The Order Dimension of the Poset of Regions of a Hyperplane Arrangement

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Summary

We show that the order dimension of the weak order on a Coxeter group of type A, B or D is equal to the rank of the Coxeter group, and give bounds on the order dimensions for the other finite types. This result arises from a unified approach which, in particular, leads to a simpler treatment of the previously known cases, types A and B. The result for weak orders follows from an upper bound on the dimension of the poset of regions of an arbitrary hyperplane arrangement. In some cases, including the weak orders, the upper bound is the chromatic number of a certain graph. For the weak orders, this graph has the positive roots as its vertex set, and the edges are related to the pairwise inner products of the roots.

Résumé

Nous prouvons que la dimension d'ordre de l'ordre faible sur un groupe de Coxeter de type A, B ou D est égale au rang du groupe de Coxeter. Pour les autres groupes finis de Coxeter, nous donnons des bornes inférieures et supérieures sur la dimension d'ordre. Ce résultat découle d'une approche unifiée qui, en particulier, nous permet de traiter les cas deja connus des types A et B d'une maniere plus simple. Le résultat concernant les ordres faibles découle d'une borne supérieure sur la dimension du poset des régions d'un arrangement arbitraire d'hyperplans. Dans certains cas, incluant les ordre faibles, la borne supérieure est le nombre chromatique d'un certain graphe. Pour les ordres faibles, l'ensemble des sommets de ce graph correspond aux racines positives tandis que les arcs sont reliés aux produits scalaires entre les racines.

Main Result

Theorem 1. *The order dimension of the weak order on an irreducible finite Coxeter group has the following value or bounds:*

$$\begin{array}{rcccl} & & \dim(A_n) & = & n \\ & & \dim(B_n) & = & n \\ & & \dim(D_n) & = & n \\ 6 & \leq & \dim(E_6) & \leq & 9 \\ 7 & \leq & \dim(E_7) & \leq & 11 \\ 8 & \leq & \dim(E_8) & \leq & 19 \\ 4 & \leq & \dim(F_4) & \leq & 5 \\ & & \dim(H_3) & = & 3 \\ 4 & \leq & \dim(H_4) & \leq & 6 \\ & & \dim(I_2(m)) & = & 2 \end{array}$$

• Order dimension of a poset P: The smallest n so that P can be embedded as an induced subposet of the componentwise order on $\mathbb{R}^n$.

• Order dimension for reducible Coxeter groups: The sum of the dimensions of the irreducible components.

• Lower bounds of Theorem 1: Easy (consider the atoms and coatoms of the poset).

• Upper bounds: By Theorem 2, below, which gives an upper bound on the order dimension of the poset of regions of any hyperplane arrangement.

The Poset of Regions

• A central hyperplane arrangement $\mathcal{A}$ is a collection of linear $(d-1)$-subspaces of $\mathbb{R}^d$.

• The *regions* are the connected components of $\mathbb{R}^d - \cup\mathcal{A}$.

• Fix a *base region* B.

• The *poset of regions* $\mathcal{P}(\mathcal{A}, B)$ is the adjacency graph of the regions of $\mathcal{A}$, directed away from B.

• Below, we define the *basic digraph*, $\mathcal{D}(\mathcal{A}, B)$, a directed graph with vertex set $\mathcal{A}$.

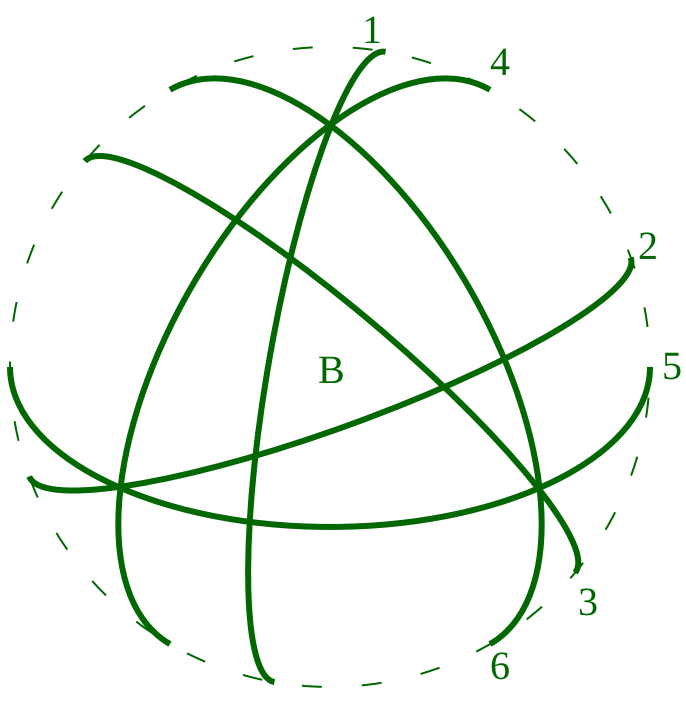
Theorem 2. *For a central hyperplane arrangement $\mathcal{A}$ with base region B, the order dimension of $\mathcal{P}(\mathcal{A}, B)$ is bounded above by the size of any covering of $\mathcal{D}(\mathcal{A}, B)$ by acyclic induced sub-digraphs.*

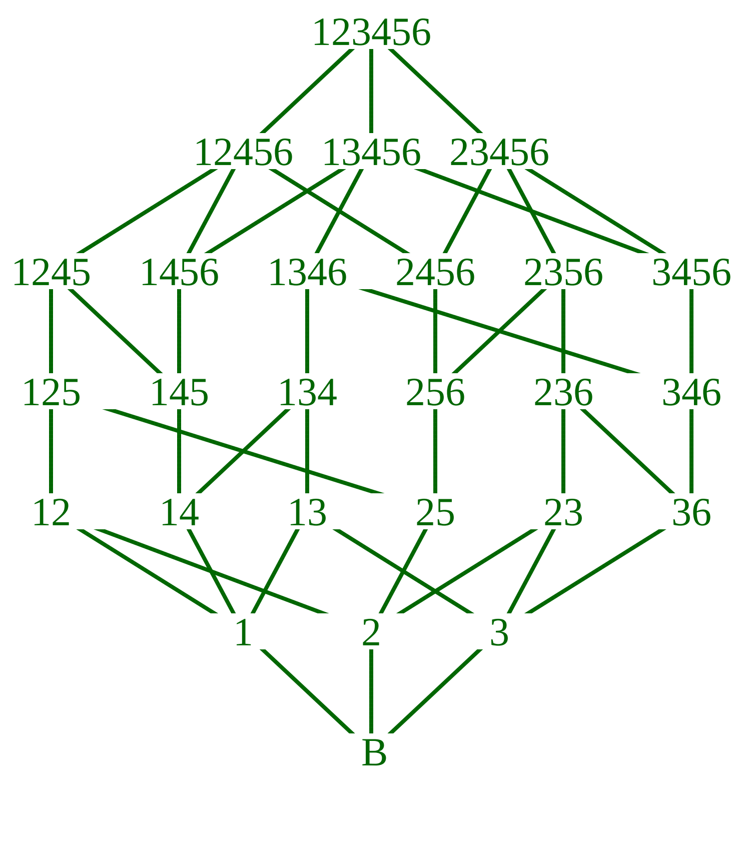
• A covering of size k of $\mathcal{D}(\mathcal{A}, B)$ by acyclic induced sub-digraphs is a partition $\mathcal{A} = I_1 \cup I_2 \cup \cdots \cup I_k$ such that each I_j induces an acyclic sub-digraph of $\mathcal{D}(\mathcal{A}, B)$.

• For a general poset P, there is a directed graph so that $\dim(P)$ is the size of the smallest covering by acyclic induced sub-digraphs. Theorem 2 gives a directed graph with much fewer vertices.

Example

In the figure below, an arrangement is represented by showing the intersection of each hyperplane with the unit sphere. The sphere is opaque so we only see its "top," and the base region is marked B. In the poset of regions, each region R is represented by its *separating set* $S(R)$, the set of hyperplanes separating R from B. The partial order $\mathcal{P}(\mathcal{A}, B)$ is containment of separating sets. This example, in green typeface, runs through the rest of the poster.





The Basic Digraph

• Given a codimension-2 subspace L of $\mathbb{R}^d$, the set $\mathcal{A}'$ of hyperplanes containing L is a *rank-two subarrangement* of $\mathcal{A}$, provided $|\mathcal{A}'| \geq 2$.

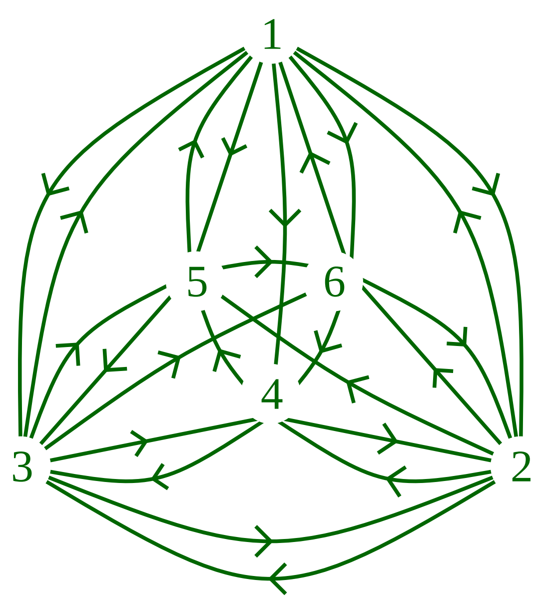
• The *basic* hyperplanes of $\mathcal{A}'$ are the two hyperplanes closest to the base region B.

• These are the rank-two subarrangements of the example, with the two basic hyperplanes underlined in each.

$$\begin{array}{ccc} \underline{1}4\underline{6} & \underline{13} & \underline{15} \\ \underline{24}5 & \underline{12} & \underline{26} \\ \underline{35}6 & \underline{23} & \underline{34} \end{array}$$

• The *basic digraph* $\mathcal{D}(\mathcal{A}, B)$ is the directed graph whose vertex set is $\mathcal{A}$, with an edge $H_1 \to H_2$ whenever H_1 is basic in the rank-two subarrangement determined by $H_1 \cap H_2$.

• This is $\mathcal{D}(\mathcal{A}, B)$ in the example.



The Embedding

- The proof of Theorem 2 gives an explicit embedding:

  - Let $I_1, I_2, \ldots, I_n$ be the covering of $\mathcal{D}(\mathcal{A}, B)$ by acyclic induced sub-digraphs.

  - Totally order each I_j compatibly with $\mathcal{D}(\mathcal{A}, B)$, and use this total order to interpret subsets of I_j as binary numbers.

  - The map is

$$\eta : R \mapsto (S(R) \cap I_1, S(R) \cap I_2, \ldots, S(R) \cap I_n).$$

- The easy direction of the proof is that $R_1 \leq R_2$ implies $\eta(R_1) \leq \eta(R_2)$.

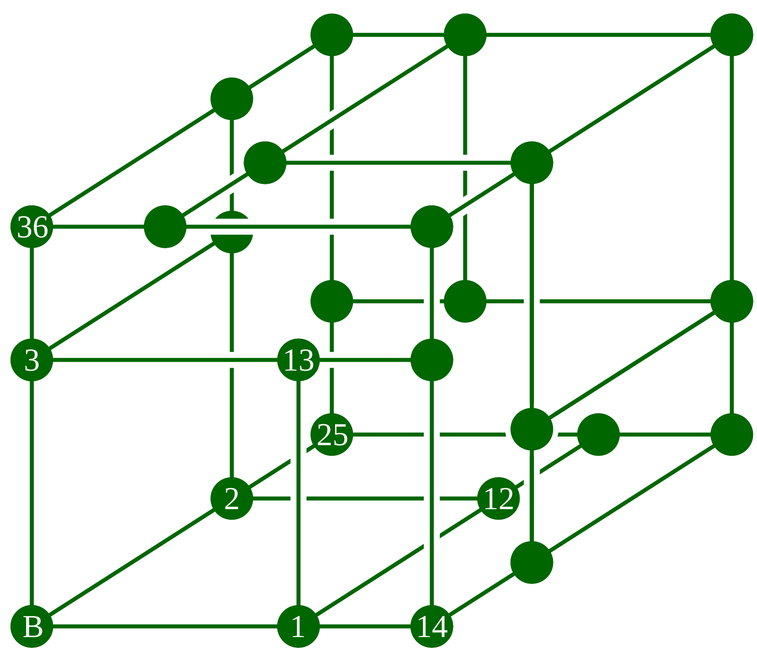
Example

In the example, there is a unique covering of $\mathcal{D}(\mathcal{A}, B)$ by three acyclic induced sub-digraphs. Thus the poset of regions in the example is 3-dimensional. The sub-digraphs are $I_1 := \{1 \to 4\}$, $I_2 := \{2 \to 5\}$ and $I_3 := \{3 \to 6\}$.

We totally order I_1 as $1, 4$. etc. So for example,

$$\begin{aligned} \eta(\{1,3,4\}) = (\{1,4\}, \emptyset, \{3\}) &= (11, 0, 10) \text{ binary,} \\ &= (3, 0, 2) \text{ decimal.} \end{aligned}$$

The embedding given by the proof of Theorem 2 is shown below. All elements lie on the surface of a cube, with 13 on the front face.



Coxeter Arrangements

• *Coxeter arrangement*: The set of reflecting hyperplanes of a finite group generated by reflections (a *Coxeter group*). These are classified, and the types are named with capital letters.

• There is a nice way to chose normal vectors (called *positive roots*) to the hyperplanes of a Coxeter arrangement.

• For Coxeter arrangements, the minimal cycles in the basic digraph $\mathcal{D}(\mathcal{A}, B)$ have cardinality two. Specifically if $H_1 \to H_2$ but $H_2 \not\to H_1$, then $l(H_1) < l(H_2)$. Here "$l(H)$" is the length of H considered as an element (a reflection) in the Coxeter group.

• Thus we can rephrase the upper bound as the chromatic number of the graph $G(\mathcal{A}, B)$ whose vertex set is $\mathcal{A}$ and whose edges are the two-cycles of $\mathcal{D}(\mathcal{A}, B)$. In other words, the edges in $G(\mathcal{A}, B)$ are the pairs H_1, H_2 of hyperplanes which are both basic in the rank-two subarrangement determined by $H_1 \cap H_2$.

• The edges of $G(\mathcal{A}, B)$ can be determined by considering inner products of pairs of roots in the corresponding root system. This leads to straightforward colorings of the graphs for Coxeter arrangements of types A, B and D. Chromatic numbers or bounds for types E, F and H were computed by John Stembridge.

Coloring Root Systems

The Coxeter arrangement A_{n-1} consists of the hyperplanes whose normals ("positive roots") are

$$\{\epsilon_i - \epsilon_j : 1 \leq j < i \leq n\}.$$

The Coxeter arrangement B_n has positive roots

$$\{\epsilon_i : 1 \leq i \leq n\} \cup \{\epsilon_i \pm \epsilon_j : 1 \leq j < i \leq n\}.$$

The Coxeter arrangement D_n has positive roots

$$\{\epsilon_i \pm \epsilon_j : 1 \leq j < i \leq n\}.$$

Key Point: In each of these cases, the inner product of the two basic roots in a rank-two subarrangement is 0 or -1.

There are many ways to n-color $G(\mathcal{A}, B)$ when $\mathcal{A}$ is the Coxeter arrangement A_n, B_n or D_n. We illustrate one nice coloring scheme which works for any n. In this example, the columns form a 6-coloring of B_6, and restrict to a 6-coloring of D_6 or a 5-coloring of A_5.

$\epsilon_6 + \epsilon_1$	$\epsilon_6 + \epsilon_2$	$\epsilon_6 + \epsilon_3$	$\epsilon_6 + \epsilon_4$	$\epsilon_6 + \epsilon_5$	ϵ_6
$\epsilon_5 + \epsilon_1$	$\epsilon_5 + \epsilon_2$	$\epsilon_5 + \epsilon_3$	$\epsilon_5 + \epsilon_4$	ϵ_5	$\epsilon_6 - \epsilon_5$
$\epsilon_4 + \epsilon_1$	$\epsilon_4 + \epsilon_2$	$\epsilon_4 + \epsilon_3$	ϵ_4	$\epsilon_5 - \epsilon_4$	$\epsilon_6 - \epsilon_4$
$\epsilon_3 + \epsilon_1$	$\epsilon_3 + \epsilon_2$	ϵ_3	$\epsilon_4 - \epsilon_3$	$\epsilon_5 - \epsilon_3$	$\epsilon_6 - \epsilon_3$
$\epsilon_2 + \epsilon_1$	ϵ_2	$\epsilon_3 - \epsilon_2$	$\epsilon_4 - \epsilon_2$	$\epsilon_5 - \epsilon_2$	$\epsilon_6 - \epsilon_2$
ϵ_1	$\epsilon_2 - \epsilon_1$	$\epsilon_3 - \epsilon_1$	$\epsilon_4 - \epsilon_1$	$\epsilon_5 - \epsilon_1$	$\epsilon_6 - \epsilon_1$

Other chromatic numbers were computed (or bounded) by John Stembridge. We summarize below:

Type	Chromatic number	Order dimension
A_n, B_n or D_n	n	n
E_6	9	$6 \leq \dim \leq 9$
E_7	11	$7 \leq \dim \leq 11$
E_8	$16 \leq \chi \leq 19$	$8 \leq \dim \leq 19$
F_4	5	$4 \leq \dim \leq 5$
H_3	3	3
H_4	6	$4 \leq \dim \leq 6$

Open Question

What are the order dimensions for the groups E_6, E_7, E_8, F_4, and H_4?

• If any of the dimensions exceeds the number of generators, it would be the first example known to the author of a simplicial arrangement in which the dimension of the poset of regions exceeds the rank.

• If each dimension is equal to the rank, is there a uniform proof of that fact (i.e. not relying on the classification of finite Coxeter groups)?

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