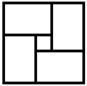
Generic rectangulations and pattern-avoiding permutations

Nathan Reading

AMS Special Session on Species and Hopf-Algebraic Combinatorics
Cornell University, Sept. 10, 2011

Generic rectangulations

Rectangulation: a tiling of a rectangle by rectangles.



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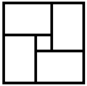

We'll consider them up to combinatorial equivalence.

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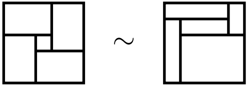

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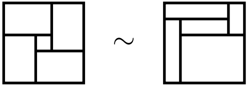

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n	g. rects. w/ n tiles	#

Recursive formula: Conant and Michaels, 2011.

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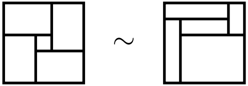

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n	g. rects. w/ n tiles	#
1		1

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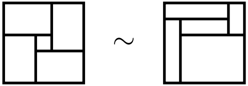

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n	g. rects. w/ n tiles	#
1		1
2	 	2

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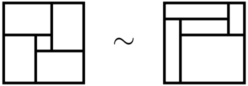

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n	g. rects. w/ n tiles	#
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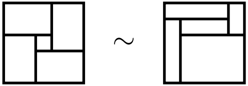

We'll consider them up to combinatorial equivalence.

n	g. rects. w/ n tiles	#
1		1
2		2
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4		24

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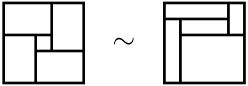

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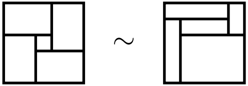

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$\vdots$		$\vdots$
27		$53845049871942333501408 \sim 5 \cdot 10^{22}$

Recursive formula: Conant and Michaels, 2011.

A general lattice-theoretic construction for Hopf subalgebras of MR

pattern-avoidance
A general ~~lattice-theoretic~~ construction for Hopf subalgebras of MR

A general ~~lattice-theoretic~~ pattern-avoidance construction for Hopf subalgebras of MR

Let y be a permutation of the form $\underset{}{\overset{\text{increasing}}{\rule{2cm}{0.4pt}}}\ k1\ \overset{\text{increasing}}{\rule{2cm}{0.4pt}} \in S_k$.

A scramble of y is $\overset{\text{any order}}{\rule{2cm}{0.4pt}}\ k1\ \overset{\text{any order}}{\rule{2cm}{0.4pt}}$.

Example: Scrambles of 256134: 256134, 256143, 526134, 526143.

Define $\text{Av}_n(y) =$
$\{x \in S_n : x \text{ avoids instances of scrambles of } y \text{ with } k1 \text{ adjacent}\}$

Example: $613982574 \notin \text{Av}_9(256134)$.

A general ~~lattice-theoretic~~ pattern-avoidance construction for Hopf subalgebras of MR

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Let y be a permutation of the form $\overset{\text{increasing}}{\rule{3em}{0.4pt}}\; k1 \;\overset{\text{increasing}}{\rule{3em}{0.4pt}} \in S_k$.

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Example: $613982574 \notin \mathrm{Av}_9(256134)$.

Define $\mathrm{Av}_n(y_1, \ldots, y_m) = \mathrm{Av}_n(y_1) \cap \cdots \cap \mathrm{Av}_n(y_m)$.

Write Av_n when y or $y_1, \ldots, y_m$ is understood.

A downward projection

Fix y or $y_1, \ldots, y_m$ as before.

Define $\pi_\downarrow : S_n \to \mathrm{Av}_n$:

- If $x \in \mathrm{Av}_n$ then $\pi_\downarrow(x) = x$.
- Otherwise, find an instance of a scramble of y (or some y_i) in x and swap "k" and "1" to get x'. Define $\pi_\downarrow(x) = \pi_\downarrow(x')$.

Example: Take $y = 256134$.

$\pi_\downarrow(613982574)$

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$\pi_\downarrow(613982574) = \pi_\downarrow(613928574)$

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Example: Take $y = 256134$.

$$\pi_\downarrow(613982574) = \pi_\downarrow(613\textcolor{red}{92}8574) = \pi_\downarrow(613\textcolor{red}{29}8574)$$

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A downward projection (hiding a lattice congruence)

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The fibers of $\pi_\downarrow$ are the congruence classes of a lattice congruence of the weak order on S_n. In particular, the fibers are intervals. The quotient lattice is a lattice structure on Av_n.

The Hopf algebra

Define a graded vector space $\mathbb{K}[\mathrm{Av}_\infty] = \bigoplus_{n\geq 0} \mathbb{K}[\mathrm{Av}_n]$.

Define $c : \mathbb{K}[\mathrm{Av}_\infty] \to \mathbb{K}[S_\infty]$ by

$$c(z) = \sum_{x \colon \pi_\downarrow(x)=z} x, \quad \text{for } z \in \mathrm{Av}_n\,.$$

Define $r : \mathbb{K}[S_\infty] \to \mathbb{K}[Av_\infty]$ by

$$r(x) = \begin{cases} x & \text{if } x \in \mathrm{Av}_n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x \in S_n.$$

Theorem (R., 2005)

$(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ *is a graded Hopf algebra, where*

$$x \bullet_{\mathrm{Av}} y = r(c(x) \bullet_S c(y)) = r(x \bullet_S y), \text{ and}$$
$$\Delta_{\mathrm{Av}}(z) = (r \otimes r)(\Delta_S(c(z))).$$

Corollary

The map c embeds $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ as a Hopf subalgebra of $(\mathbb{K}[S_\infty], \bullet_S, \Delta_S)$.

Motivating examples

$\mathbb{K}[\mathrm{Av}_\infty(21)]$

$\mathbb{K}[\mathrm{Av}_\infty(312, 231)]$

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Motivating examples

$\mathbb{K}[\mathsf{Av}_\infty(21)]$ One-dimensional graded pieces.

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In every case above, the pattern-avoidance condition defines the Hopf algebra "extrinsically," but there is a much simpler "intrinsic" description in terms of some combinatorial object. In other specific examples, we would like to discover the intrinsic combinatorics.

Motivating examples and new examples

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$\mathbb{K}[\mathrm{Av}_\infty(231, 4123)]$	Hopf algebra of sashes (Law's talk)

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$\vdots$

A map from permutations to diagonal rectangulations

$\rho : S_n \to \{\text{rectangulations}\}$.

Example: $\rho(467198352)$

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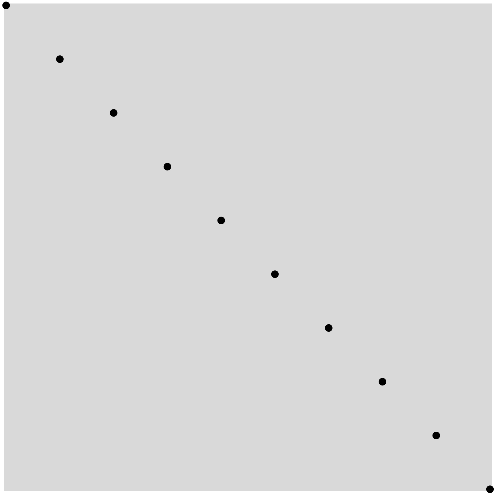

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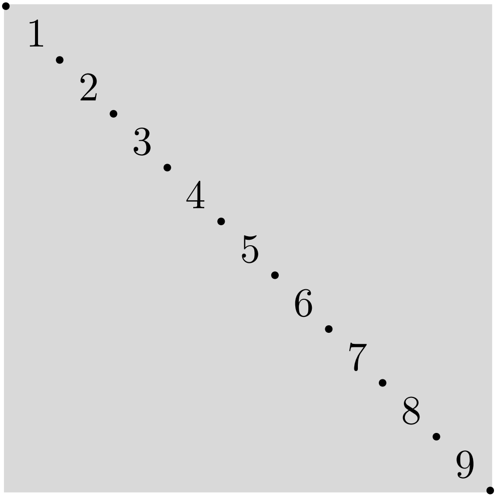

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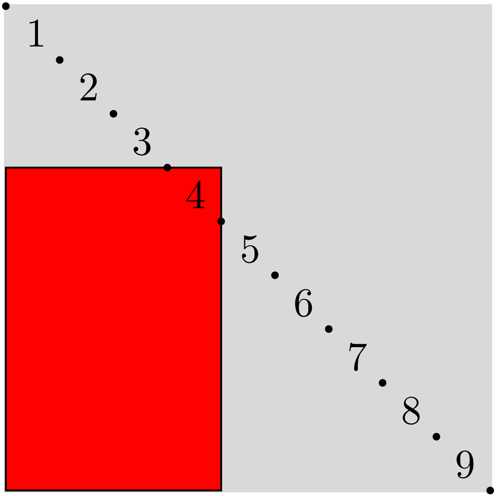

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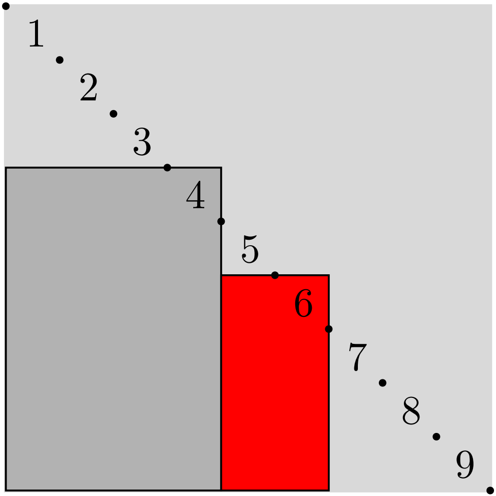

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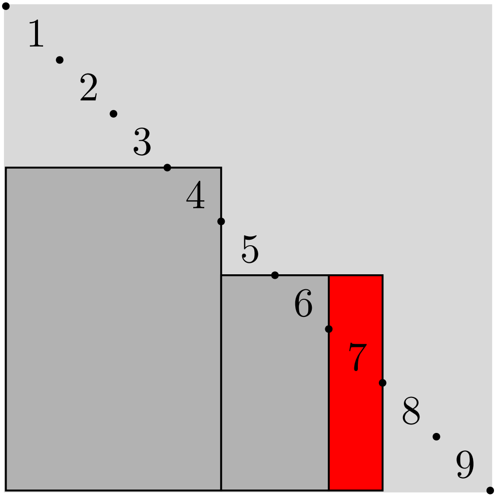

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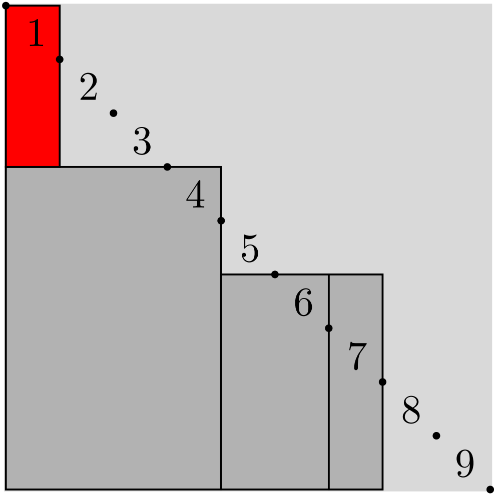
Example: $\rho(467\textcolor{red}{1}98352)$



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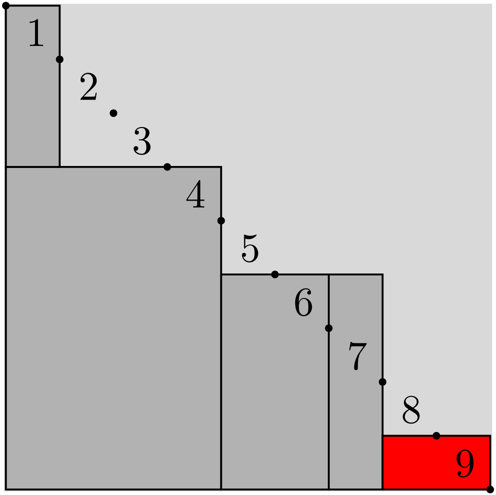

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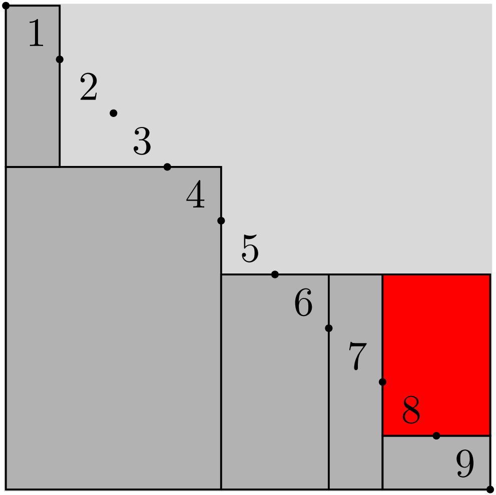

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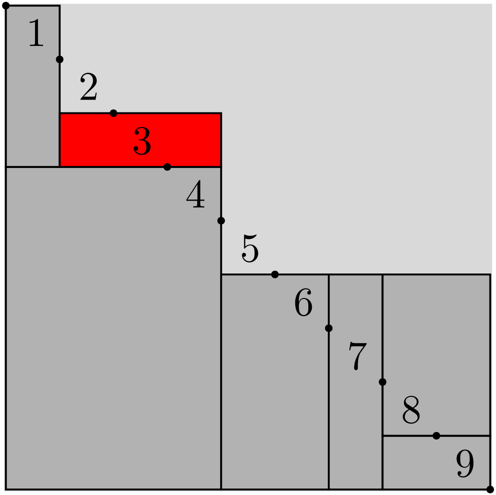
Example: $\rho(467198{\color{red}3}52)$



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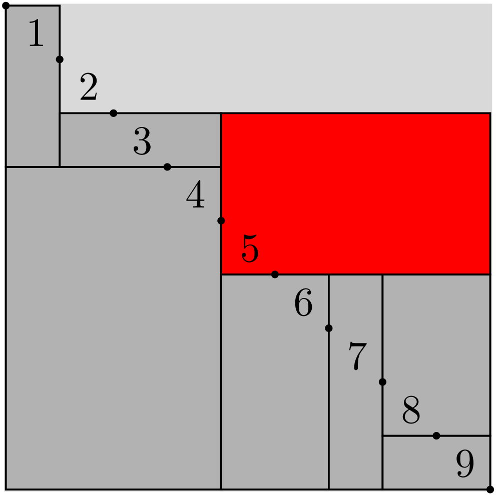

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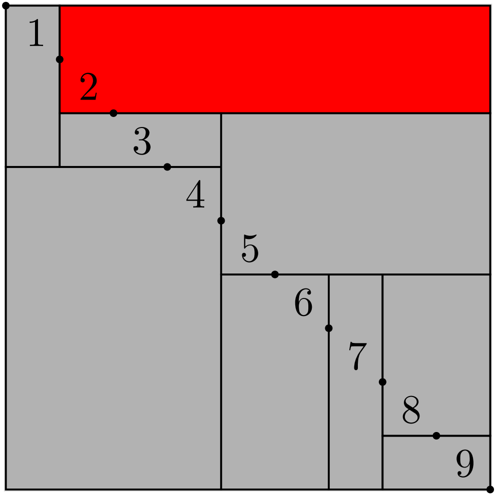
Example: $\rho(46719835\textcolor{red}{2})$



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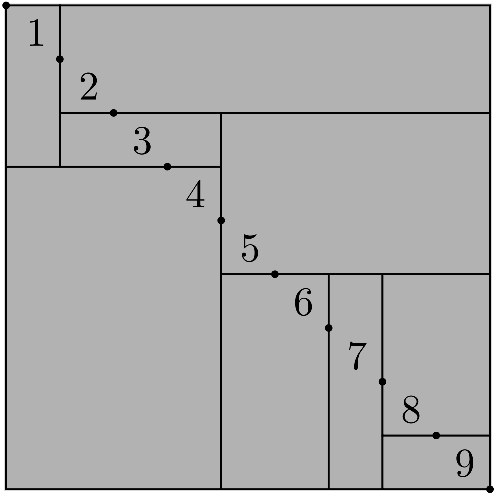
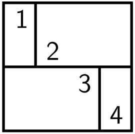
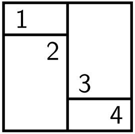
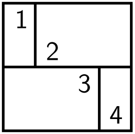


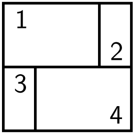
$\rho(3142) =$  $= \rho(3412)$

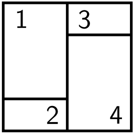
$\rho(2143) =$  $= \rho(2413)$

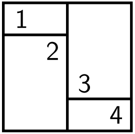
These two examples are the essence of the reason why $\pi_{\downarrow}(x)$ is the smallest permutation (in weak order) with $\rho(\pi_{\downarrow}(x)) = \rho(x)$.

A map from permutations to generic rectangulations

$\gamma(3142) =$ 

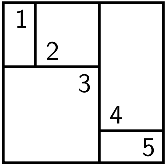
$\gamma(3412) =$ 

$\gamma(2143) =$ 

$\gamma(2413) =$ 

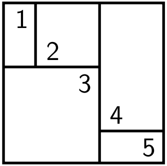
Generic rectangulations and pattern-avoidance

$\rho(31524) =$



$= \rho(35124)$

$\gamma(31524) =$



$= \gamma(35124)$

Similarly, $\gamma(24153) = \gamma(24513)$.

These examples are the essence of the proof that generic rectangulations are in bijection with $\mathsf{Av}_n(24513, 35124)$.