Mutation-linear algebra and universal geometric cluster algebras

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Mutation-linear ("μ-linear") algebra

Universal geometric cluster algebras

The mutation fan

Universal geometric cluster algebras from surfaces

Mutation maps $\eta_{\mathbf{k}}^{B}$

Let B be a skew-symmetrizable exchange matrix.

Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $\widetilde{B}$ be B, extended with one coefficient row $\mathbf{a}$.

For a sequence $\mathbf{k} = k_q, k_{q-1}, \ldots, k_1$, define $\eta_{\mathbf{k}}^{B}(\mathbf{a})$ to be the coefficient row of $\mu_{\mathbf{k}}(\widetilde{B})$. Concretely, for $\mathbf{k} = k$:

$\eta_k^B(\mathbf{a}) = (a'_1, \ldots, a'_n)$, where

$$a'_j = \begin{cases} -a_k & \text{if } j = k; \\ a_j + a_k b_{kj} & \text{if } j \neq k,\ a_k \geq 0 \text{ and } b_{kj} \geq 0; \\ a_j - a_k b_{kj} & \text{if } j \neq k,\ a_k \leq 0 \text{ and } b_{kj} \leq 0; \\ a_j & \text{otherwise.} \end{cases}$$

The mutation maps $\eta_{\mathbf{k}}^{B}$ are piecewise-linear homeomorphisms of $\mathbb{R}^n$. Their inverses are also mutation maps.

B-coherent linear relations

Let S be a finite set, let $(\mathbf{v}_i : i \in S)$ be vectors in $\mathbb{R}^n$ and let $(c_i : i \in S)$ be real numbers.

The formal expression $\sum_{i \in S} c_i \mathbf{v}_i$ is a B-coherent linear relation if

$$\sum_{i \in S} c_i \eta_{\mathbf{k}}^B(\mathbf{v}_i) = \mathbf{0}, \text{ and} \tag{1}$$

$$\sum_{i \in S} c_i \mathbf{min}(\eta_{\mathbf{k}}^B(\mathbf{v}_i), \mathbf{0}) = \mathbf{0} \tag{2}$$

hold for every finite sequence $\mathbf{k} = k_q, \ldots, k_1$.

In particular, $\sum_{i \in S} c_i \mathbf{v}_i$ is a linear relation in the usual sense.

Example: For $B = [0]$, a B-coherent relation is a linear relation among vectors in $\mathbb{R}^n$ all agreeing in sign.

Basis for B

Let R be $\mathbb{Z}$ or a field between $\mathbb{Q}$ and $\mathbb{R}$. Usually $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

The vectors $(\mathbf{b}_i : i \in I)$ in R^n are an R-basis for B if and only if the following two conditions hold.

(i) Spanning: For all $\mathbf{a} \in R^n$, there exists a finite subset S of I and coefficients c_i in R such that $\mathbf{a} - \sum_{i \in S} c_i \mathbf{b}_i$ is a B-coherent linear relation.

(ii) Independence: If S is a finite subset of I and $\sum_{i \in S} c_i \mathbf{b}_i$ is a B-coherent linear relation, then $c_i = 0$ for all $i \in S$.

Example: For any R, $\{\pm 1\} \subset R^1$ is an R-basis for $[0]$.

Theorem

Every skew-symmetrizable B has an R-basis.

Proof: Zorn's lemma, same argument that shows that every vector space has a (Hamel) basis. The proof is non-constructive.

Why do μ-linear algebra?

- Universal geometric cluster algebras
- The mutation fan

Tropical Semifield (broadly defined)

I: an indexing set (no requirement on cardinality).

$(u_i : i \in I)$: formal symbols (tropical variables).

Tropical semifield $\mathrm{Trop}(u_i : i \in I)$:

Elements are products $\prod_{i \in I} u_i^{a_i}$ with $a_i \in R$.

Multiplication as usual

$\mathrm{Trop}(u_i : i \in I)$ is the module R^I, written multiplicatively.

Auxiliary addition $\quad \prod_{i \in I} u_i^{a_i} \oplus \prod_{i \in I} u_i^{b_i} = \prod_{i \in I} u_i^{\min(a_i, b_i)}$.

Topology on R: discrete.

Topology on $\mathrm{Trop}(u_i : i \in I)$: product topology as R^I.

I finite: discrete.

I countable: FPS.

Cluster algebra of geometric type (broadly defined)

Extended exchange matrix: a collection of rows indexed by $[n] \cup I$.

In matrix notation, $\widetilde{B} = [b_{ij}]$.

Rows of $\widetilde{B}$ indexed by $[n]$ are the matrix B.

Other rows are coefficient rows. These are vectors in R^n.

Coefficients: $y_j = \prod_{i \in I} u_i^{b_{ij}} \in \mathrm{Trop}(u_i : i \in I)$ for each $j \in [n]$.

Cluster algebra of geometric type:

$$\mathcal{A}(\mathbf{x}, \widetilde{B}) = \mathcal{A}(\mathbf{x}, B, \{y_1, \ldots, y_n\}).$$

(The usual definition: take I finite and $R = \mathbb{Z}$.)

Coefficient specialization

$\widetilde{B}$ and $\widetilde{B}'$: extended exchange matrices of rank n.

For each sequence $\mathbf{k}$ and $j \in [n]$, write $y_{\mathbf{k},j}$ for the j^{th} coefficient defined by $\mu_{\mathbf{k}}(\widetilde{B})$ and $y'_{\mathbf{k},j}$ for the j^{th} coefficient defined by $\mu_{\mathbf{k}}(\widetilde{B}')$.

A ring homomorphism $\varphi : \mathcal{A}(\mathbf{x}, \widetilde{B}) \to \mathcal{A}(\mathbf{x}', \widetilde{B}')$ is a coefficient specialization if

(i) the exchange matrices B and B' coincide;

(ii) $\varphi(x_j) = x'_j$ for all $j \in [n]$ (i.e. initial cluster variables coincide);

(iii) φ restricts to a continuous R-linear map on tropical semifields.

(iv) $\varphi(y_{\mathbf{k},j}) = y'_{\mathbf{k},j}$ and $\varphi(y_{\mathbf{k},j} \oplus 1) = y'_{\mathbf{k},j} \oplus 1$ for each $\mathbf{k}$ and each j.

Coefficient specialization

$\widetilde{B}$ and $\widetilde{B}'$: extended exchange matrices of rank n.

For each sequence $\mathbf{k}$ and $j \in [n]$, write $y_{\mathbf{k},j}$ for the j^{th} coefficient defined by $\mu_{\mathbf{k}}(\widetilde{B})$ and $y'_{\mathbf{k},j}$ for the j^{th} coefficient defined by $\mu_{\mathbf{k}}(\widetilde{B}')$.

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(iv) $\varphi(y_{\mathbf{k},j}) = y'_{\mathbf{k},j}$ and $\varphi(y_{\mathbf{k},j} \oplus 1) = y'_{\mathbf{k},j} \oplus 1$ for each $\mathbf{k}$ and each j.

Continuity refers to the product topology on tropical semifields.

An R-linear map is just a multiplicative homomorphism of semifields when $R = \mathbb{Z}$ or $\mathbb{Q}$.

Universal geometric cluster algebras

Fix an exchange matrix B.

Category: geometric cluster algebras with exchange matrix B; coefficient specializations. (Depends on R.)

Universal geometric cluster algebra: universal object in category.

Isomorphism type of a geometric cluster algebra depends only on the extended exchange matrix $\widetilde{B}$. So we are really looking for universal geometric exchange matrices.

We'll use the term universal geometric coefficients for the coefficient rows of a universal geometric exchange matrix.

Previous results (for B bipartite, finite type):

Fomin-Zelevinsky, 2006:
Coefficient rows of universal extended exchange matrix are indexed by almost positive co-roots $\beta^\vee$. The row for $\beta^\vee$ has i^{th} entry $\varepsilon(i)[\beta^\vee : \alpha_i^\vee]$ for $i = 1, \ldots, n$.

Reading-Speyer, 2007:
The linear map taking a positive root α_i to $-\varepsilon(i)\omega_i$ maps almost positive roots into rays of (bipartite) Cambrian fan.

Yang-Zelevinsky, 2008:
Rays of the Cambrian fan are spanned by vectors whose fundamental weight coordinates are $\mathbf{g}$-vectors of cluster variables.

Putting this all together (including sign change and "$^\vee$"):
Universal extended exchange matrix for B has coefficient rows given by $\mathbf{g}$-vectors for B^T.

(Works for any R or for a much larger category)

Universal geometric coefficients and μ-linear algebra

Theorem (R., 2012)

A collection $(\mathbf{b}_i : i \in I)$ *are universal coefficients for* B *over* R *if and only if they are an* R*-basis for* B*.*

(That is, to make a universal extended exchange matrix, extend B with coefficient rows forming an R-basis for B.)

Example: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is universal for any R.

Why universal coefficients correspond to bases

Recall: a coefficient specialization is a continuous R-linear map between tropical semifields taking coefficients to coefficients everywhere in the Y-pattern.

Proposition.
Let $\mathrm{Trop}(u_i : i \in I)$ and $\mathrm{Trop}(v_j : j \in J)$ be tropical semifields and fix a family $(p_{ij} : i \in I, j \in J)$ of elements of R. Then the following are equivalent.

(i) $\exists$ continuous R-linear map φ from $\mathrm{Trop}(u_i : i \in I)$ to $\mathrm{Trop}(v_j : j \in J)$, with $\varphi(u_i) = \prod_{j \in J} v_j^{p_{ij}}$ for all $i \in I$.

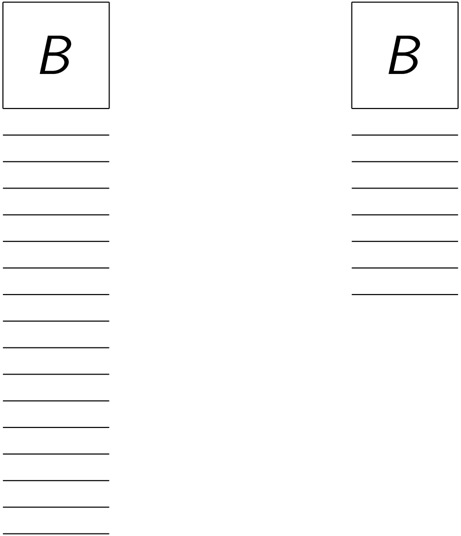
(ii) For each $j \in J$, there are only finitely many indices $i \in I$ such that p_{ij} is nonzero.

When these conditions hold, the unique map φ is

$$\varphi\Big(\prod_{i \in I} u_i^{a_i}\Big) = \prod_{j \in J} v_j^{\sum_{i \in I} p_{ij} a_j}.$$

Why universal coefficients correspond to bases (continued)

A picture suitable for hand-waving:



The mutation fan

One "easy" way to get B-coherent linear relations: Find vectors in the same domain of linearity of all mutation maps and make a linear relation among them.

Define an equivalence relation $\equiv^B$ on $\mathbb{R}^n$ by setting

$$\mathbf{a}_1 \equiv^B \mathbf{a}_2 \quad \iff \quad \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_1)) = \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_2)) \quad \forall \mathbf{k}.$$

$\mathbf{sgn}(\mathbf{a})$ is the vector of signs $(-1, 0, +1)$ of the entries of $\mathbf{a}$.

B-classes: equivalence classes of $\equiv^B$.

B-cones: closures of B-classes.

Facts:

B-classes are cones.
B-cones are closed cones.
Each map $\eta_{\mathbf{k}}^B$ is linear on each B-cone.

The mutation fan (continued)

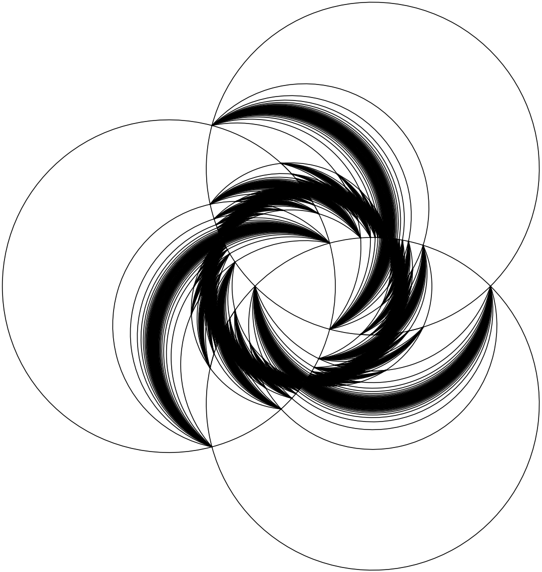
Mutation fan for B:
The collection $\mathcal{F}_B$ of all B-cones and all faces of B-cones.

Theorem (R., 2011)

$\mathcal{F}_B$ is a complete fan.

Example: For $B = [0]$:
μ_1 is negation and $\mathcal{F}_B = \{\{\mathbf{0}\}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}\}$.

Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ (Markov quiver)



Positive bases

An R-basis for B is positive if, for every $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$, the unique B-coherent linear relation $\mathbf{a} - \sum_{i \in I} c_i \mathbf{b}_i$ has all $c_i \geq 0$.

In many examples, positive R-bases exist, but not always, and it depends on R.

There is at most one positive R-basis for B, up to scaling each basis element by a positive unit in R.

Theorem (R., 2012)

If a positive $\mathbb{R}$-basis for B exists, then $\mathcal{F}_B$ is simplicial. The basis consists of exactly one vector in each ray of $\mathcal{F}_B$.

For $R \neq \mathbb{R}$, there is a similar statement in terms of the "R-part" of $\mathcal{F}_B$ (e.g. the "rational part") of $\mathcal{F}_B$.

$\mathbf{g}$-Vectors enter the picture

Conjecture

The nonnegative orthant $(\mathbb{R}_{\geq 0})^n$ is a B-cone. (Equivalently, principal coefficients are sign-coherent.)

Theorem (R., 2011)

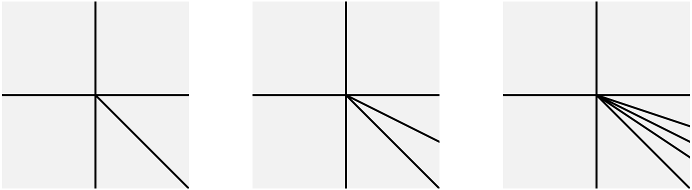
Assume the above conjecture holds for every exchange matrix that is mutation-equivalent to B or to $-B$. Then the $\mathbf{g}$-vector fan for B^T is a subfan of $\mathcal{F}_B$.

Proof uses a Nakanishi-Zelevinsky result.

One interpretation of this theorem: The mutation fan is the natural generalization of the $\mathbf{g}$-vector fan of a cluster algebra of finite type. This leads to lots of speculative ideas, e.g. about "nice" additive bases for cluster algebras or virtual semi-invariants of quivers, or indices of "unreachable" objects in $\mathcal{C}_{Q,W}$ (Plamondon's talk).

Rank 2 finite type

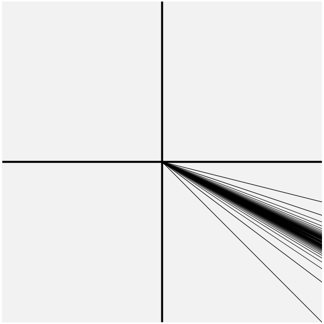
$\mathcal{F}_B$ is finite and rational. Taking a smallest integer vector in each ray, we get a positive R-basis for B for any R. (These are the $\mathbf{g}$-vectors for B^T.)



$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -3 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 3 & -2 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}$$

Rank 2 affine type

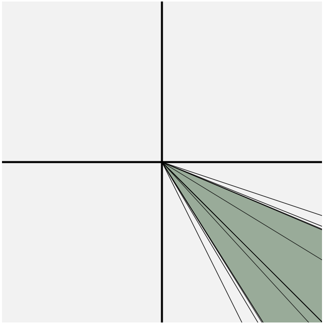
$\mathcal{F}_B$ is infinite but still rational. Again a positive R-basis for B for any R. (These are the $\mathbf{g}$-vectors for B^T with an additional limiting vector.)



$$\left[\begin{array}{rr} 0 & 1 \\ -4 & 0 \\ \hline -1 & 0 \\ 1 & -1 \\ 3 & -2 \\ \vdots & \vdots \\ \hline 0 & -1 \\ 4 & -3 \\ 8 & -5 \\ \vdots & \vdots \\ \hline 0 & 1 \\ 4 & -1 \\ 8 & -3 \\ \vdots & \vdots \\ \hline 1 & 0 \\ 3 & -1 \\ 5 & -2 \\ \vdots & \vdots \\ \hline 2 & -1 \end{array}\right]$$

Rank 2 wild type

For $B = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$, the fan $\mathcal{F}_B$ is infinite and algebraic, not rational.



All cones are rational except the shaded cone, which is the non-negative span of

$$[\sqrt{6}, \quad -\sqrt{6} - \sqrt{2}]$$

and

$$[3(\sqrt{6} + \sqrt{2}), \quad -2\sqrt{6}].$$

A positive R-basis for B if $R = \mathbb{R}$:

Take a nonzero vector in each ray of $\mathcal{F}_B$.

A $\mathbb{Q}$-basis for B:

Take a rational vector in each ray except the irrational rays. Then take any two linearly independent rational vectors in the shaded cone. This is not a positive basis.

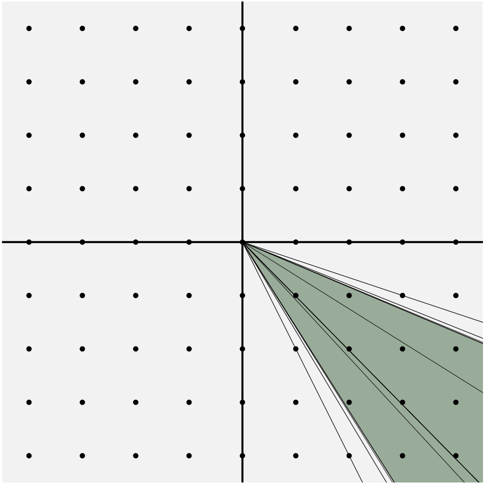
Rank 2 wild type (continued)... the case $R = \mathbb{Z}$

$B = \left[\begin{smallmatrix} 0 & 2 \\ -3 & 0 \end{smallmatrix}\right]$, C is shaded cone.

A $\mathbb{Z}$-basis for B:

• The smallest integer point in each rational ray, and

• Two integer points in C whose $\mathbb{Z}$-linear span contains all integer points in C.



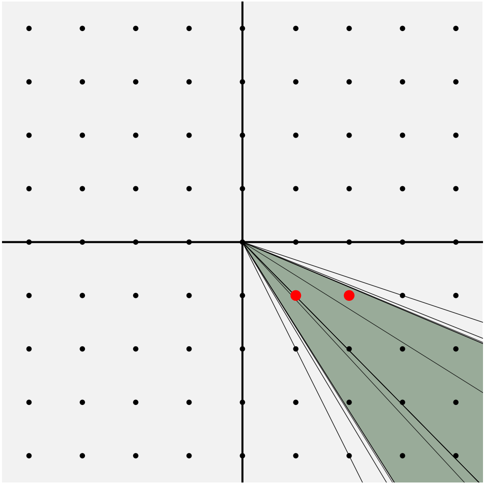
Rank 2 wild type (continued)... the case $R = \mathbb{Z}$

$B = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$, C is shaded cone.

A $\mathbb{Z}$-basis for B:

- The smallest integer point in each rational ray, and
- Two integer points in C whose $\mathbb{Z}$-linear span contains all integer points in C. The two red points work because they are a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

This is not a positive basis.



Finite and affine type

As mentioned earlier: In finite type, $\mathbf{g}$-vectors for B^T are a basis for B.

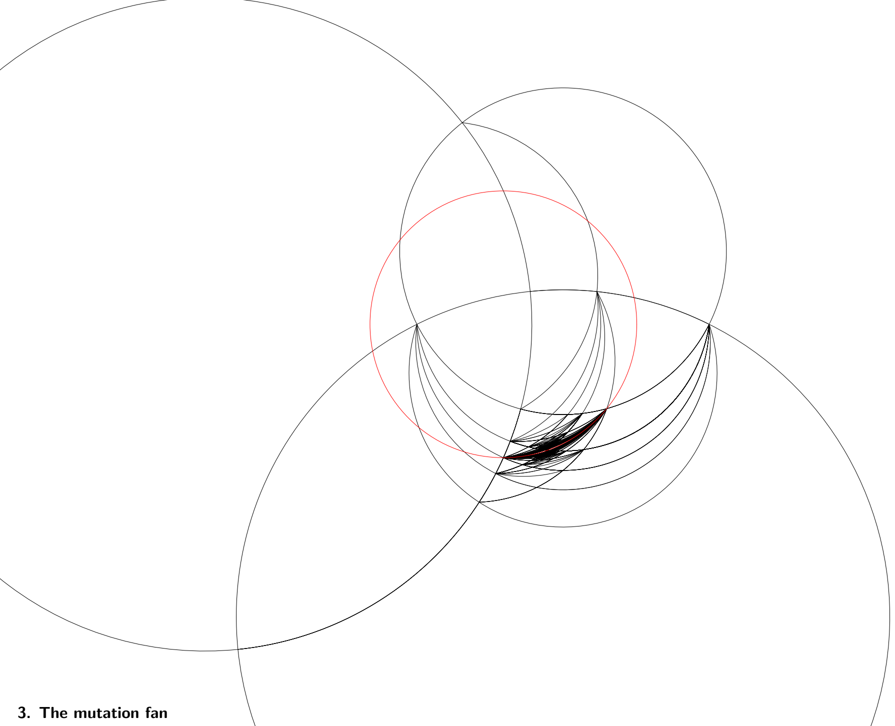
This can be proved directly and uniformly using the geometry of root systems.

Similar arguments can be made in affine type. Using results of a paper in progress (joint with Speyer) on $\mathbf{g}$-vectors in affine type, we should be able to prove the following conjecture:

Conjecture

For B of affine type, a positive R-basis for B (over any R) consists of the $\mathbf{g}$-vectors for B^T, plus one additional integer vector in the boundary of the Tits cone.

This is not surprising in light of Sherman-Zelevinsky and in light of the surfaces case. (In fact, I have a proof in the surfaces case.)



Universal geometric cluster algebras from surfaces

Allowable curves: Almost, but not quite, the set of curves that appear in (unbounded measured) laminations.

Compatibility of allowable curves: Almost but not quite that they don't intersect. (One kind of intersection is allowed.)

What should happen:

1. $\mathbb{Z}$-basis or $\mathbb{Q}$-basis for $B(T)$ consists of shear coordinates of allowable curves.
2. (Rational part of) the mutation fan consists of cones spanned by shear coordinates of pairwise compatible sets of allowable curves.

What happens depends on two key properties of the surface:

The Curve Separation Property and the Null Tangle Property.

(Properties are proved in some cases. No counterexamples.)

The Null Tangle Property

A tangle is a finite weighted collection Ξ of distinct allowable curves, with no requirement of compatibility. The shear coordinates of Ξ are the weighted sum of the shear coordinates of its curves. A null tangle has shear coordinates zero with respect to every tagged triangulation.*

The Null Tangle Property: A null tangle has all weights zero.

Theorem (R., 2012)

The shear coordinates of allowable curves are a (positive) $\mathbb{Z}$- or $\mathbb{Q}$-basis for $B(T)$ if and only if the Null Tangle Property holds.

Theorem (R., 2012)

The Null Tangle Property holds for surfaces with polynomial growth and for the once-punctured torus.

The Curve Separation Property

The Curve Separation Property: If λ and ν are incompatible allowable curves, then there exists a tagged triangulation* T and a tagged arc $\gamma \in T$ such that the shear coordinates of λ and ν with respect to T have strictly opposite signs in the entry indexed by γ.

Null Tangle Property $\implies$ Curve Separation Property

Theorem (R., 2012)

The rational part of the mutation fan consists of cones spanned by shear coordinates of pairwise compatible sets of allowable curves if and only if the Curve Separation Property holds.

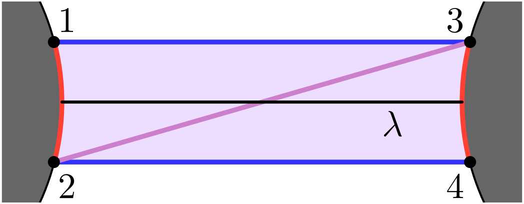
Theorem (R., 2012)

The Curve Separation Property holds except possibly for surfaces of genus > 1 with no boundary components and exactly one puncture.

Proving the properties

Curve Separation Property: For each incompatible λ and ν, construct a tagged triangulation T. Many cases.

For example, if λ connects two distinct boundary segments:



Similarly, in any null tangle, a curve λ connecting two distinct boundary segments occurs with weight zero.

Thanks for listening.

References:

Universal geometric cluster algebras. arXiv:1209.3987

Universal geometric cluster algebras from surfaces. arXiv:1209.4095

Universal geometric coefficients for the once-punctured torus. In preparation.