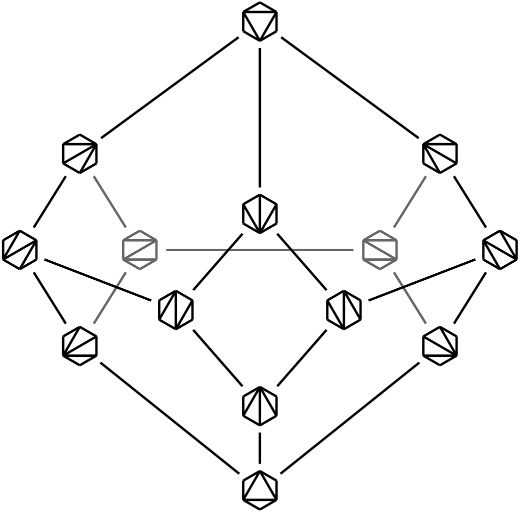
Dual combinatorics of clusters

Nathan Reading and David Speyer

NC State University

AMS Sectional Meeting
Davidson, NC, March 3, 2007

Associahedron (Haiman, Lee, Milnor, Stasheff, 1963–1989)



Triangulations of a polygon. (Think: maximal collections of noncrossing diagonals.)

Edges connecting triangulations are "diagonal flips."
This is a regular graph.

The associahedron is a simple convex polytope whose 1-skeleton is this graph.

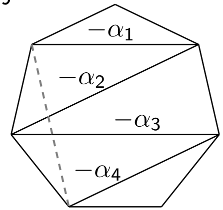
Generalized associahedron (Fomin, Zelevinsky, 2003)

Combinatorial datum: a finite crystallographic root system.

Almost positive roots: Positive roots & negatives of simple roots.
Clusters: max'l sets of pairwise "compatible" almost positive roots.
Compatibility graph: vertices are clusters, edges delete one root and replace it with the unique other root that makes a new cluster.
This graph is the 1-skeleton of the generalized associahedron.

Example (The root system of type A_n)

Diagonals of $(n+3)$-gon $\leftrightarrow$ "almost positive roots." $\binom{n+2}{2} - 1$.
Simples: $\alpha_1, \ldots \alpha_n$, Positives: $\alpha_i + \cdots + \alpha_i$, $i \leq j$
"Compatible" $=$ "noncrossing"
Exchanges $=$ diagonal flips.



Motivation for generalized associahedra

Fomin and Zelevinsky's definition arose from their study of cluster algebras. Specifically, each cluster algebra $\mathcal{A}$ of finite type corresponds to a crystallographic root system Φ. The generalized associahedron for Φ encodes the combinatorics of $\mathcal{A}$. In particular:

1. The cluster variables of $\mathcal{A}$ are indexed by the almost-positive roots of Φ.
2. Compatibility graph for almost-positive roots $\leftrightarrow$ exchange graph for cluster variables.
3. Denominator vector of a cluster variable $\leftrightarrow$ simple-root coordinates of the almost positive root.

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4$, $\quad c = s_1 s_2 s_3$, $\quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4, \quad c = s_1 s_2 s_3, \quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4, \quad c = s_1 s_2 s_3, \quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	$s_1 s_2$	4123

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4$, $c = s_1 s_2 s_3$, $\pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	$s_1 s_2$	4123
3	s_3	$s_1 s_2 s_3$	3124

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4, \quad c = s_1s_2s_3, \quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	s_1s_2	4123
3	s_3	$s_1s_2s_3$	3124
4	s_1	$s_1s_2s_3\vert$	3124

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4, \quad c = s_1 s_2 s_3, \quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	$s_1 s_2$	4123
3	s_3	$s_1 s_2 s_3$	3124
4	s_1	$s_1 s_2 s_3 \vert$	3124
5	s_2	$s_1 s_2 s_3 \vert s_2$	2134

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4, \quad c = s_1s_2s_3, \quad \pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	s_1s_2	4123
3	s_3	$s_1s_2s_3$	3124
4	s_1	$s_1s_2s_3\vert$	3124
5	s_2	$s_1s_2s_3\vert s_2$	2134
6	s_3	$s_1s_2s_3\vert s_2$	2134

Sorting words in the Coxeter group $W = S_n$

Multiplying a permutation π on the left by an adjacent transposition $s_i := (i\ i{+}1)$ swaps the entries i and $i+1$ in π.

Do this repeatedly, always putting entries into numerical order, and record the sequence of s_i's. Result: a reduced word for π.

Fix an order on the adjacent transpositions, and write a reduced word for π by trying the adjacent transpositions in that order, cyclically. Result: a sorting word for π. (C.f. "bubble sort.")

Example ($W = S_4$, $c = s_1 s_2 s_3$, $\pi = 4231$)

Step	s_i tried	Sorting word	Permutation
0			4231
1	s_1	s_1	4132
2	s_2	$s_1 s_2$	4123
3	s_3	$s_1 s_2 s_3$	3124
4	s_1	$s_1 s_2 s_3 \vert$	3124
5	s_2	$s_1 s_2 s_3 \vert s_2$	2134
6	s_3	$s_1 s_2 s_3 \vert s_2$	2134
7	s_1	$s_1 s_2 s_3 \vert s_2 \vert s_1$	1234

Sortable elements of a Coxeter group W (R., 2005)

In general, write a c-sorting word for $w \in W$ by trying the generators cyclically according to some order on the simple reflections S. (This order also defines a Coxeter element c.) Place a divider "|" every time a pass through S is completed.
A c-sorting word can be interpreted as a sequence of sets (sets of letters between dividers "|").
If the sequence is nested then w is c-sortable.

Example ($\pi = 4231$ with c-sorting word $s_1 s_2 s_3 | s_2 | s_1$)

π is not c-sortable because $\{s_1\} \not\subseteq \{s_2\}$.

Example ($W = S_3, \quad c = s_1 s_2$)

c-sortable:	$1, \; s_1, \; s_1 s_2, \; s_1 s_2 | s_1, \; s_2$
not c-sortable:	$s_2	s_1$

Example ($W = S_n, \quad c = s_1 s_2 \cdots s_n$)

c-sortables $\leftrightarrow$ "231-avoiding" or "stack-sortable" permutations. (C.f. Björner and Wachs, 1997.) For another c, "312-avoiding."

Results on sortable elements

1. For finite W, any c, bijection to W-noncrossing partitions. $w \mapsto$ reflections associated to "descents." (R., 2005)
2. For finite W, any c, bijection to vertices of the generalized associahedron—i.e. clusters. (R., 2005)
3. Thus, bijective explanation of why clusters and noncrossing partitions are equinumerous. (A different explanation: Athanasiadis, Brady, McCammond, Watt, 2005–2006)
4. Deep connection to the lattice theory of the weak order on W, specifically Cambrian lattices. (R., 2005)
5. Sortable elements lead to Cambrian fans, a novel (combinatorial) construction of the generalized associahedron. (R., Speyer, 2006)

Coxeter fan, Cambrian fan, cluster fan

The Coxeter arrangement: the set of reflecting hyperplanes of reflections of W. The hyperplanes cut space into simplicial cones (the Coxeter fan $\mathcal{F}$). Elements of $W \leftrightarrow$ maximal cones of $\mathcal{F}$.

$\pi_\downarrow^c$: the unique longest c-sortable element below w.
(In S_n "length" $=$ "number of inversions.")
Define $x \equiv_c y$ if $\pi_\downarrow^c(x) = \pi_\downarrow^c(y)$.

The Cambrian fan $\mathcal{F}_c$: Maximal cones are unions (over $\equiv_c$-classes) of maximal cones of the Coxeter fan. (Why is $\mathcal{F}_c$ a fan? Because $\equiv_c$ is a lattice congruence of the weak order.)

The cluster fan: Each cluster defines a maximal cone—the positive linear span of the roots in the cluster.

Combinatorial isomorphism

Theorem (R., Speyer, 2006)

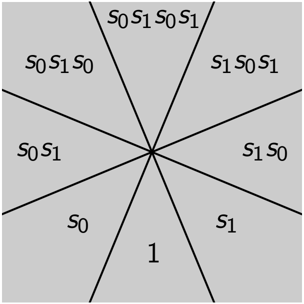
The bijection c-sortables $\leftrightarrow$ clusters induces a combinatorial isomorphism between the Cambrian fan and the cluster fan.

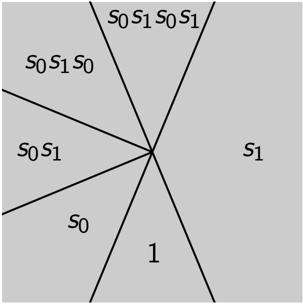
Consequences for cluster algebras

1. Constructs the combinatorial backbone of cluster algebras of finite type in a new way.
2. Some cluster algebra constructions are more natural in the Cambrian setting. For example, g-vectors of cluster variables are obtained as fundamental-weight coordinates. Also, this setting offers some insight into Fomin and Zelevinsky's sign-coherence conjecture.
3. Suggests a way to generalize the combinatorics of generalized associahedra to infinite Coxeter groups (work in progress).

Example ($W = B_2$, $c = s_0 s_1$)

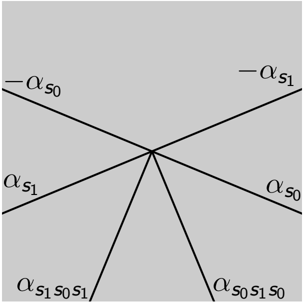
$\mathcal{F}$

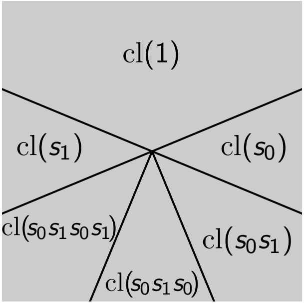




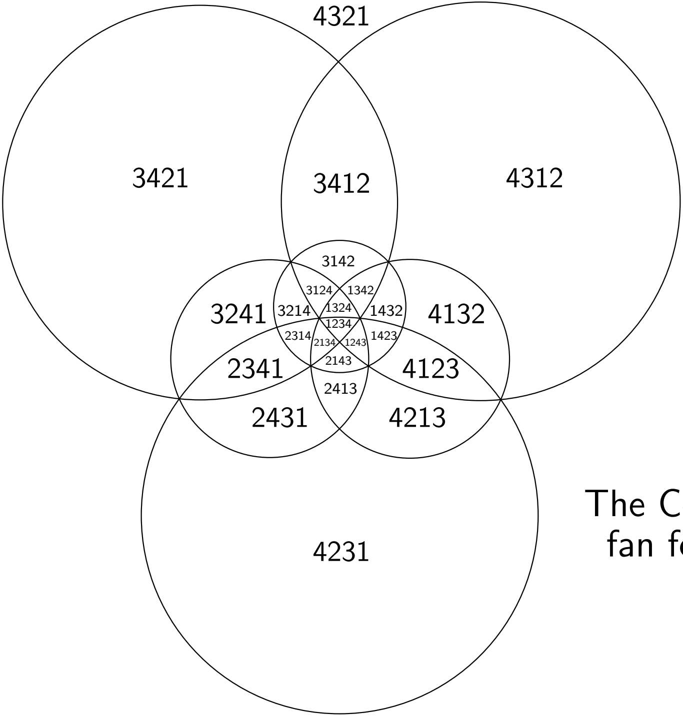
$\mathcal{F}_c$

Cluster fan

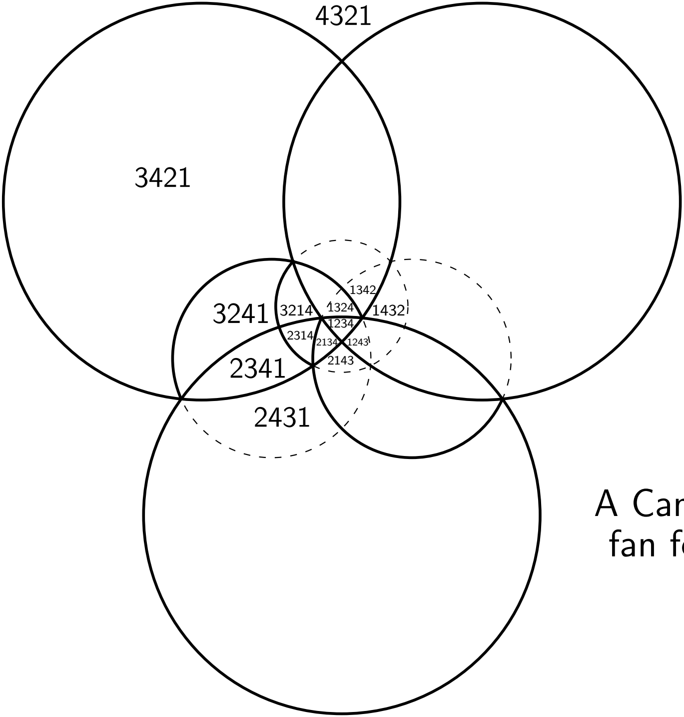




The bijection



The Coxeter fan for S_4



A Cambrian fan for S_4