Mutation-linear maps

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FPSAC 2014, July 2, 2014

Mutation-linear algebra

The mutation fan

When is the identity map mutation-linear?
Linear algebra = the study of linear relations:

formal finite linear combinations $\sum c_i v_i$ that evaluate to 0.

- $U \subseteq \mathbb{R}^n$ is independent if there are no nontrivial linear relations supported in $U$.
- $U \subseteq \mathbb{R}^n$ is spanning if for every $a \in \mathbb{R}^n$, there exists a linear relation $a - \sum c_i v_i$ with $\{v_i\} \subseteq U$.
- A map $\eta : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for every linear relation $\sum c_i v_i$ in $\mathbb{R}^n$, the expression $\sum c_i \eta(v_i)$ is a linear relation in $\mathbb{R}^m$. 

1. Mutation-linear algebra
Partial linear structures

Distinguish a subset $\mathcal{A}$ (the “active” linear relations) of all linear relations. Reformulate linear algebra in terms of $\mathcal{A}$. 

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• $\eta: (\mathbb{R}^n, \mathcal{A}) \rightarrow (\mathbb{R}^m, \mathcal{A}')$ is linear if for each active linear relation $\sum c_i v_i$ in $\mathbb{R}^n$, the expression $\sum c_i \eta(v_i)$ is an active linear relation. That is, $\sum c_i v_i \in \mathcal{A} \implies \sum c_i \eta(v_i) \in \mathcal{A}'$.

Comments:
• Some conditions $\implies \exists$ basis (usual Zorn’s lemma argument).
• Generality here for clarity only. I have exactly 1 kind of example.
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- Some conditions $\implies \exists$ basis (usual Zorn’s lemma argument).
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Let $B = (b_{ij})$ be $n \times n$ skew-symmetrizable integer matrix. (An exchange matrix.) Let $a \in \mathbb{R}^n$ and let $\tilde{B}$ be $[\begin{array}{c} B \end{array} a]$ (i.e. $B$ with an extra row $a$). For $k \in \{1, \ldots, n\}$, the mutation of $B$ in direction $k$ is $B' = \mu_k(B)$ with entries given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}; \\ b_{ij} + |b_{ik}|b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik}b_{kj} > 0; \\ b_{ij} & \text{otherwise.} \end{cases}$$

Example:

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} \xrightarrow{\mu_3} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & 3 & 0 \\ -1 & -3 & 1 & 1 \end{bmatrix}$$

For a sequence $k = k_q, k_{q-1}, \ldots, k_1$, similarly define $\mu_k(B)$. 

1. Mutation-linear algebra
We continue with $\tilde{B} = [B_a]$ and $k = k_q, k_{q-1}, \ldots, k_1$. Define $\eta^B_k(a)$ to be the coefficient row of $\mu_k(\tilde{B})$. Concretely, for $k = k$:

$\eta^B_k(a) = (a'_1, \ldots, a'_n)$, where

$$a'_j = \begin{cases} 
-a_k & \text{if } j = k; \\
ak + ak b_{kj} & \text{if } j \neq k, ak \geq 0 \text{ and } b_{kj} \geq 0; \\
ak - ak b_{kj} & \text{if } j \neq k, ak \leq 0 \text{ and } b_{kj} \leq 0; \\
a_j & \text{otherwise.}
\end{cases}$$

$\eta^B_k$ is linear in $\{a \in \mathbb{R}^n : a_k \geq 0\}$ and linear in $\{a \in \mathbb{R}^n : a_k \leq 0\}$.

The maps $\eta^B_k$ are the mutation maps associated to $B$. They are piecewise-linear homeomorphisms of $\mathbb{R}^n$. Their inverses are also mutation maps.
Example: \( B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \)

\[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}
\xrightarrow{\mu_1}
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\begin{bmatrix} -a_1 \\ ? \end{bmatrix}
\]

\[
? = \begin{cases} 
  a_2 & \text{if } a_1 \leq 0 \\
  a_2 + a_1 & \text{if } a_1 \geq 0
\end{cases}
\]

1. Mutation-linear algebra
Let $S$ be a finite set, let $(v_i : i \in S)$ be vectors in $\mathbb{R}^n$ and let $(c_i : i \in S)$ be real numbers.

The formal expression $\sum_{i \in S} c_i v_i$ is a $B$-coherent linear relation if

$$\sum_{i \in S} c_i \eta^B_k(v_i) = 0.$$

holds for every finite sequence $k = k_q, \ldots, k_1$.

In particular, $\sum_{i \in S} c_i v_i$ is a linear relation in the usual sense.
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\(^1\)I’m fibbing.
Write $\mathbb{R}^B$ for the partial linear structure on $\mathbb{R}^n$ whose active linear relations are the $B$-coherent linear relations. The study of $\mathbb{R}^B$ is mutation-linear algebra.

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- $U \subseteq \mathbb{R}^B$ is spanning if for every $a \in \mathbb{R}^n$, there exists a $B$-coherent linear relation $a - \sum c_i v_i$ with $\{v_i\} \subseteq U$.

What I’m not going to talk about: Basis for $\mathbb{R}^B = \text{Universal geometric coefficients for cluster algebras associated to } B$. 

What I am going to talk about: Mutation-linear maps. 

$\eta: \mathbb{R}^B \rightarrow \mathbb{R}^B$ is mutation-linear if for every $B$-coherent linear relation $\sum c_i v_i$, the expression $\sum c_i \eta(v_i)$ is a $B'$-coherent linear relation.
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What I am going to talk about: Mutation-linear maps.

- A map $\eta : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear if for every $B$-coherent linear relation $\sum c_i v_i$, the expression $\sum c_i \eta(v_i)$ is a $B'$-coherent linear relation.
Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
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Each map is linear on each colored region.

A basis for $B$: A nonzero vector in each of the 5 rays defining regions.

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The mutation fan

The example suggests an “easy” way to get $B$-coherent linear relations: Find vectors in the same domain of linearity of all mutation maps and make a linear relation among them.

Define an equivalence relation $\equiv^B$ on $\mathbb{R}^n$ by setting

$$a_1 \equiv^B a_2 \iff \text{sgn}(\eta^B_k(a_1)) = \text{sgn}(\eta^B_k(a_2)) \quad \forall k.$$ 

$\text{sgn}(a)$ is the vector of signs ($-1, 0, +1$) of the entries of $a$.

$B$-classes: equivalence classes of $\equiv^B$.

$B$-cones: closures of $B$-classes.

Mutation fan for $B$:

The collection $\mathcal{F}_B$ of all $B$-cones and all faces of $B$-cones.

**Theorem** (R., 2011). $\mathcal{F}_B$ is a complete fan (possibly with infinitely many cones).
Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$
A basis for $\mathbb{R}^B$ is positive if, for every $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the unique $B$-coherent linear relation $\mathbf{a} - \sum_{i \in I} c_i \mathbf{b}_i$ has all $c_i \geq 0$.

**Theorem** (R., 2014). If a positive basis exists for $\mathbb{R}^B$, then a map $\eta : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear if and only if for every $B$-cone $C$, the restriction $\eta|_{C}$ is a linear map into some $B'$-cone.
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*Feel free to not worry about positive bases for $\mathbb{R}^B$. Just think of this as “If $B$ is well-behaved, then…”*

*We’ll stay in the well-behaved case.*
We’ll restrict our focus to bijective mutation-linear maps. (These are not necessarily mutation-linear isomorphisms.)

We’ll restrict our focus further to the question:

> When is the identity map $\mathbb{R}^B \rightarrow \mathbb{R}^{B'}$ mutation-linear?

By the previous theorem:

$$\text{id} : \mathbb{R}^B \rightarrow \mathbb{R}^{B'}$$ is mutation-linear if and only if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$.

This can fail. The surprise is that this actually happens often.

A necessary* condition: $B$ dominates $B'$.

That is, $b_{ij}$ and $b'_{ij}$ weakly agree in sign and $|b_{ij}| \geq |b'_{ij}|$ for all $i, j$. 

3. When is the identity map mutation-linear?
Example: \( B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \quad B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \)

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Non-Example: \[ B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad B' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \]

3. When is the identity map mutation-linear?
So far

- Mutation maps: a family of piecewise linear maps given by matrix mutation.
- $B$-coherent linear relations: Linear relations preserved by all mutation maps.
- Mutation-linear maps $\mathbb{R}^B \to \mathbb{R}^{B'}$: Send $B$-coherent relations to $B'$-coherent relations.
- Mutation fan $\mathcal{F}_B$: Common domains of linearity of all mutation maps. For well-behaved $B$, mutation-linear maps are closely tied to mutation fans.
- We are focusing on when $\text{id} : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear. This is if and only if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$. Necessary: $B$ dominates $B'$. 

3. When is the identity map mutation-linear?
The rest of the talk

Some specific cases where id : $\mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear (i.e. $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$).

- Finite type ($\mathcal{F}_B$ is finite) acyclic:
  - $\mathcal{F}_B$ is dual fan to generalized associahedron (the Cambrian fan, or equivalently the $g$-vector fan).
  - $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ whenever $B$ dominates $B'$.
  - Refinement relations arises from a lattice-quotient relationship between Cambrian lattices.
  - There is a related lattice-quotient relationship between weak orders.

- Refinement relation suggests an algebraic relation between cluster algebras.

- “Affine associahedron fans.”

- “Resection” of triangulated surfaces.

3. When is the identity map mutation-linear?
Cambrian fans (finite type)

Each $B$ defines a Cartan matrix $A$.

E.g. \[ B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \]

**Coxeter fan**: Defined by the reflecting hyperplanes of the Coxeter group $W$ associated to $A$. Maximal cones $\leftrightarrow$ elements of $W$.

**Cambrian fan**: A certain coarsening of the Coxeter fan. Depends on the extra sign information that’s in $B$ (or equivalently, depends on a Coxeter element, or equivalently an orientation of the Coxeter diagram). Two ways to look at this:

- Coarsen according to a certain lattice congruence on $W$.
- Coarsen according to the combinatorics of “sortable elements.”

For $S_n$, the normal fan to the usual associahedron. (In general, generalized associahedron.)

3. When is the identity map mutation-linear?
For $B$ acyclic of finite type, $\mathcal{F}_B$ is a Cambrian fan. (Key technical point: identify fundamental weights with standard basis vectors.)

**Theorem** (R., 2013). For $B$ acyclic of finite type, $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ if and only if $B$ dominates $B'$.

Domination relations among exchange matrices imply domination relations among Cartan matrices. So the theorem is a statement that refinement relations exist among Cambrian fans when we decrease edge-labels (or erase edges) on Coxeter diagrams.

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3. When is the identity map mutation-linear?
Cluster variables: Generate the cluster algebra (not freely!).

Rays of the mutation fan $\mathcal{F}_B$ are in bijection with cluster variables.

Therefore, if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$, there is an inclusion

$$\{\text{rays of } \mathcal{F}_{B'}\} \hookrightarrow \{\text{rays of } \mathcal{F}_B\}$$

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Therefore, if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$, there is an inclusion

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Therefore there is a natural injective map on cluster variables. This extends (in all cases we have checked) to an embedding of $\mathcal{A}_0(B')$ as a subring of $\mathcal{A}_0(B)$. (You have to deal correctly with coefficients—make the map preserve $\mathbf{g}$-vectors).

Close algebraic relationships between different cluster algebras of the same rank are surprising $a$ $priori$.  

3. When is the identity map mutation-linear?
The **Cambrian lattice** $\text{Camb}_B$ is:

- A partial order on maximal cones in the Cambrian fan $\mathcal{F}_B$. The fan and the order interact very closely.

- A lattice quotient—and a sublattice—of the weak order on the finite Coxeter group associated to $B$.

One way to prove the refinement of fans is to show that there is a surjective lattice homomorphism from $\text{Camb}_B$ to $\text{Camb}_{B'}$.

**Theorem** (R., 2012). This happens for all acyclic, finite-type $B, B'$ with $B$ dominating $B'$.
Example: $A_3$ Tamari is a lattice quotient of $B_3$ Tamari.

3. When is the identity map mutation-linear?
One way to prove that there is a surjective lattice homomorphism from $\text{Camb}_B$ to $\text{Camb}_{B'}$:

Prove that there is a surjective lattice homomorphism between the corresponding weak orders.

**Theorem** (R., 2012). If $(W, S)$ and $(W', S)$ are finite Coxeter systems such that $W$ dominates $W'$, then the weak order on $W'$ is a lattice quotient of the weak order on $W$.

**Domination** here means that the diagram of $W'$ is obtained from the diagram of $W$ by reducing edge-labels and/or erasing edges.
Example: $A_3$ as a lattice quotient of $B_3$

(This is not $S_3$ as a lattice quotient of $B_3$. It's $S_4$.)
Affine associahedron fan

This is the “dual fan of affine associahedron” (except we don’t have an affine associahedron).

Joint with David Speyer: a Cambrian (\text{g-vector}) model of affine associahedron fan.

Joint with Salvatore Stella: an almost-positive roots (\text{d-vector}) model of affine associahedron fan.

**Observed and expected to be proved soon:** For $B$ acyclic of affine Cartan type, $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ if and only if $B$ dominates $B'$. (Necessarily in this case, $B'$ is of finite type.)
Example: \( B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \)

\[ \mathcal{F}_B \]

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3. When is the identity map mutation-linear?
Triangulated surfaces

Start with an orientable surface with boundary and some number of marked points. (Interior marked points called punctures.)

Arcs are non-self-intersecting curves connecting marked points (up to isotopy).

A triangulation $T$ is a maximal set of non-intersecting arcs. This cuts the surface into triangles. Number the arcs $1, \ldots, n$.

Signed adjacency matrix $B(T) = [b_{ij}]$ of a triangulation: A triangle with arc $i$ preceding arc $j$ clockwise around a triangle contributes $+1$ to $b_{ij}$. Counterclockwise contributes $-1$.

(Fomin, Shapiro, Thurston)
3. When is the identity map mutation-linear?
Resecting a triangulated surface on an edge

$B$: the signed adjacency matrix of a triangulated surface. $B'$: the signed adjacency matrix for a surface obtained by resection,

**Proposition.** $B$ dominates $B'$.

**Theorem.** (R., 2013) Assuming the Null Tangle Property, $\text{id} : \mathbb{R}^B \rightarrow \mathbb{R}^{B'}$ is mutation-linear.

Null Tangle Property: Probably true in many cases but not in general. Known for “polynomial growth” cases, for 1-punctured torus and 4-punctured sphere (the latter is joint with Barnard, Meehan, Polster).

3. When is the identity map mutation-linear?
Example

This is a resection on arc 1.

Torus:

\[
\begin{bmatrix}
0 & 2 & -2 \\
-2 & 0 & 2 \\
2 & -2 & 0
\end{bmatrix}
\]

Annulus:

\[
\begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 2 \\
1 & -2 & 0
\end{bmatrix}
\]

3. When is the identity map mutation-linear?
This is a resection on arc 1.

Torus:

\[
\begin{bmatrix}
0 & 2 & -2 \\
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2 & -2 & 0
\end{bmatrix}
\]

This is the mutation-fan refinement example from earlier.

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Example: \[ B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \quad B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \]

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\( \mathcal{F}_B \)

\( \mathcal{F}_{B'} \)

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3. When is the identity map mutation-linear?
Curves and shear coordinates

**Allowable curves**: Closed curves or curves that on each end, either spiral in to a marked point, or hit the boundary, with some technical conditions. (Cf. unbounded measured laminations.)

![Diagram of curves and arcs]

Given a triangulation with arcs numbered 1, \ldots, \(n\), each allowable curve \(\lambda\) has **shear coordinates**, a vector in \(\mathbb{R}^n\). For the \(i\)\(^{th}\) entry, we consider intersections of \(\lambda\) with the \(i\)\(^{th}\) arc. Nonzero contributions:

![Diagram of shear coordinates]

3. When is the identity map mutation-linear?
The Null Tangle Property

A tangle: finite weighted collection $\Xi$ of distinct allowable curves. Shear coordinates of $\Xi$: weighted sum of the shear coordinates. Null tangle: shear coordinates zero with respect to every triangulation*.

The Null Tangle Property: A null tangle has all weights zero.

Theorem (R., 2012). The shear coordinates of allowable curves are a (positive, integral) basis for $B(T)$ if and only if the Null Tangle Property holds.

Theorem (R., 2012). The Null Tangle Property holds for a disk with $\leq 2$ punctures, for an annulus with $\leq 1$ puncture, for a sphere with three boundary components and no punctures, and for the once-punctured torus.

Theorem (Barnard, Meehan, Polster, R., 2014). Also for a 4-punctured sphere.
Thanks for listening.