Mutation-linear maps

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Mutation-linear algebra

The mutation fan

When is the identity map mutation-linear?

Linear algebra

Linear algebra = the study of linear relations:

formal finite linear combinations $\sum c_i \mathbf{v}_i$ that evaluate to $\mathbf{0}$.

- $U \subseteq \mathbb{R}^n$ is independent if there are no nontrivial linear relations supported in U.
- $U \subseteq \mathbb{R}^n$ is spanning if for every $\mathbf{a} \in \mathbb{R}^n$, there exists a linear relation $\mathbf{a} - \sum c_i \mathbf{v}_i$ with $\{\mathbf{v}_i\} \subseteq U$.
- A map $\eta : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for every linear relation $\sum c_i \mathbf{v}_i$ in $\mathbb{R}^n$, the expression $\sum c_i \eta(\mathbf{v}_i)$ is a linear relation in $\mathbb{R}^m$.

Partial linear structures

Distinguish a subset $\mathcal{A}$ (the "active" linear relations) of all linear relations. Reformulate linear algebra in terms of $\mathcal{A}$.

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- $\eta : (\mathbb{R}^n, \mathcal{A}) \to (\mathbb{R}^m, \mathcal{A}')$ is linear if for each active linear relation $\sum c_i \mathbf{v}_i$ in $\mathbb{R}^n$, the expression $\sum c_i \eta(\mathbf{v}_i)$ is an active linear relation. That is, $\sum c_i \mathbf{v}_i \in \mathcal{A} \implies \sum c_i \eta(\mathbf{v}_i) \in \mathcal{A}'$.

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- Some conditions $\implies$ $\exists$ basis (usual Zorn's lemma argument).
- Generality here for clarity only. I have exactly 1 kind of example.

Matrix mutation

Let $B = (b_{ij})$ be $n \times n$ skew-symmetrizable integer matrix. (An exchange matrix.) Let $\mathbf{a} \in \mathbb{R}^n$ and let $\widetilde{B}$ be $\left[\begin{smallmatrix} B \\ \mathbf{a} \end{smallmatrix}\right]$ (i.e. B with an extra row $\mathbf{a}$). For $k \in \{1, \ldots, n\}$, the mutation of B in direction k is $B' = \mu_k(B)$ with entries given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\}; \\ b_{ij} + |b_{ik}|b_{kj} & \text{if } k \notin \{i,j\} \text{ and } b_{ik}b_{kj} > 0; \\ b_{ij} & \text{otherwise.} \end{cases}$$

Example: $$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} \overset{\mu_3}{\longleftrightarrow} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 1 & 0 & -1 \\ -2 & -3 & 3 & 0 \\ -1 & -3 & 1 & 1 \end{bmatrix}$$

For a sequence $\mathbf{k} = k_q, k_{q-1}, \ldots, k_1$, similarly define $\mu_{\mathbf{k}}(B)$.

Mutation maps $\eta_{\mathbf{k}}^B$

We continue with $\widetilde{B} = \left[\begin{smallmatrix} B \\ \mathbf{a} \end{smallmatrix}\right]$ and $\mathbf{k} = k_q, k_{q-1}, \ldots, k_1$. Define $\eta_{\mathbf{k}}^B(\mathbf{a})$ to be the coefficient row of $\mu_{\mathbf{k}}(\widetilde{B})$. Concretely, for $\mathbf{k} = k$:

$\eta_k^B(\mathbf{a}) = (a_1', \ldots, a_n')$, where

$$a_j' = \begin{cases} -a_k & \text{if } j = k; \\ a_j + a_k b_{kj} & \text{if } j \neq k,\ a_k \geq 0 \text{ and } b_{kj} \geq 0; \\ a_j - a_k b_{kj} & \text{if } j \neq k,\ a_k \leq 0 \text{ and } b_{kj} \leq 0; \\ a_j & \text{otherwise.} \end{cases}$$

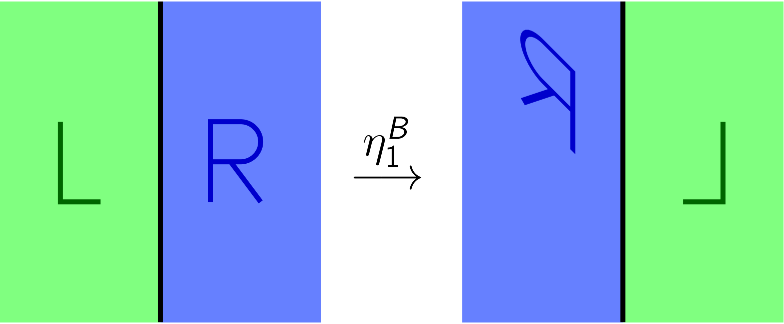
η_k^B is linear in $\{\mathbf{a} \in \mathbb{R}^n : a_k \geq 0\}$ and linear in $\{\mathbf{a} \in \mathbb{R}^n : a_k \leq 0\}$.

The maps $\eta_{\mathbf{k}}^B$ are the mutation maps associated to B.
They are piecewise-linear homeomorphisms of $\mathbb{R}^n$.
Their inverses are also mutation maps.

Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ a_1 & a_2 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -a_1 & ? \end{bmatrix}$$

$$? = \begin{cases} a_2 & \text{if } a_1 \leq 0 \\ a_2 + a_1 & \text{if } a_1 \geq 0 \end{cases}$$



B-coherent linear relations

Let S be a finite set, let $(\mathbf{v}_i : i \in S)$ be vectors in $\mathbb{R}^n$ and let $(c_i : i \in S)$ be real numbers.

The formal expression $\sum_{i \in S} c_i \mathbf{v}_i$ is a B-coherent linear relation if

$$\sum_{i \in S} c_i \eta_{\mathbf{k}}^B(\mathbf{v}_i) = \mathbf{0}.$$

holds for every finite sequence $\mathbf{k} = k_q, \ldots, k_1$.

In particular, $\sum_{i \in S} c_i \mathbf{v}_i$ is a linear relation in the usual sense.

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[1] I'm fibbing.

Mutation-linear algebra

Write $\mathbb{R}^B$ for the partial linear structure on $\mathbb{R}^n$ whose active linear relations are the B-coherent linear relations. The study of $\mathbb{R}^B$ is mutation-linear algebra.

- $U \subseteq \mathbb{R}^B$ is independent if there are no nontrivial B-coherent linear relations supported in U.
- $U \subseteq \mathbb{R}^B$ is spanning if for every $\mathbf{a} \in \mathbb{R}^n$, there exists a B-coherent linear relation $\mathbf{a} - \sum c_i \mathbf{v}_i$ with $\{\mathbf{v}_i\} \subseteq U$.

What I'm **not** going to talk about: Basis for $\mathbb{R}^B$ = Universal geometric coefficients for cluster algebras associated to B.

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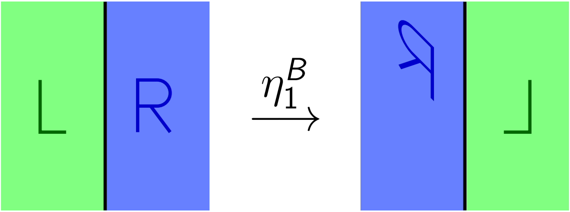
What I **am** going to talk about: Mutation-linear maps.

- A map $\eta : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear if for every B-coherent linear relation $\sum c_i \mathbf{v}_i$, the expression $\sum c_i \eta(\mathbf{v}_i)$ is a B'-coherent linear relation.

Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



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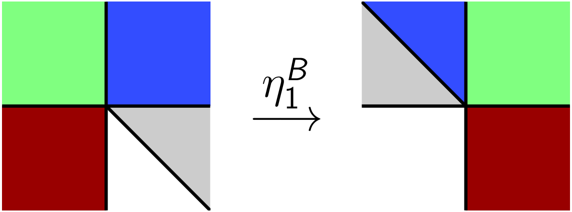

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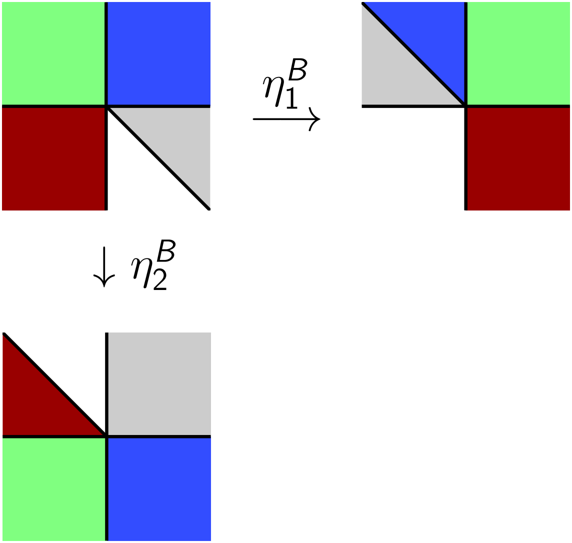

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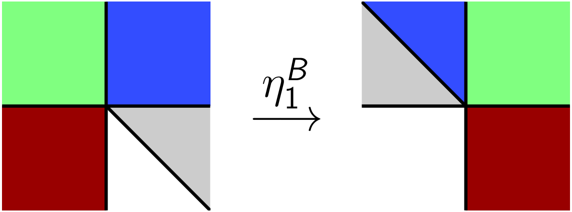

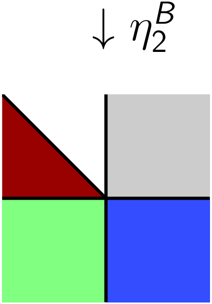

Each map is linear on each colored region.

A basis for B: A nonzero vector in each of the 5 rays defining regions.

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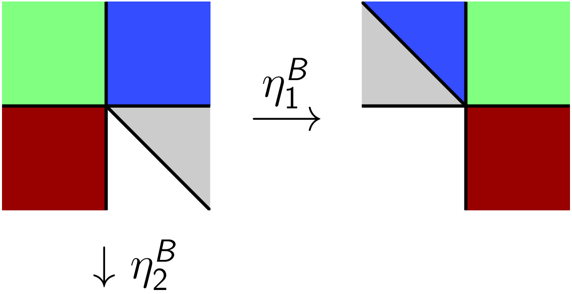
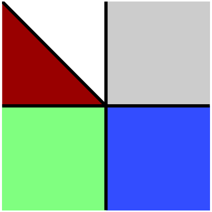




Each map is linear on each colored region.

A basis for B: A nonzero vector in each of the 5 rays defining regions. This is a positive basis!

The mutation fan

The example suggests an "easy" way to get B-coherent linear relations: Find vectors in the same domain of linearity of all mutation maps and make a linear relation among them.

Define an equivalence relation $\equiv^B$ on $\mathbb{R}^n$ by setting

$$\mathbf{a}_1 \equiv^B \mathbf{a}_2 \quad \Longleftrightarrow \quad \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_1)) = \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_2)) \quad \forall \mathbf{k}.$$

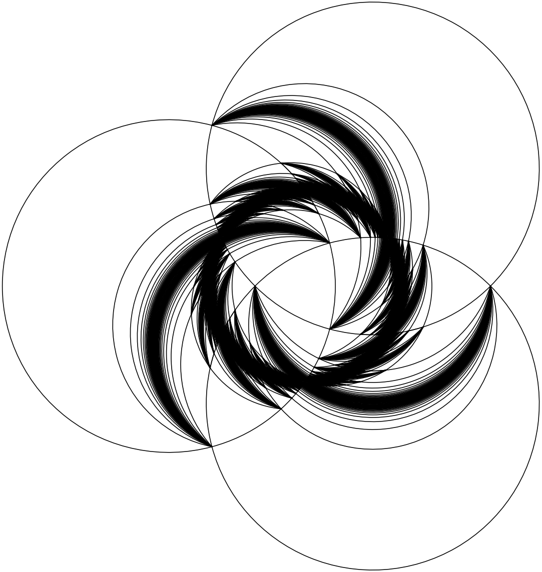
$\mathbf{sgn}(\mathbf{a})$ is the vector of signs $(-1, 0, +1)$ of the entries of $\mathbf{a}$.

B-classes: equivalence classes of $\equiv^B$.
B-cones: closures of B-classes.

Mutation fan for B:
The collection $\mathcal{F}_B$ of all B-cones and all faces of B-cones.

Theorem (R., 2011). $\mathcal{F}_B$ is a complete fan (possibly with infinitely many cones).

Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$



The mutation fan and mutation-linear algebra

A basis for $\mathbb{R}^B$ is positive if, for every $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the unique B-coherent linear relation $\mathbf{a} - \sum_{i \in I} c_i \mathbf{b}_i$ has all $c_i \geq 0$.

Theorem (R., 2014). If a positive basis exists for $\mathbb{R}^B$, then a map $\eta : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear if and only if for every B-cone C, the restriction $\eta|_C$ is a linear map into some B'-cone.

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Feel free to not worry about positive bases for $\mathbb{R}^B$. Just think of this as "If B is well-behaved, then..."

We'll stay in the well-behaved case.

Bijective mutation-linear maps

We'll restrict our focus to bijective mutation-linear maps.
(These are not necessarily mutation-linear isomorphisms.)

We'll restrict our focus further to the question:

When is the identity map $\mathbb{R}^B \to \mathbb{R}^{B'}$ mutation-linear?

By the previous theorem:

id : $\mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear if and only if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$.

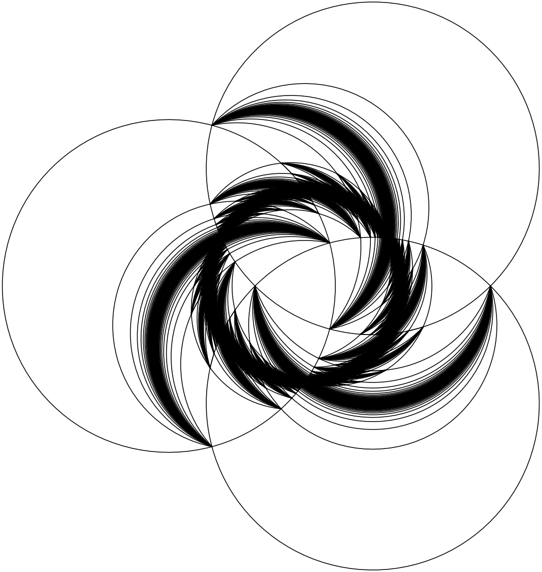
This can fail. The surprise is that this actually happens often.

A necessary* condition: B dominates B'.

That is, b_{ij} and b'_{ij} weakly agree in sign and $|b_{ij}| \geq |b'_{ij}|$ for all i, j.

Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$\mathcal{F}_B$



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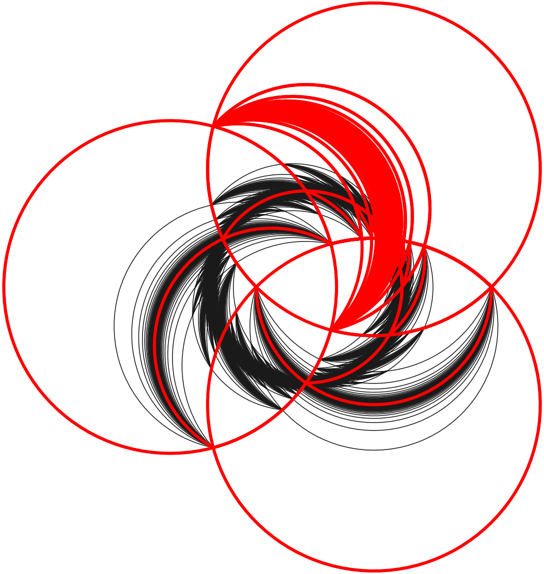

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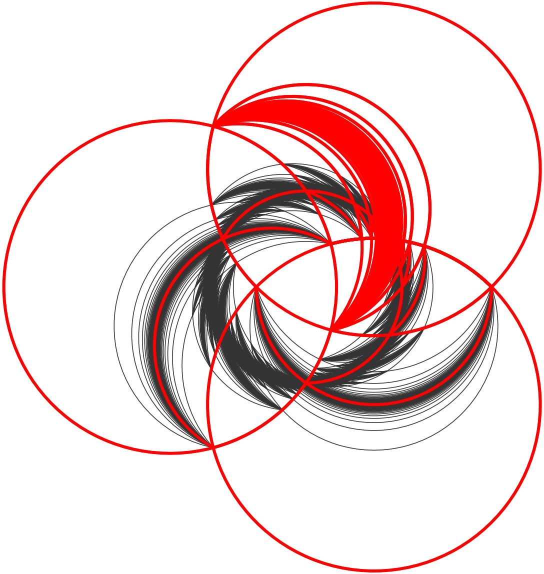

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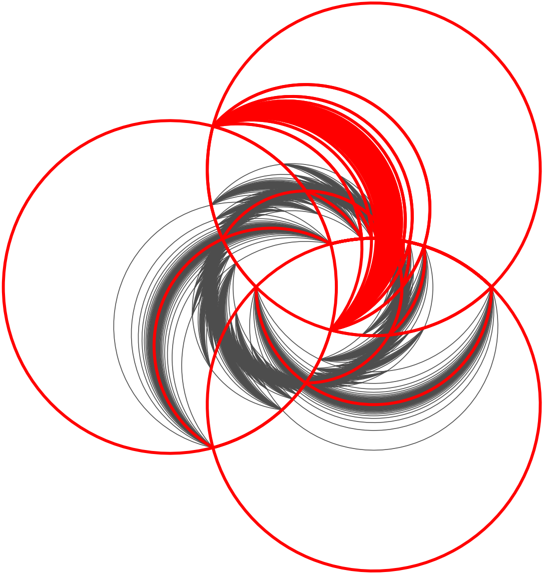

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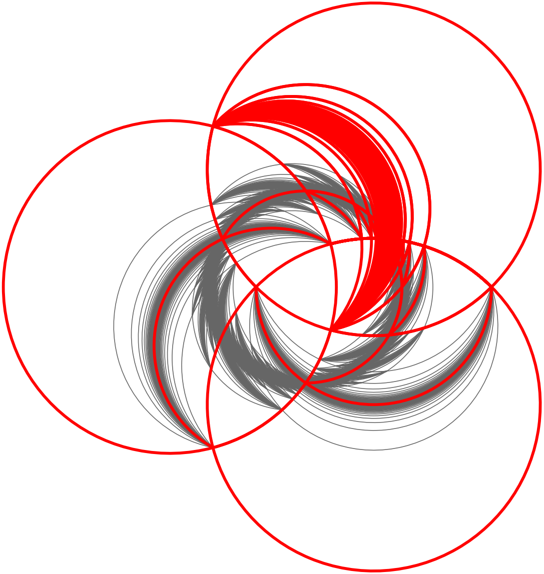

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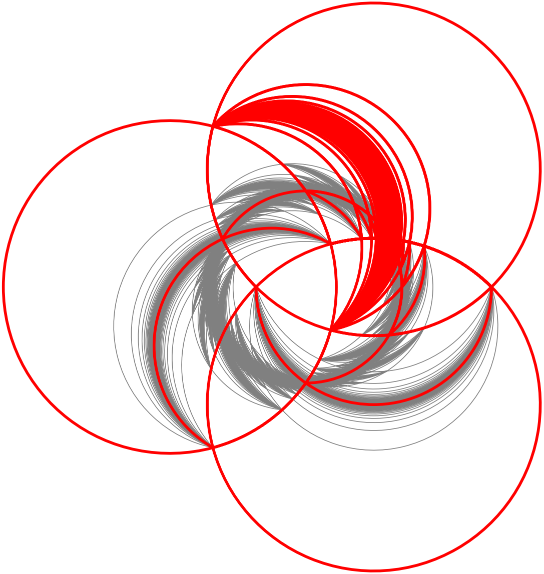

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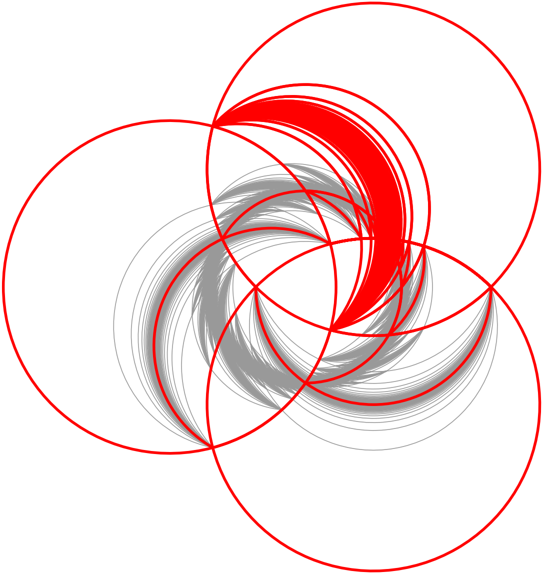

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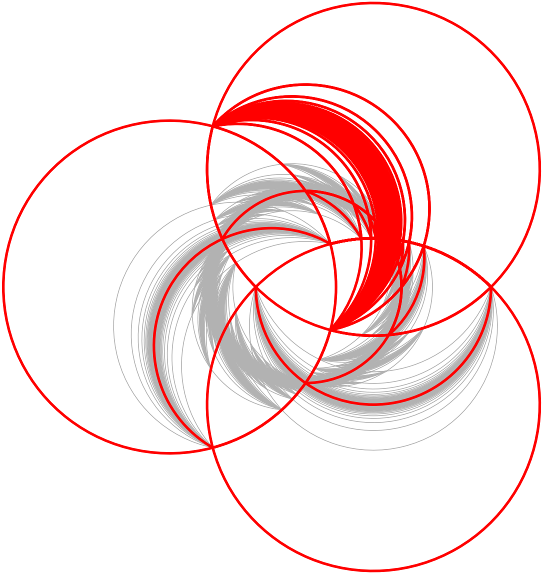

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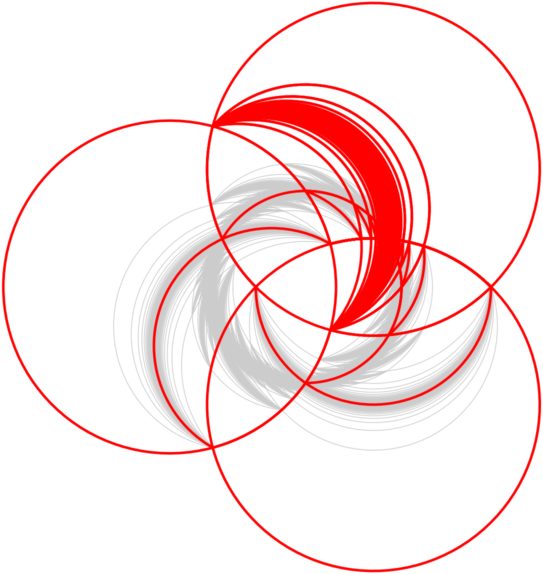

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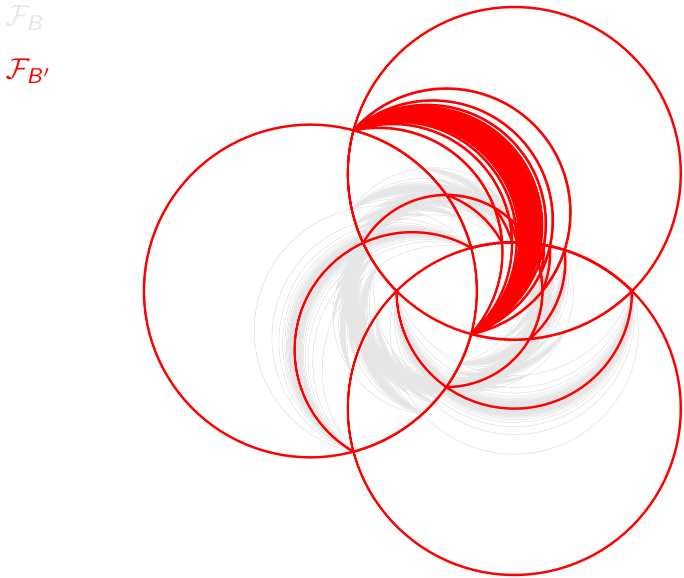

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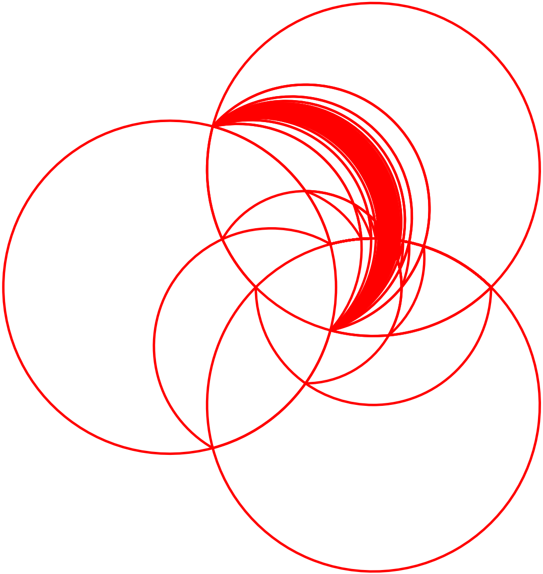

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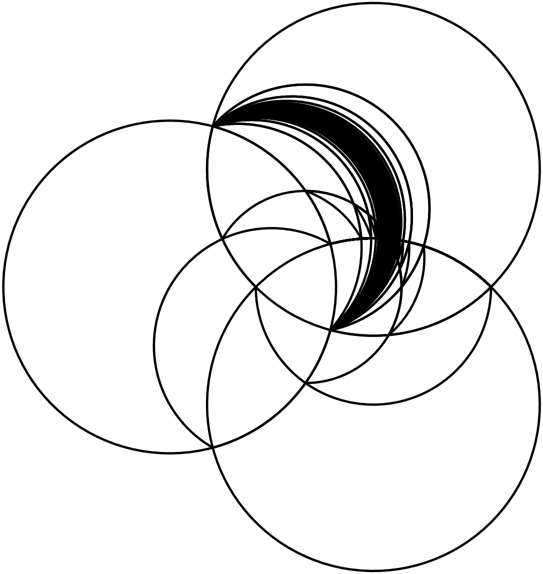

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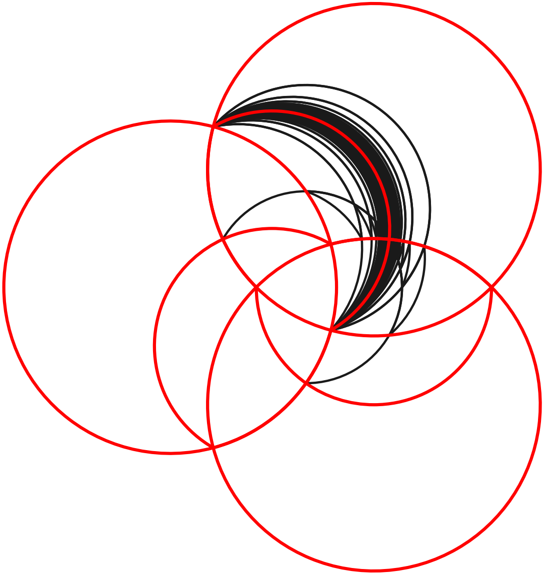

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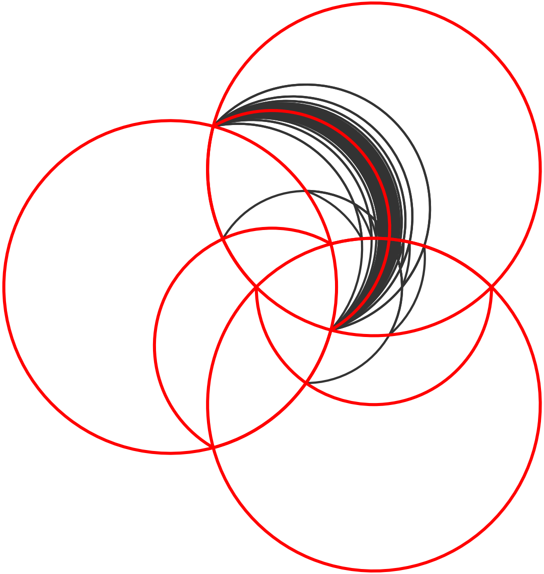

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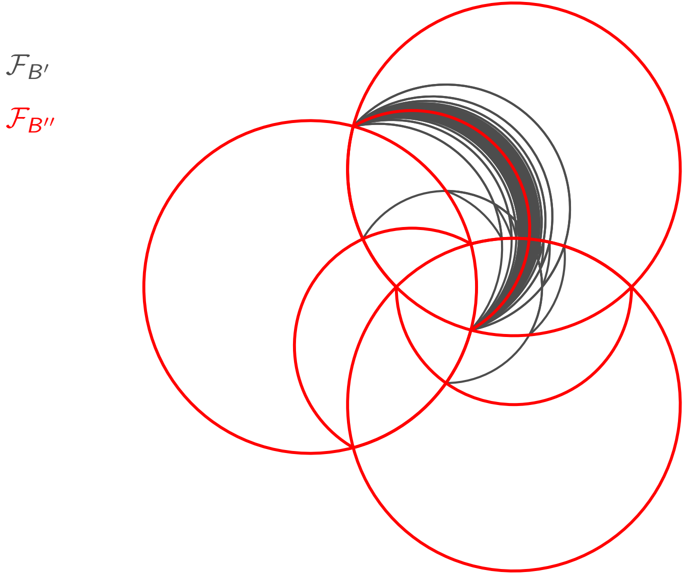

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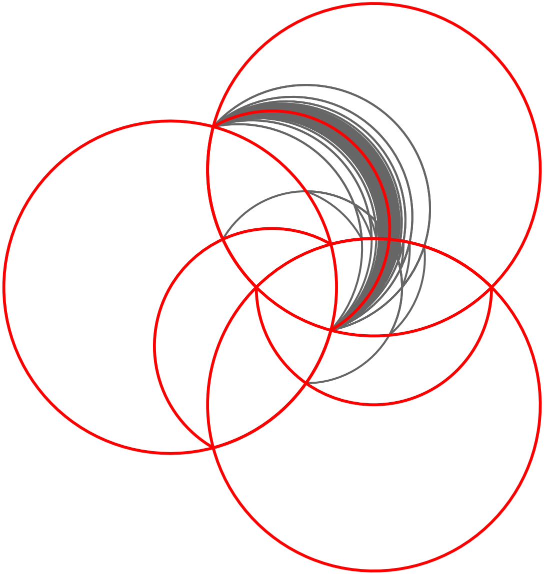

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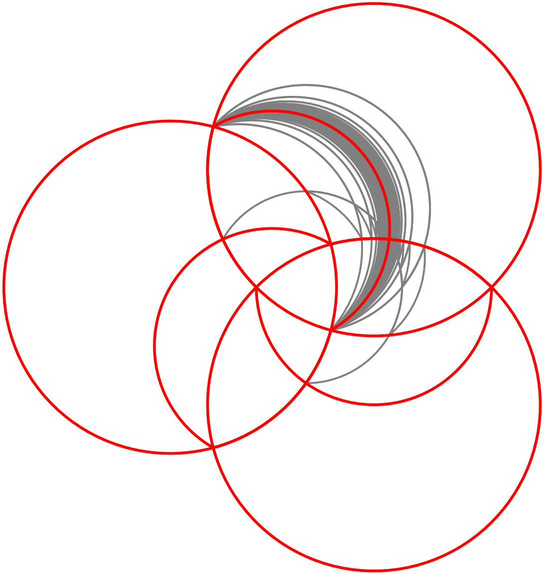

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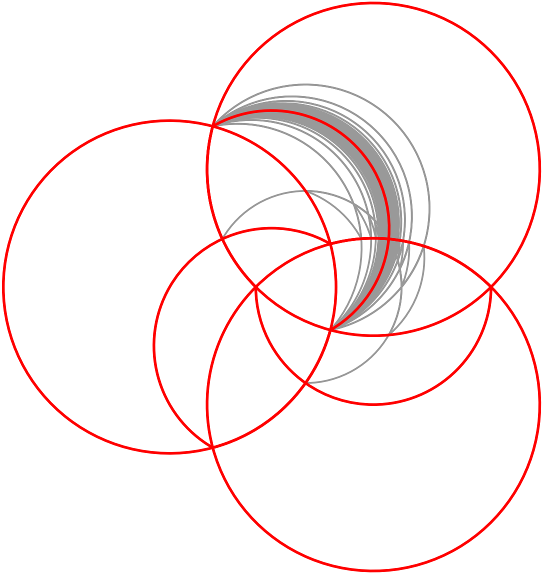

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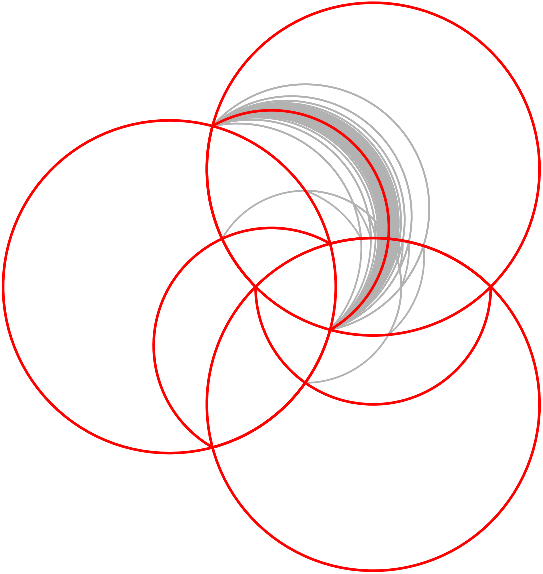

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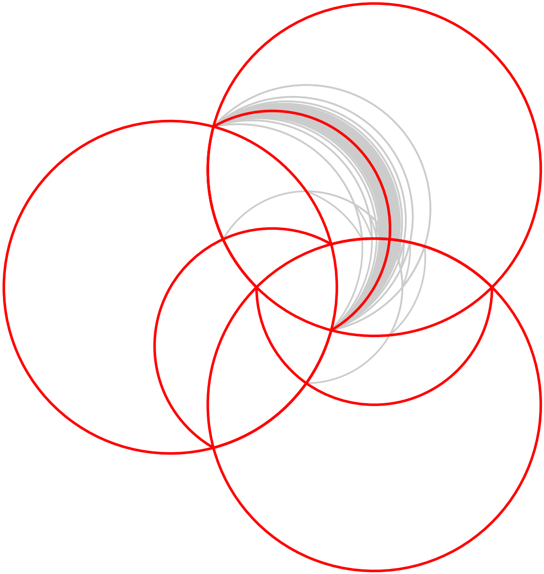

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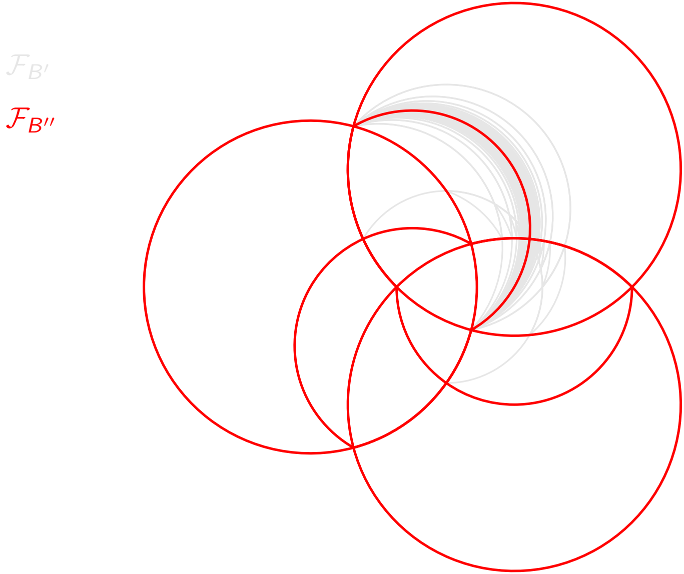

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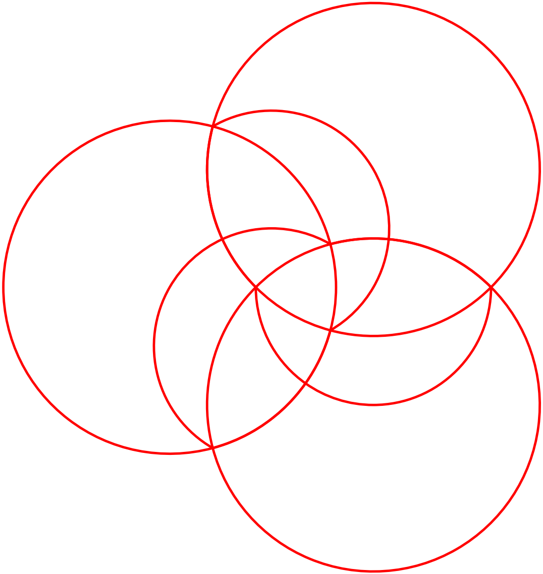

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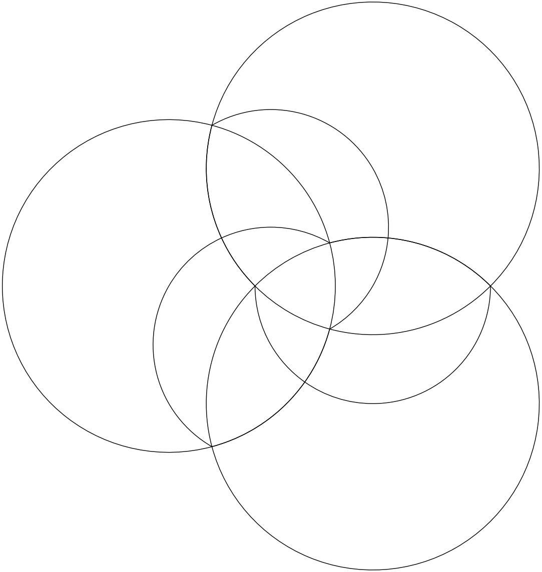
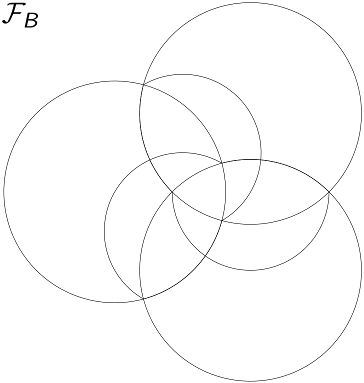
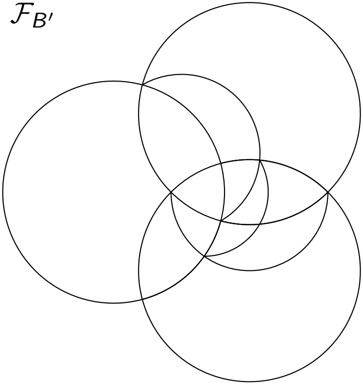


Non-Example: $B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$\mathcal{F}_B$



$\mathcal{F}_{B'}$



So far

- Mutation maps: a family of piecewise linear maps given by matrix mutation.
- B-coherent linear relations: Linear relations preserved by all mutation maps.
- Mutation-linear maps $\mathbb{R}^B \to \mathbb{R}^{B'}$: Send B-coherent relations to B'-coherent relations.
- Mutation fan $\mathcal{F}_B$: Common domains of linearity of all mutation maps. For well-behaved B, mutation-linear maps are closely tied to mutation fans.
- We are focusing on when $\mathrm{id} : \mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear. This is if and only if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$.
  Necessary: B dominates B'.

The rest of the talk

Some specific cases where id : $\mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear (i.e. $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$).

- Finite type ($\mathcal{F}_B$ is finite) acyclic:
  - $\mathcal{F}_B$ is dual fan to generalized associahedron (the Cambrian fan, or equivalently the $\mathbf{g}$-vector fan).
  - $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ whenever B dominates B'.
  - Refinement relations arises from a lattice-quotient relationship between Cambrian lattices.
  - There is a related lattice-quotient relationship between weak orders.
- Refinement relation suggests an algebraic relation between cluster algebras.
- "Affine associahedron fans."
- "Resection" of triangulated surfaces.

Cambrian fans (finite type)

Each B defines a Cartan matrix A.

$$\text{E.g.} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \to \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Coxeter fan: Defined by the reflecting hyperplanes of the Coxeter group W associated to A. Maximal cones $\leftrightarrow$ elements of W.

Cambrian fan: A certain coarsening of the Coxeter fan. Depends on the extra sign information that's in B (or equivalently, depends on a Coxeter element, or equivalently an orientation of the Coxeter diagram). Two ways to look at this:

- Coarsen according to a certain lattice congruence on W.
- Coarsen according to the combinatorics of "sortable elements."

For S_n, the normal fan to the usual associahedron. (In general, generalized associahedron.)

Cambrian fans and mutation fans

For B acyclic of finite type, $\mathcal{F}_B$ is a Cambrian fan. (Key technical point: identify fundamental weights with standard basis vectors.)

Theorem (R., 2013). For B acyclic of finite type, $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ if and only if B dominates B'.

Domination relations among exchange matrices imply domination relations among Cartan matrices. So the theorem is a statement that refinement relations exist among Cambrian fans when we decrease edge-labels (or erase edges) on Coxeter diagrams.

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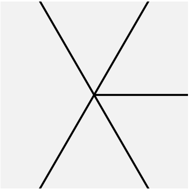

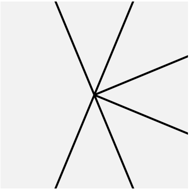

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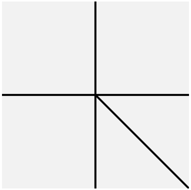

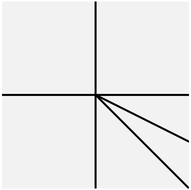

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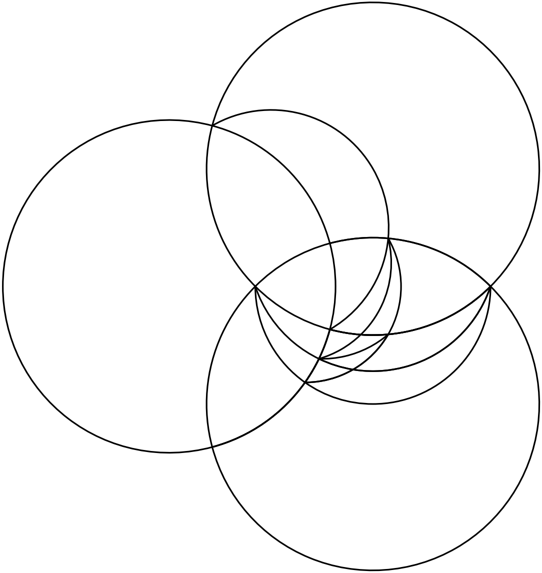
$\mathcal{F}_B$



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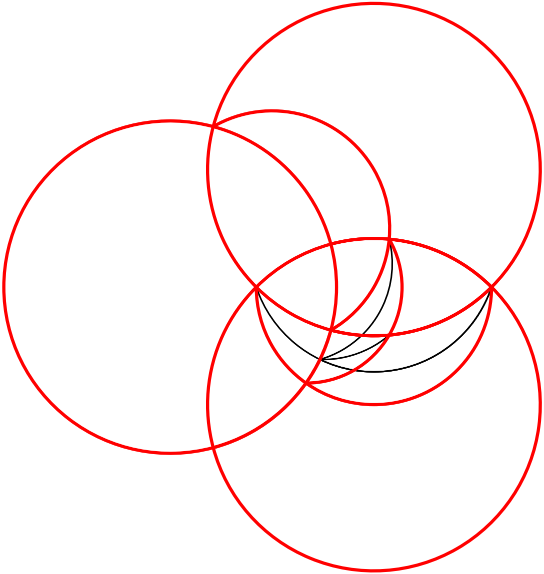
$\mathcal{F}_B$

$\mathcal{F}_{B'}$



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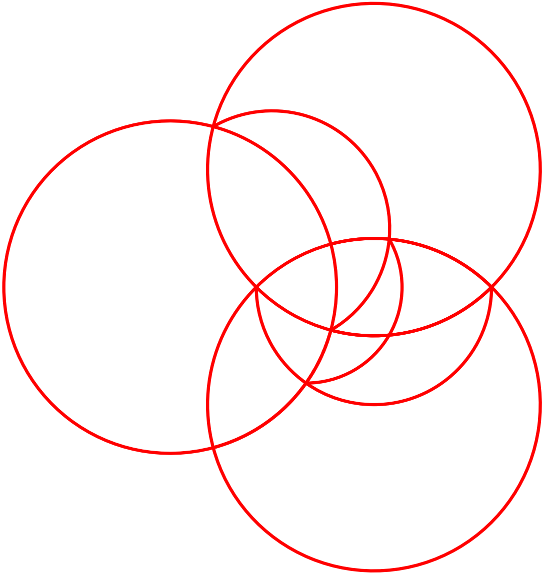

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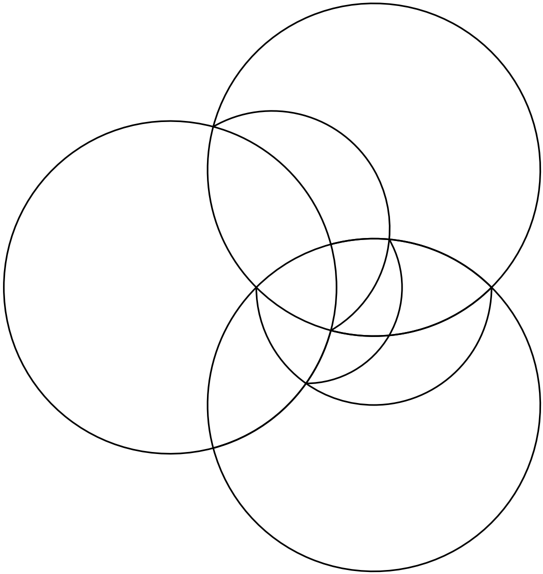

Ring homomorphisms of cluster algebras (finite type)

Cluster variables: Generate the cluster algebra (not freely!).

Rays of the mutation fan $\mathcal{F}_B$ are in bijection with cluster variables.

Therefore, if $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$, there is an inclusion

$$\{\text{rays of } \mathcal{F}_{B'}\} \hookrightarrow \{\text{rays of } \mathcal{F}_B\}$$

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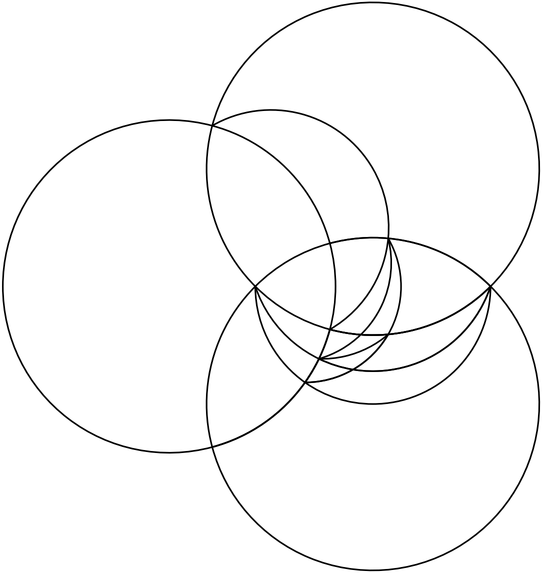

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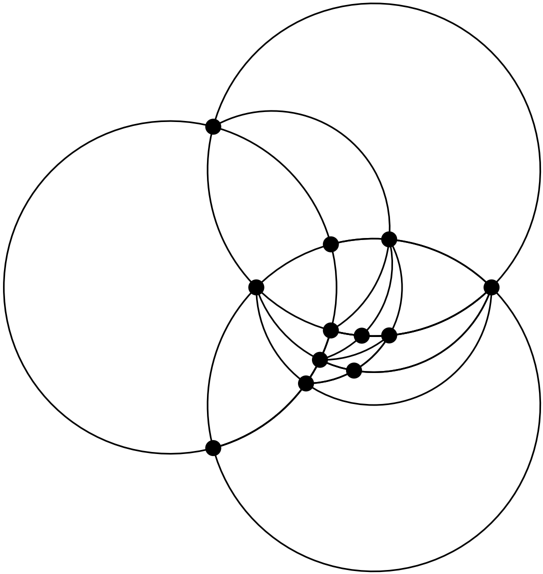

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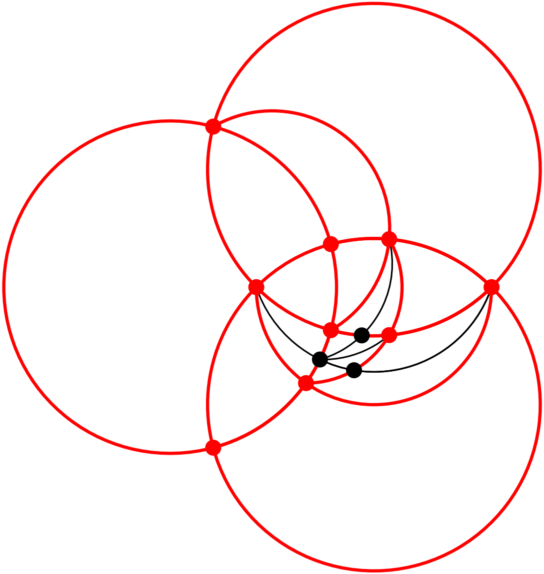

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Therefore there is a natural injective map on cluster variables. This extends (in all cases we have checked) to an embedding of $\mathcal{A}_0(B')$ as a subring of $\mathcal{A}_0(B)$. (You have to deal correctly with coefficients—make the map preserve $\mathbf{g}$-vectors).

Close algebraic relationships between different cluster algebras of the same rank are surprising *a priori*.

Lattice homomorphisms between Cambrian lattices

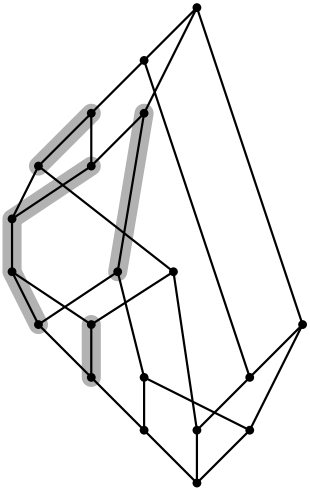
The Cambrian lattice Camb_B is:

- A partial order on maximal cones in the Cambrian fan $\mathcal{F}_B$. The fan and the order interact very closely.
- A lattice quotient—and a sublattice—of the weak order on the finite Coxeter group associated to B.

One way to prove the refinement of fans is to show that there is a surjective lattice homomorphism from Camb_B to $\mathrm{Camb}_{B'}$.

Theorem (R., 2012). This happens for all acyclic, finite-type B, B' with B dominating B'.

Example: A_3 Tamari is a lattice quotient of B_3 Tamari



Lattice homomorphisms between weak orders

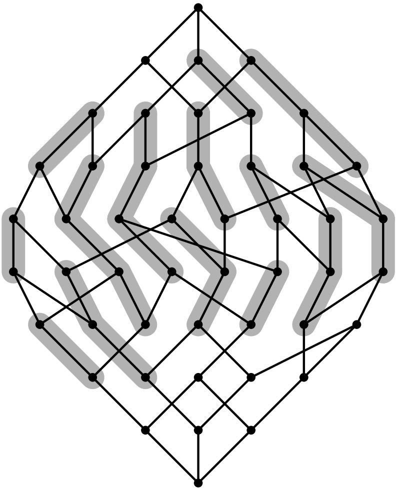
One way to prove that there is a surjective lattice homomorphism from Camb_B to $\mathrm{Camb}_{B'}$:

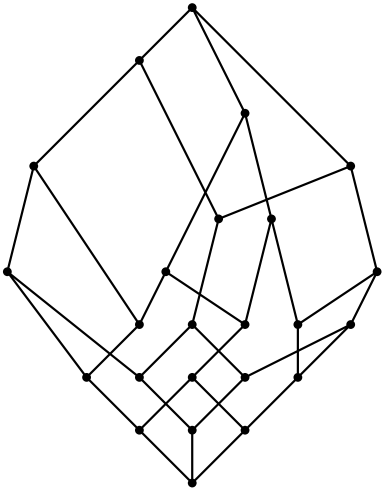
Prove that there is a surjective lattice homomorphism between the corresponding weak orders.

Theorem (R., 2012). If (W, S) and (W', S) are finite Coxeter systems such that W dominates W', then the weak order on W' is a lattice quotient of the weak order on W.

Domination here means that the diagram of W' is obtained from the diagram of W by reducing edge-labels and/or erasing edges.

Example: A_3 as a lattice quotient of B_3





(This is not S_3 as a lattice quotient of B_3. It's S_4.)

Affine associahedron fan

This is the "dual fan of affine associahedron" (except we don't have an affine associahedron).

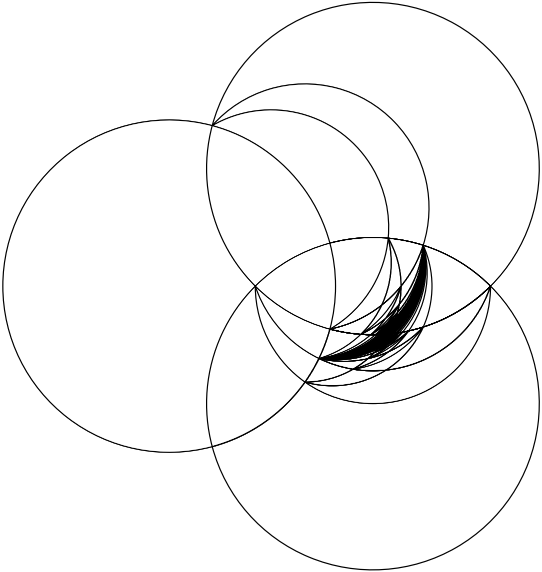
Joint with David Speyer: a Cambrian ($\mathbf{g}$-vector) model of affine associahedron fan.

Joint with Salvatore Stella: an almost-positive roots ($\mathbf{d}$-vector) model of affine associahedron fan.

Observed and expected to be proved soon: For B acyclic of affine Cartan type, $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ if and only if B dominates B'. (Necessarily in this case, B' is of finite type.)

Example: $B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$

$\mathcal{F}_B$



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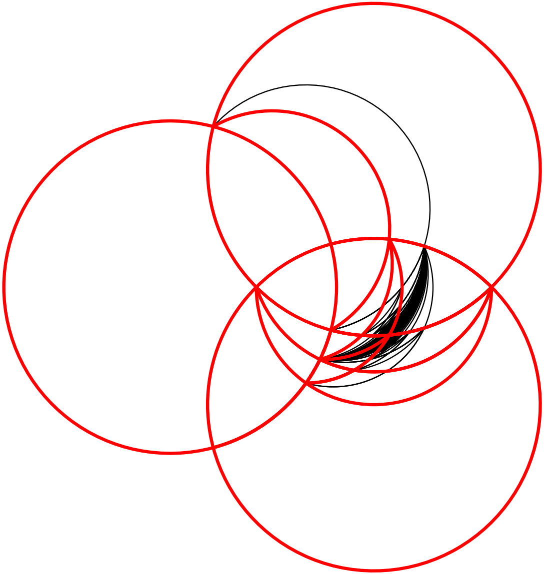
$\mathcal{F}_B$

$\mathcal{F}_{B'}$



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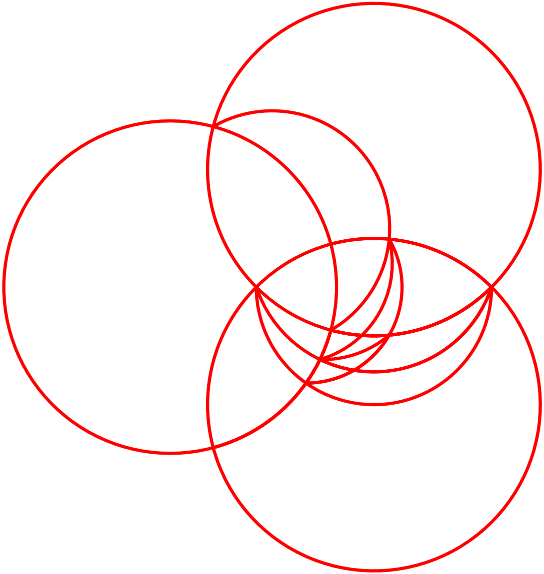

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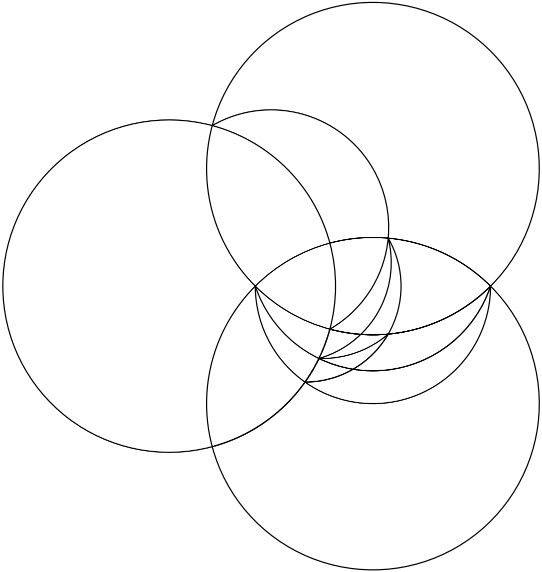


Triangulated surfaces

Start with an orientable surface with boundary and some number of marked points. (Interior marked points called punctures.)

Arcs are non-self-intersecting curves connecting marked points (up to isotopy).

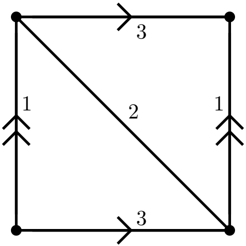
A triangulation T is a maximal set of non-intersecting arcs. This cuts the surface into triangles. Number the arcs $1, \ldots, n$.

Signed adjacency matrix $B(T) = [b_{ij}]$ of a triangulation:
A triangle with arc i preceding arc j clockwise around a triangle contributes $+1$ to b_{ij}. Counterclockwise contributes -1.

(Fomin, Shapiro, Thurston)

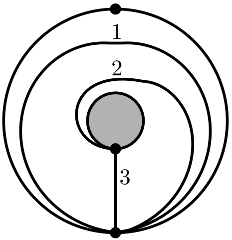
Signed adjacency matrix example

Torus:



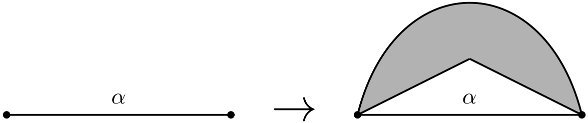
$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Annulus



$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

Resecting a triangulated surface on an edge



B: the signed adjacency matrix of a triangulated surface.
B': the signed adjacency matrix for a surface obtained by resection,

Proposition.* B dominates B'.

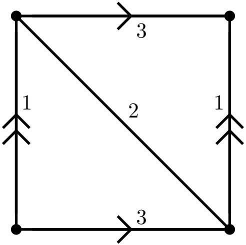
Theorem. (R., 2013) Assuming the Null Tangle Property, id : $\mathbb{R}^B \to \mathbb{R}^{B'}$ is mutation-linear.

Null Tangle Property: Probably true in many cases but not in general. Known for "polynomial growth" cases, for 1-puctured torus and 4-punctured sphere (the latter is joint with Barnard, Meehan, Polster).

Example

This is a resection on arc 1.

Torus:



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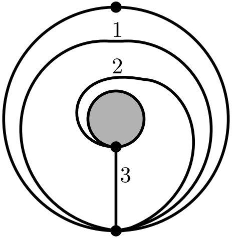

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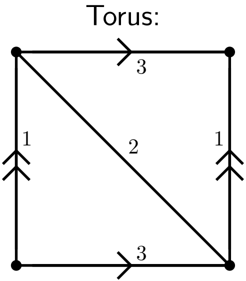

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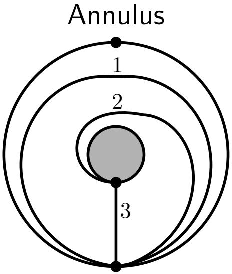

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

This is the mutation-fan refinement example from earlier.

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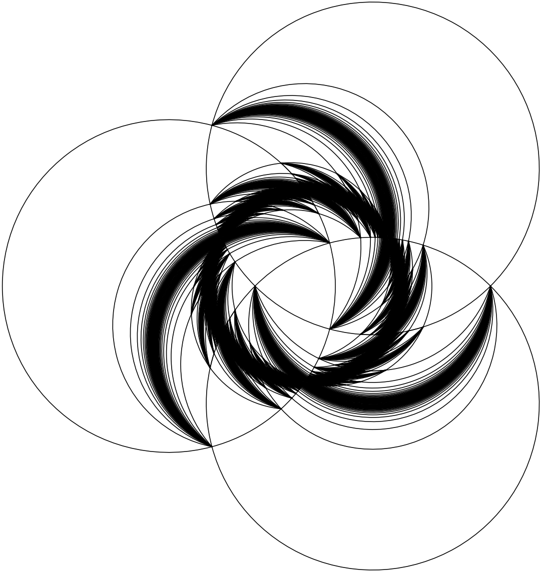
$\mathcal{F}_B$



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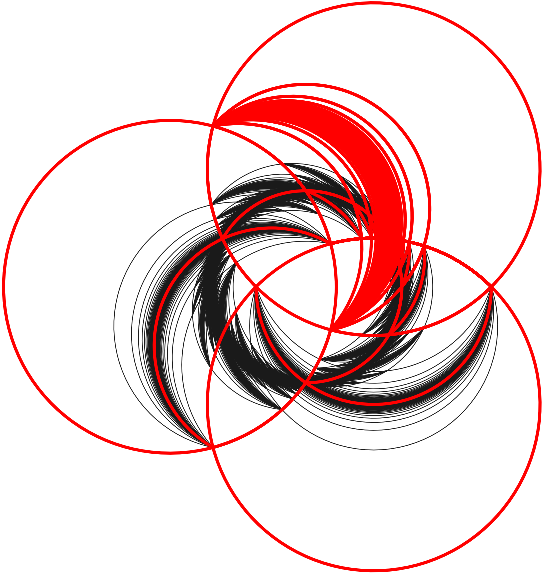
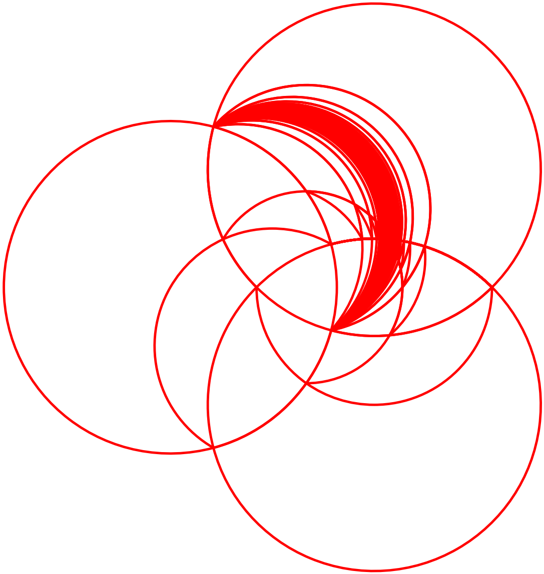
$\mathcal{F}_{B'}$



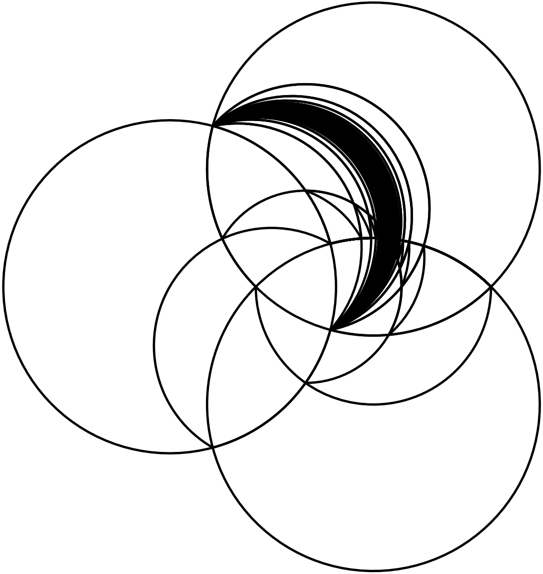
Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$\mathcal{F}_{B'}$



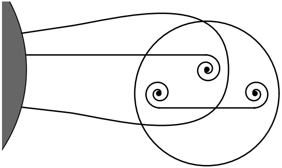
Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$\mathcal{F}_{B'}$

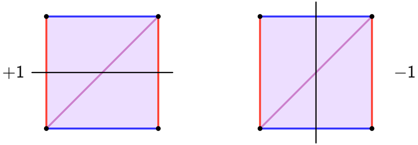


Curves and shear coordinates

Allowable curves: Closed curves or curves that on each end, either spiral in to a marked point, or hit the boundary, with some techical conditions. (Cf. unbounded measured laminations.)



Given a triangulation with arcs numbered $1, \ldots, n$, each allowable curve λ has shear coordinates, a vector in $\mathbb{R}^n$. For the i^{th} entry, we consider intersections of λ with the i^{th} arc. Nonzero contributions:



The Null Tangle Property

A tangle: finite weighted collection Ξ of distinct allowable curves.
Shear coordinates of Ξ: weighted sum of the shear coordinates.
Null tangle: shear coordinates zero with respect to every triangulation*.
The Null Tangle Property: A null tangle has all weights zero.

Theorem (R., 2012). The shear coordinates of allowable curves are a (positive, integral) basis for $B(T)$ if and only if the Null Tangle Property holds.

Theorem (R., 2012). The Null Tangle Property holds for a disk with ≤ 2 punctures, for an annulus with ≤ 1 puncture, for a sphere with three boundary components and no punctures, and for the once-punctured torus.

Theorem (Barnard, Meehan, Polster, R., 2014). Also for a 4-punctured sphere.

Thanks for listening.