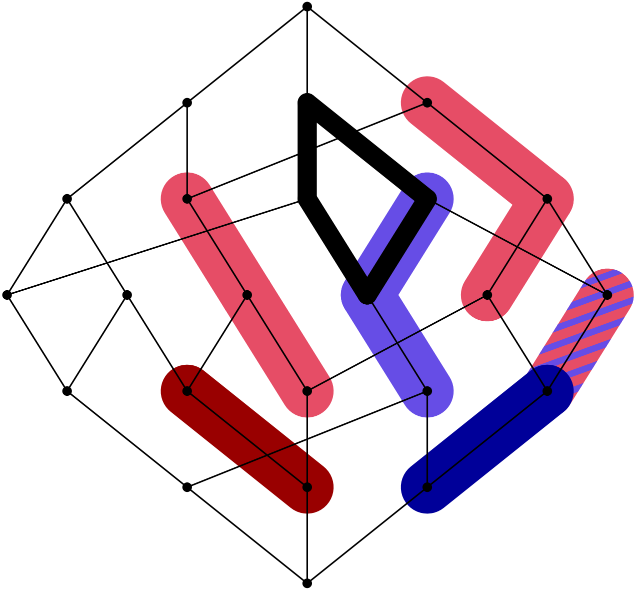
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



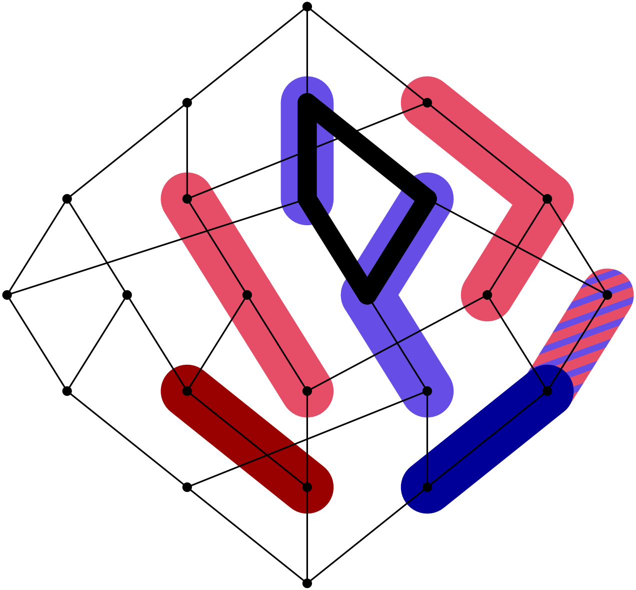
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



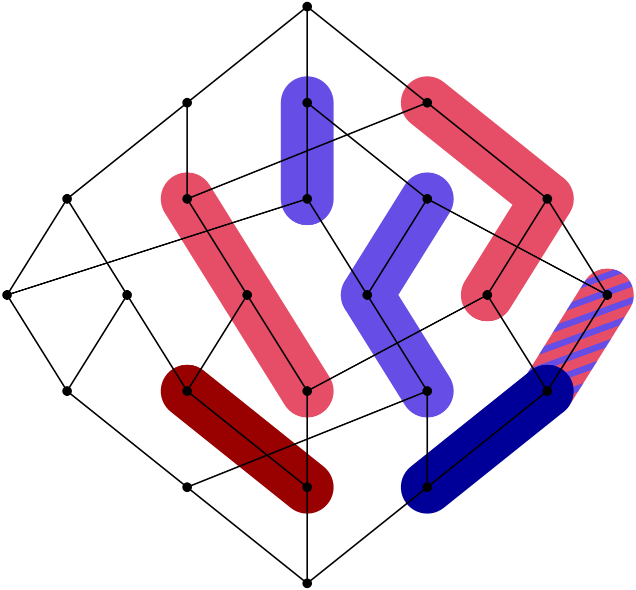
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



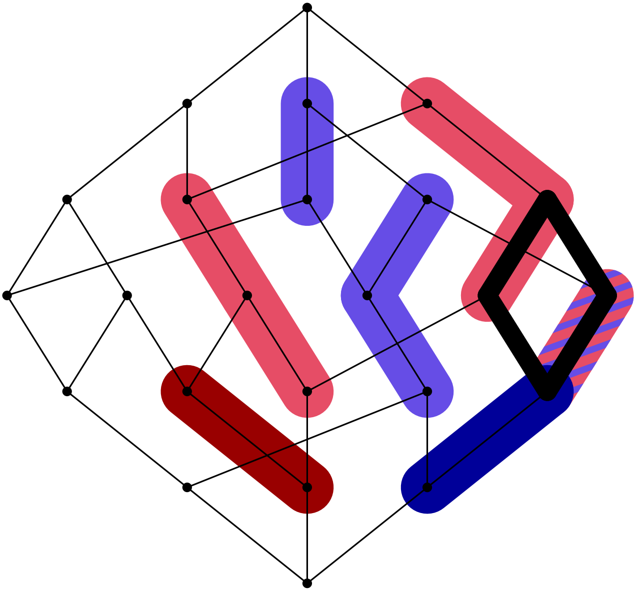
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



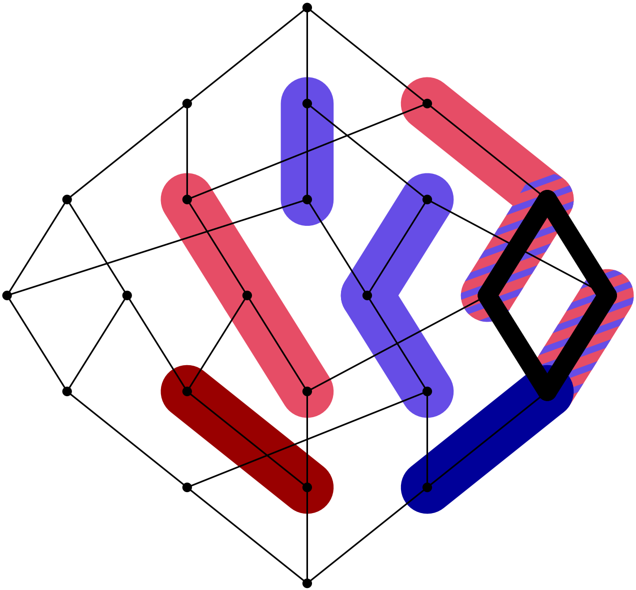
Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.



Forcing example

The congruence generated by the **red** and **blue** edge-equivalences.