Cluster algebras and infinite associahedra

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CombinaTexas 2008

Coxeter groups

Associahedra and cluster algebras

Sortable elements/Cambrian fans

Infinite type

Much of the work described here is joint with David Speyer.

Coxeter groups

A Coxeter group is a group with a certain presentation. Choose a finite generating set $S = \{s_1, \ldots, s_n\}$ and for every $i < j$, choose an integer $m(i,j) \geq 2$. Define:

$$W = \left\langle S \mid s_i^2 = 1,\ \forall\, i \ \text{ and } \ (s_i s_j)^{m(i,j)} = 1,\ \forall\, i < j \right\rangle.$$

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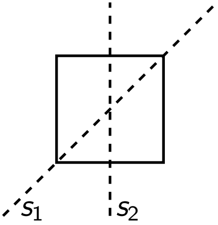
Why would anyone write this down?

Finite (Real!) reflection groups

A hyperplane in $\mathbb{R}^d$ is a subspace of dimension $d-1$.
An orthogonal reflection is a linear transformation which fixes a hyperplane with a (-1)-eigenspace orthogonal to the hyperplane.
A finite reflection group is a finite group generated by reflections.

Example (Symmetries of a square)



All symmetries of the square are compositions of the reflections s_1 and s_2. This is a finite reflection group.

The composition $s_1 s_2$ is a 90° rotation, so $(s_1 s_2)^4 = 1$. Abstractly, this group is

$$\left\langle \{s_1, s_2\} \mid s_1^2 = s_2^2 = (s_1 s_2)^4 \right\rangle .$$

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Finite (Real!) reflection groups (continued)

Theorem. A finite group is a Coxeter group if and only if it is a finite reflection group.

Finite reflection groups are interesting for many reasons, including applications to Lie theory and algebraic geometry. Finite reflection groups enjoy very pretty invariant theory. (Today at 1:45.)

Besides the generators S, other elements act as reflections. The collection $\mathcal{A}$ of reflecting hyperplanes for all these reflections cuts space into "regions." The generators S are the reflections in the walls of some region D. Identify D with the identity element 1. The map $w \mapsto wD$ is a bijection from W to $\mathcal{A}$-regions.

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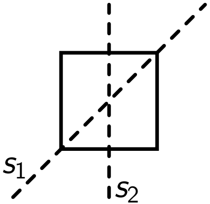

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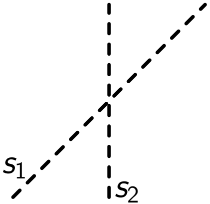

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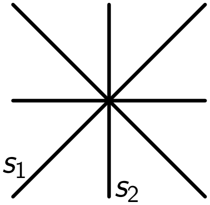

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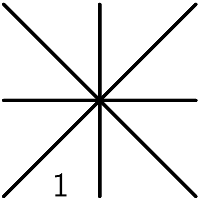

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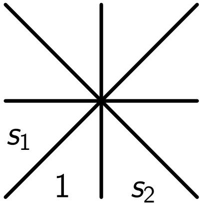

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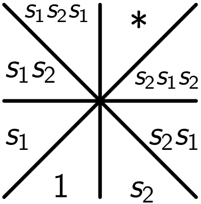
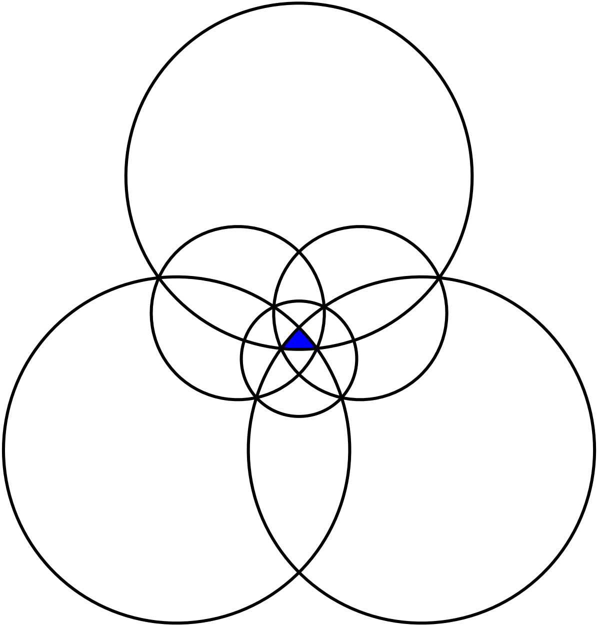


$$* = s_1s_2s_1s_2 = s_2s_1s_2s_1$$

Symmetric group S_4 (symmetries of regular tetrahedron)



Blue region is 1.

Regions $\leftrightarrow$ elements.

Largest circles: hyperplanes for s_1, s_2, and s_3. (s_2 on top.)

$m(s_1, s_2) = 3.$
$m(s_2, s_3) = 3.$
$m(s_1, s_3) = 2.$

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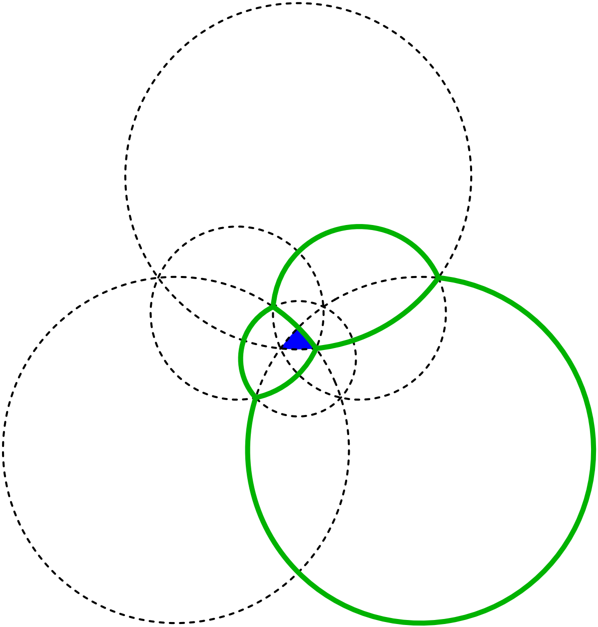

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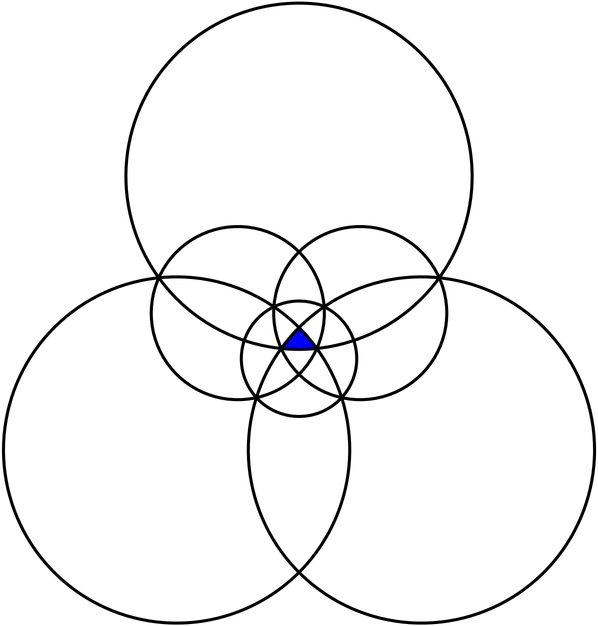

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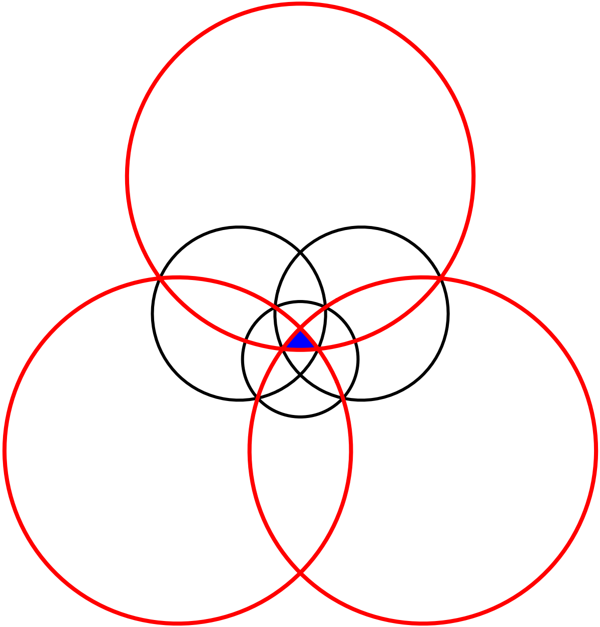

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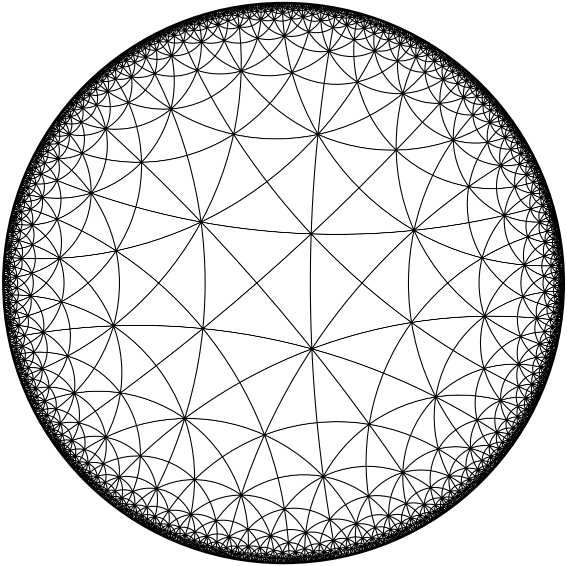
Infinite groups

Theorem. Any discrete group generated by reflections in a (constant curvature) spherical, Euclidean or hyperbolic space is a Coxeter group.

Theorem. For any Coxeter group W, one can define a symmetric bilinear form B on $\mathbb{R}^S$ and construct W as a discrete subgroup of the orthogonal group $O(\mathbb{R}^S, B)$ generated by reflections (i.e. reflections that are orthogonal with respect to B).

Again, a particular region (connected component of $\mathbb{R}^S$ minus the reflecting hyperplanes) represents the identity. Important point: The Tits cone is defined to be the union of closures of all regions in the W-orbit of the region representing 1. When W is infinite, the Tits cone is not all of $\mathbb{R}^S$.

A reflection group in the Poincare disk



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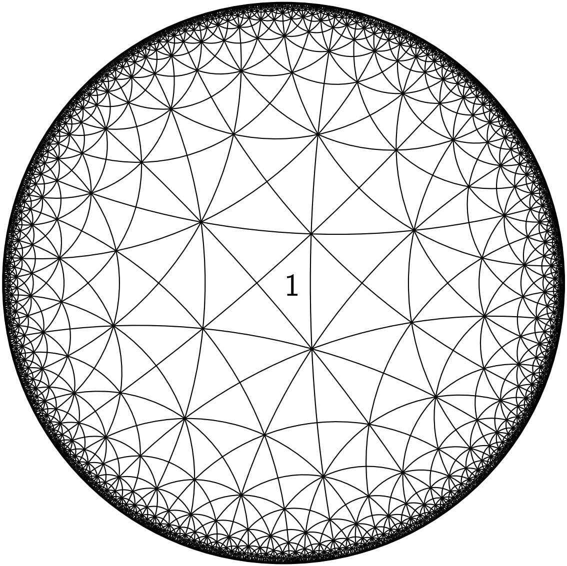

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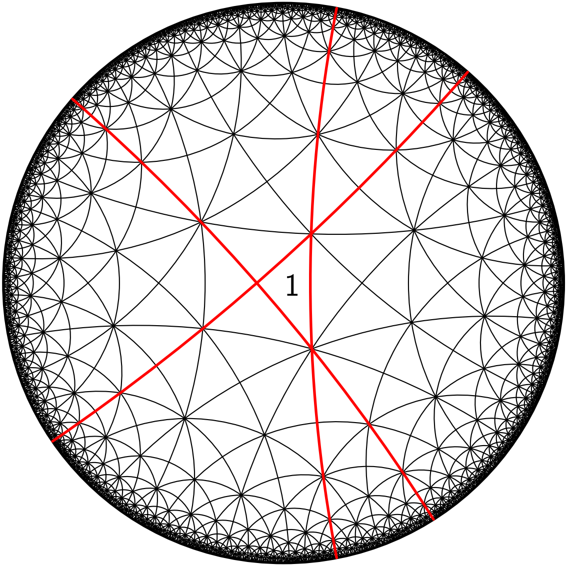

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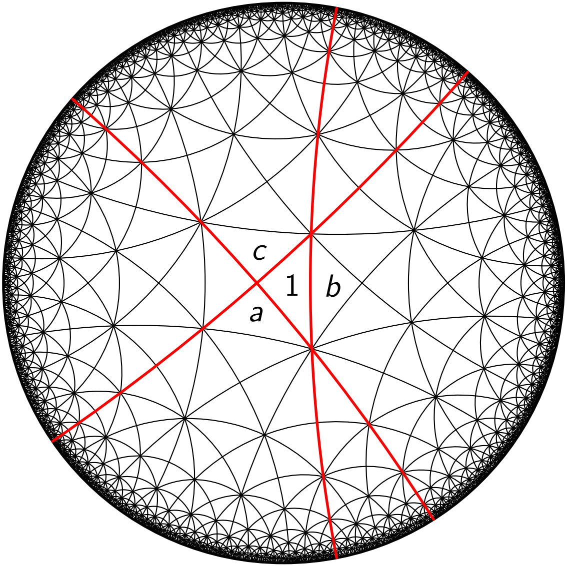
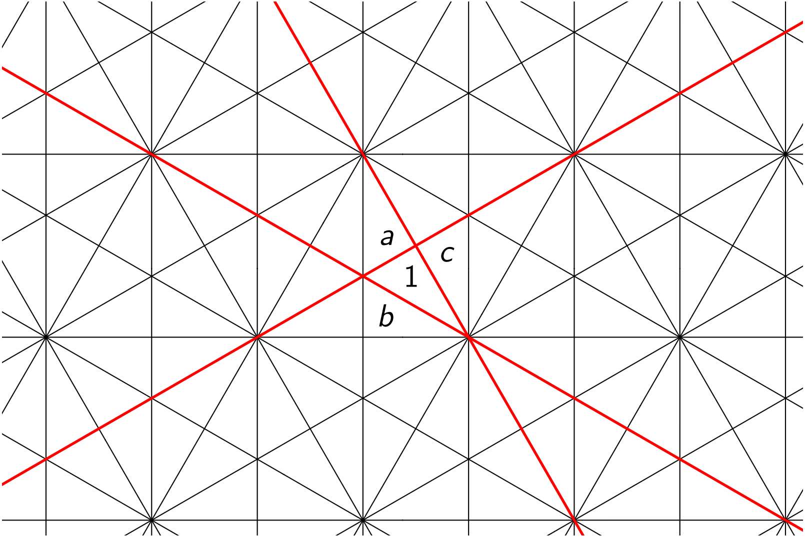


$m(a, b) = 5, \ m(b, c) = 4, \ m(a, c) = 2$

A reflection group in the Euclidean plane



$$m(a,b) = 3,\ m(b,c) = 6,\ m(a,c) = 2$$

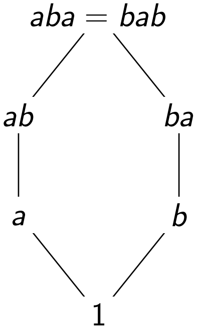
The weak order on a Coxeter group

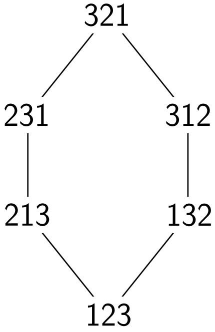
A reduced word for $w \in W$: a shortest possible sequence of elements of S such that the product of the sequence is w.
The weak order on W: the poset generated by relations $ws < w$ when $s \in S$ and w has a reduced word ending in s.
Weak order encodes the structure of reduced words.

Example: weak order on S_n

Move "up" by swapping two adjacent entries of the permutation, so as to put them out of order.

$S = \{a, b\}$
$a = (1\ 2),\ b = (2\ 3)$

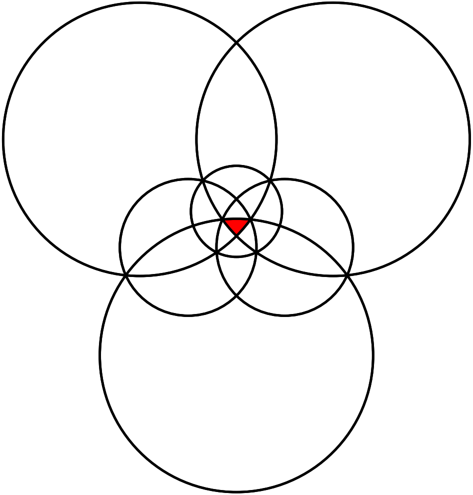




The weak order as a poset of regions (Example: S_4)

Recall: Group elements $\leftrightarrow$ regions (in Tits cone).

Geometric characterization of weak order: go "up" from 1 by crossing hyperplanes.



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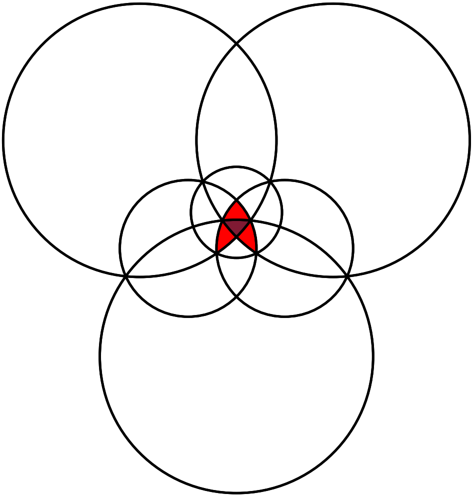

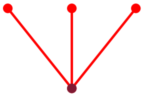

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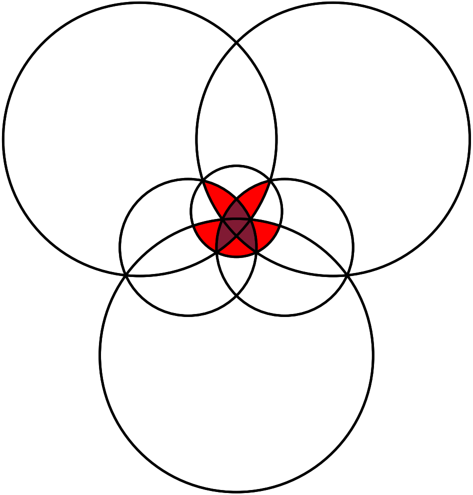

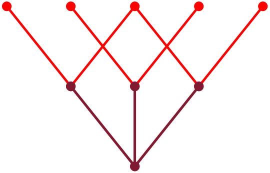

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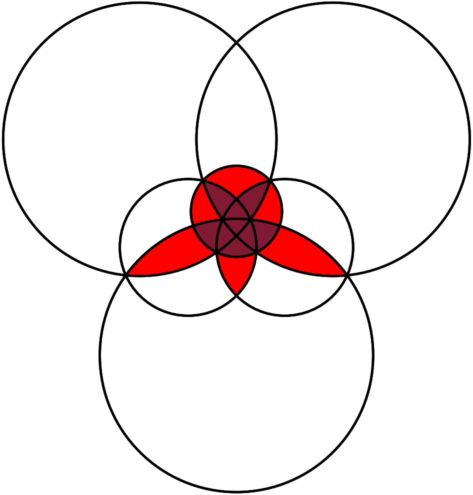

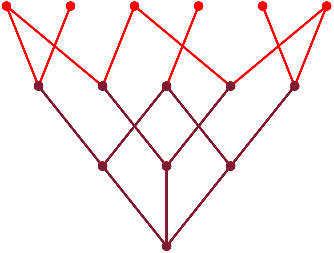

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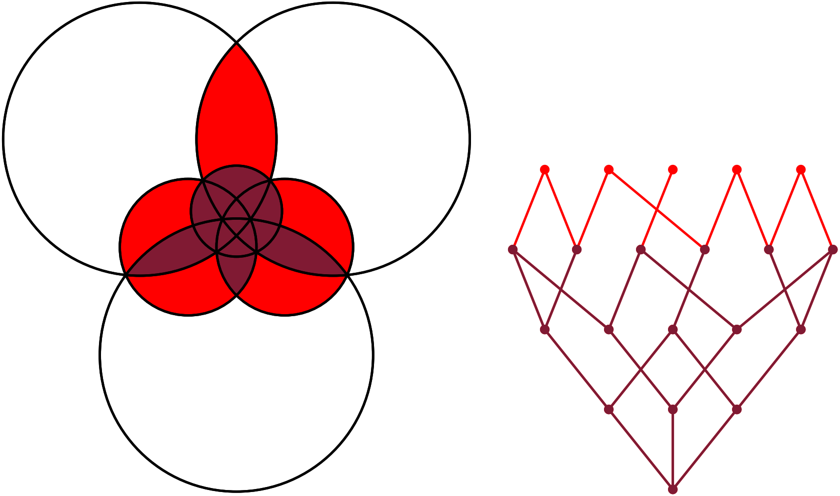

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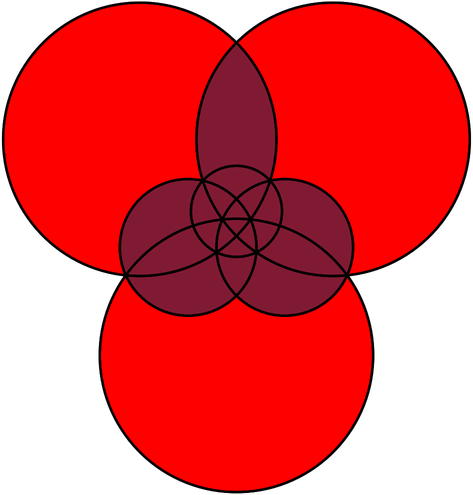

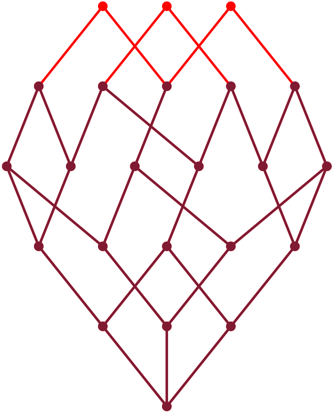

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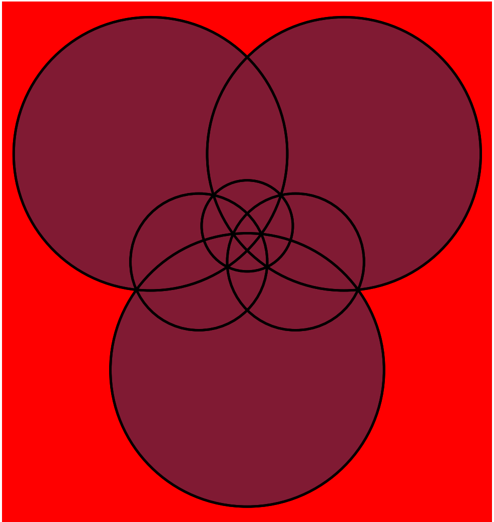

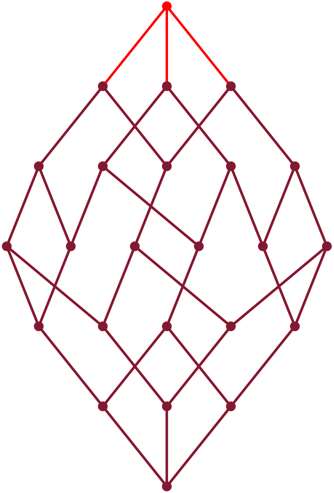

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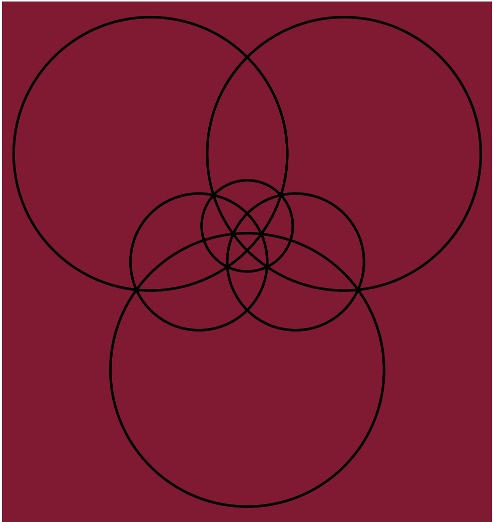
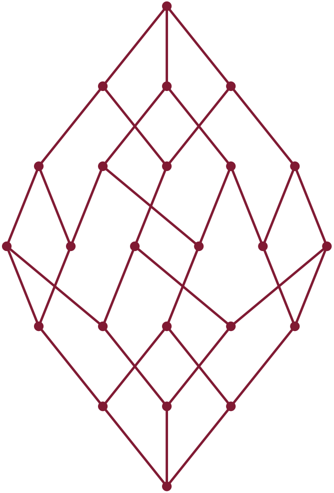




First take-home lesson

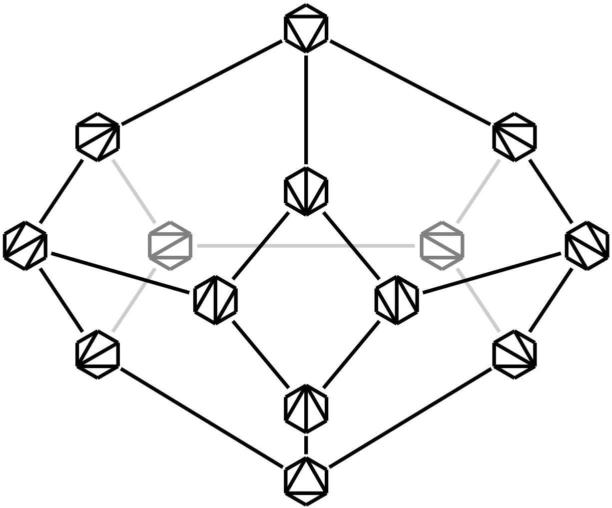
The beauty of Coxeter groups lies in the rich interplay between geometry, combinatorics and order/lattice theory.

Most proofs about Coxeter groups exploit a combination of methods:

1. Combinatorics of words in S.
2. Geometry of arrangements of reflecting hyperplanes. (Alternately, geometry of root systems. A root system is a collection of two normal vectors per hyperplane, chosen so that the root system is permuted by W.)
3. Order/lattice theory of the weak order. (Alternately, the Bruhat order.)
4. Linear algebra.
5. More...

The classical associahedron

An n-dimensional polytope whose vertices are labeled by triangulations of a convex $(n + 3)$-gon. (Counted by the Catalan number.) Edges correspond to diagonal flips. For $n = 3$:



Combinatorial clusters

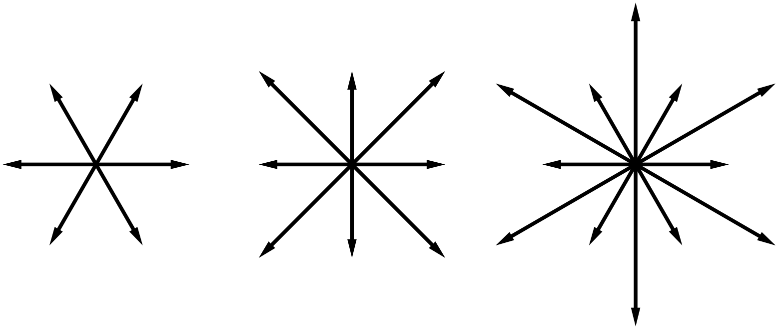
Root system Φ for W: two normals for each reflecting hyperplane.
Positive roots: nice choice of one root for each hyperplane.
Simple roots: positive roots associated to the generators S.
Almost positive roots: positive roots union negatives of simples.

Triangulations of a polygon are generalized by clusters: maximal sets of pairwise "compatible" roots in $\Phi_{\geq -1}$.

Clusters are counted by $\mathrm{Cat}(W)$, the W-Catalan number. (When W is the symmetric group, this is the usual Catalan number.)

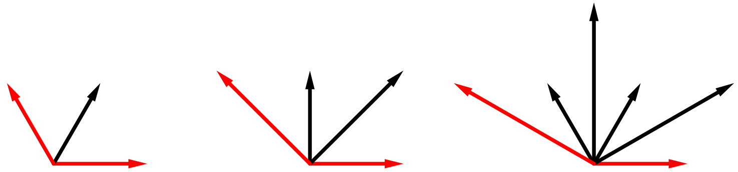
Rank-two examples (i.e. $|S| = 2$)

Root systems.



Rank-two examples (i.e. $|S| = 2$)

Positive roots. (Simple roots in red.)



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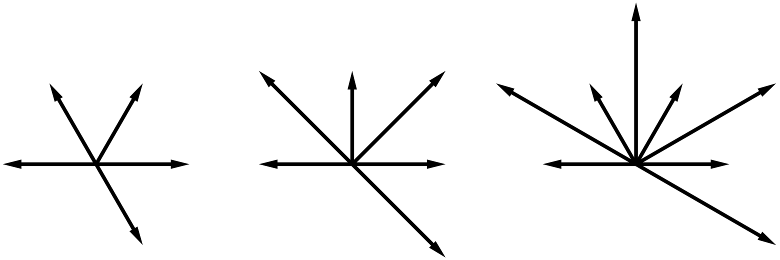
Almost positive roots.



In these examples, two almost positive roots are compatible if and only if they are "adjacent." Notice that the positive linear spans of clusters decompose space into cones. This happens in general, and the decomposition is called the cluster fan. The cluster fan is dual to a polytope called the generalized associahedron.

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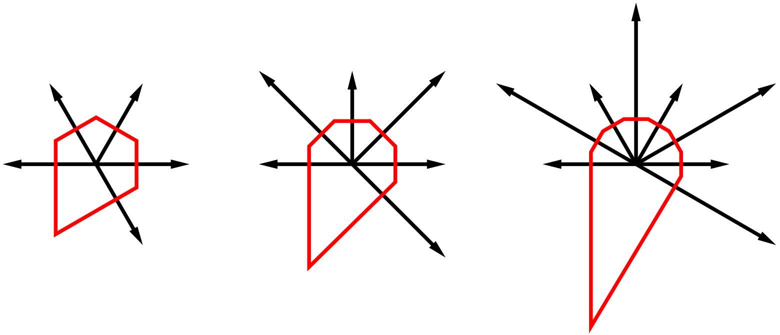

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Example ($W = S_n$)

Simple root: α_i corresponding to "reflection" $(i \;\; i{+}1)$.

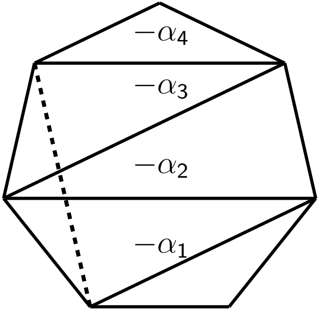
Positive root: $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots \alpha_j$.

Almost positive roots $\leftrightarrow$ diagonals of $(n{+}2)$-gon.

Compatible $\leftrightarrow$ noncrossing.

Negative simple roots $\leftrightarrow$ diagonals forming "snake."

Positive root $\alpha_{ij} \leftrightarrow$ diagonal crossing $-\alpha_i, \ldots, -\alpha_j$ and no other negative simple. (α_{23} shown dotted.)



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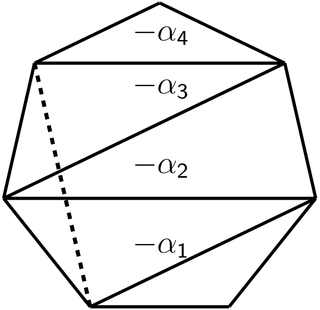


Compatibility for general W: There is a "rotation" on $\Phi_{\geq -1}$ such that every positive root can be rotated to a negative simple root. Compatibility is rotation invariant and $-\alpha$ is compatible with β is and only if α has coefficient zero in the simple-root expansion of β.

Cluster algebras (Fomin and Zelevinsky)

Generalized associahedra are the underlying combinatorial structures for cluster algebras of finite type.
Initial motivation: a framework for the study of total positivity and of Lusztig/Kashiwara's canonical bases of quantum groups. Many algebras related to reductive Lie groups have the structure of a cluster algebra.

Subsequent applications (due to many researchers):

1. Discrete dynamical systems based on rational recurrences.
2. Y-systems in thermodynamic Bethe Ansatz.
3. Quiver representations.
4. Grassmannians, projective configurations & tropical analogues.
5. Poisson geometry, Teichmüller theory.
6. Triangulations of orientable surfaces.

Definition of a cluster algebra (modulo all the details)

A seed consists of a cluster of n rational functions, called cluster variables and matrix B.

For each cluster variable x in a seed, mutation creates a new seed:

x is replaced with a new rational function x'.

The other cluster variables are unchanged.

B is replaced with a new matrix.

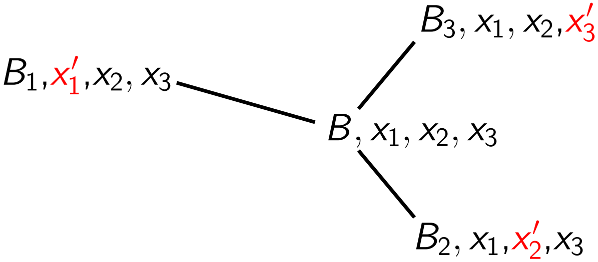
(B is the combinatorial data telling you how to do mutations.)

Start with an initial seed and let mutations propagate in all directions to obtain a (usually infinite) collection of seeds. Use the collection of all cluster variables in all these seeds to generate an algebra. This is the cluster algebra associated to the initial seed.

An example with $n = 3$

$$B, x_1, x_2, x_3$$

An example with $n = 3$



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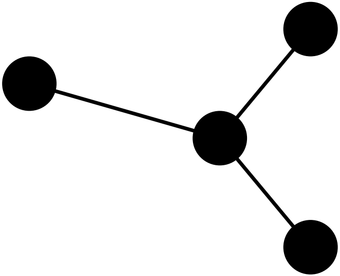

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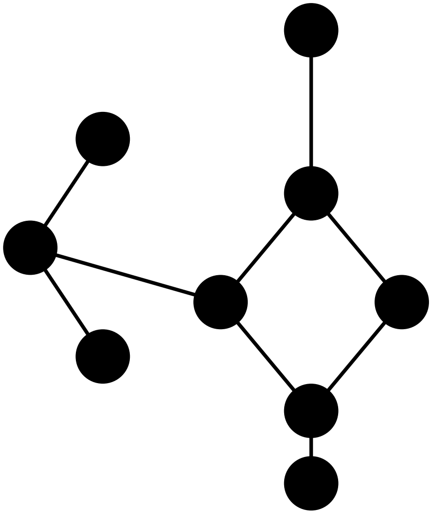

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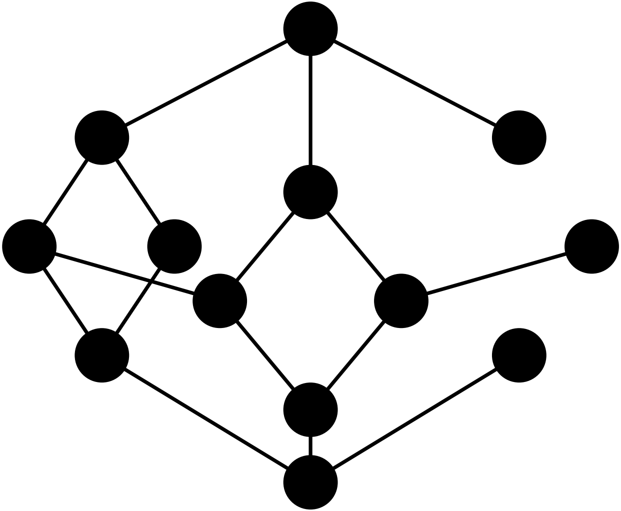

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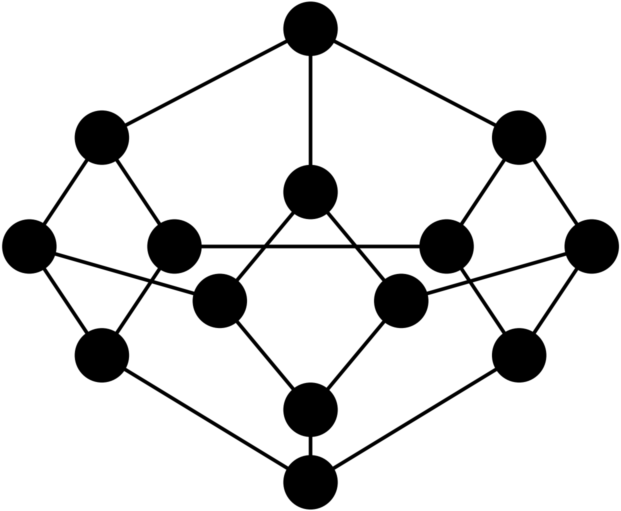

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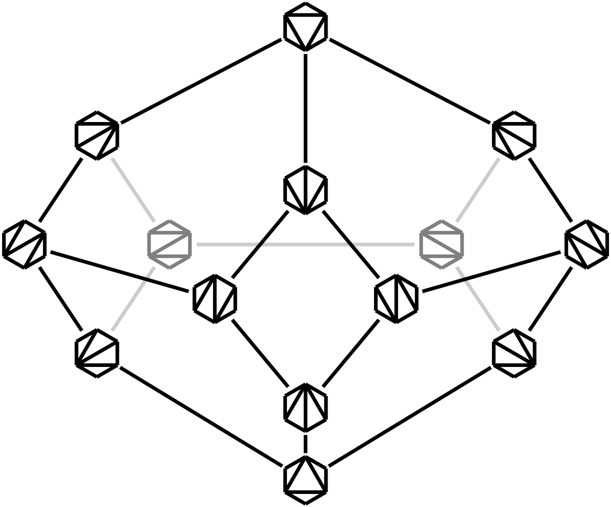

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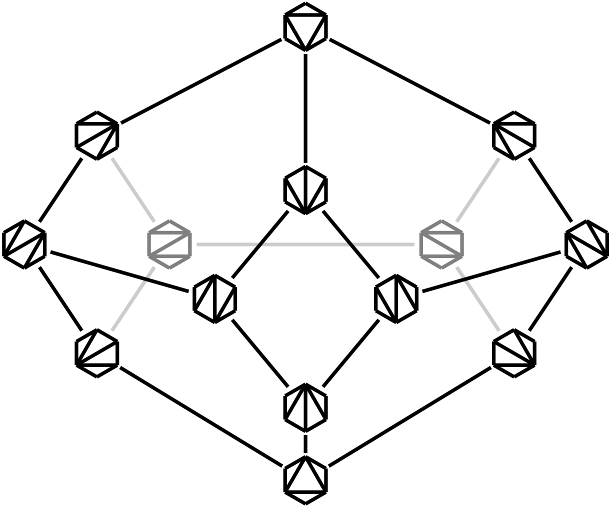


Theorem (Fomin, Zelevinsky). Cluster algebras with a finite number of seeds correspond to generalized associahedra.

Second take-home lesson

Cluster algebras (of “finite type”) are intricately connected with the combinatorics of Coxeter groups. But in most contexts, restricting to cluster algebras of finite type is very artificial.

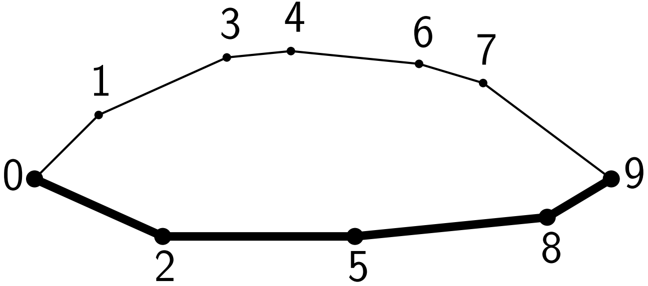
The standard construction of generalized associahedra does not generalize in an obvious way to an infinite Coxeter group. It also uses very little Coxeter theoretic machinery.

For the rest of the talk, I’ll describe an alternate approach to generalized associahedra that uses a wide range of Coxeter-theoretic tools and which generalizes naturally to infinite Coxeter groups.

A map from permutations to triangulations

Arrange the numbers from 0 to $n+1$ on a polygon such that numbers strictly increase left to right. Begin with a path along the bottom. Modify the path by removing/adding vertices in the order given by π. The triangulation is the union of the paths.

Example: $\pi = 42783165$



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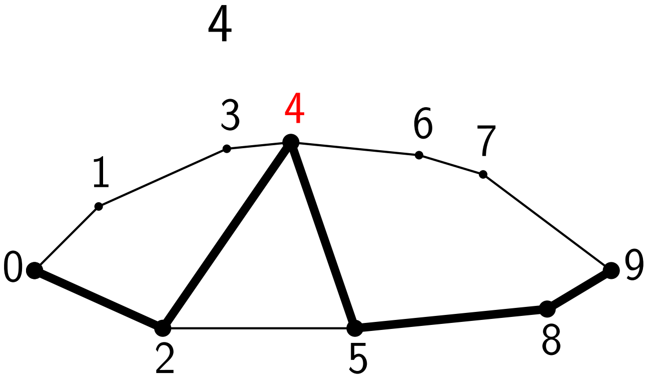

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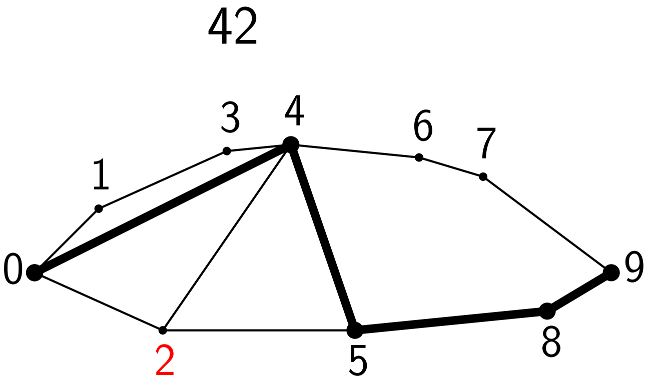

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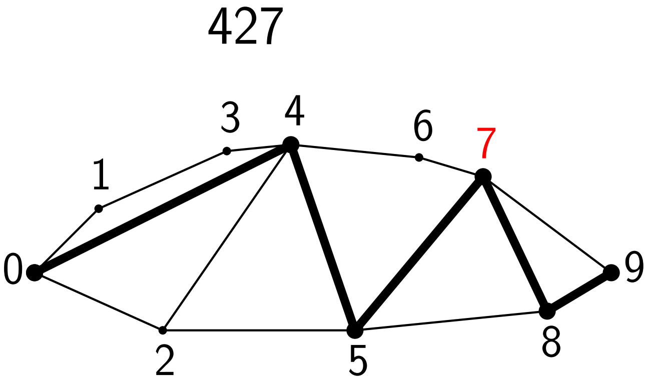

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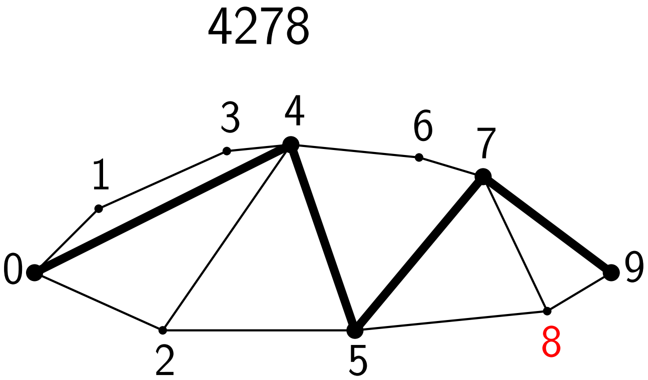

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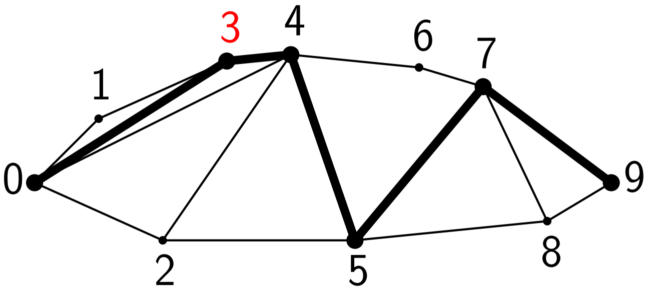

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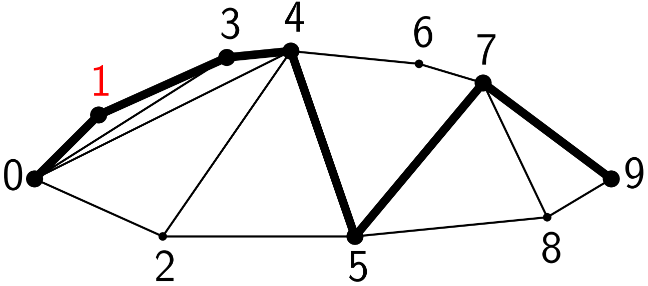

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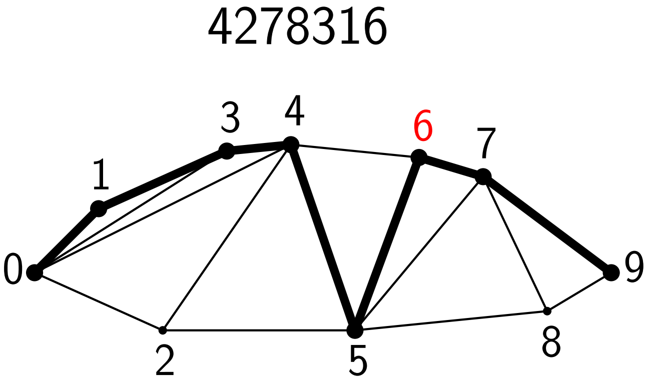

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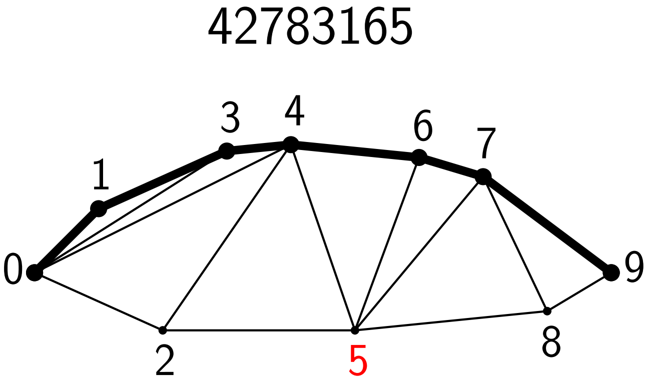

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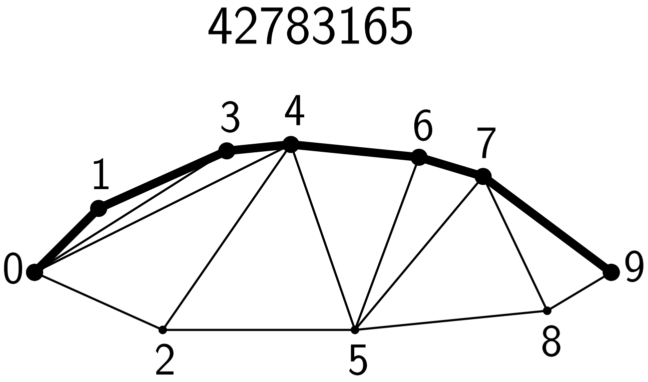
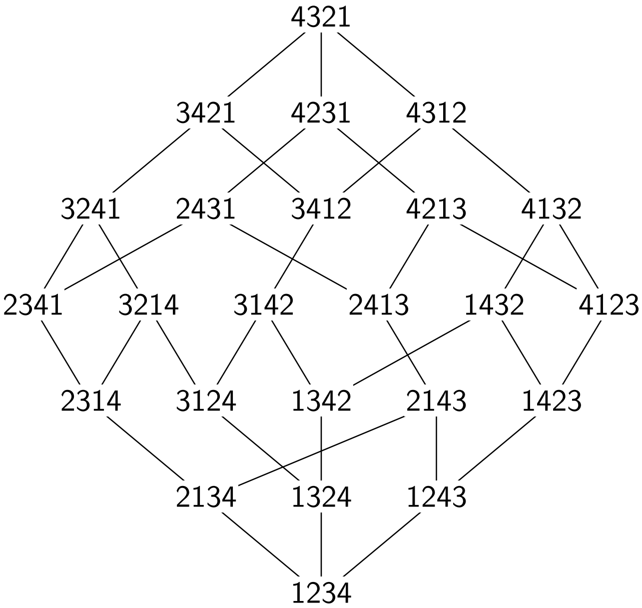


S_4 to triangulations



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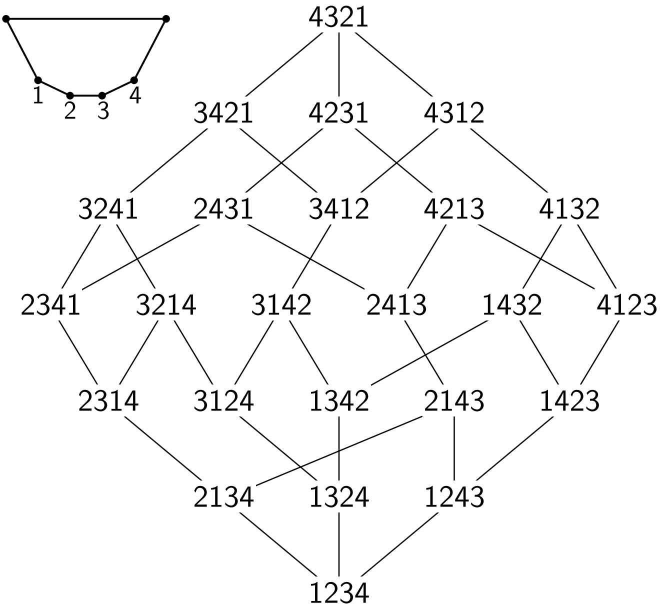

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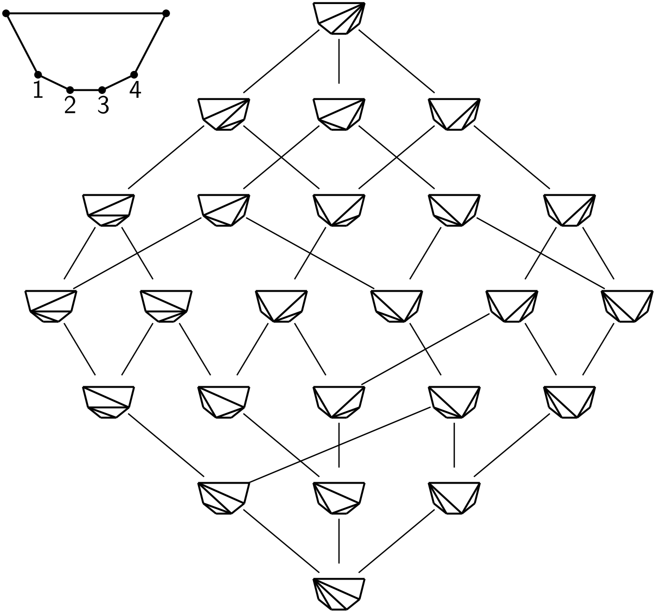

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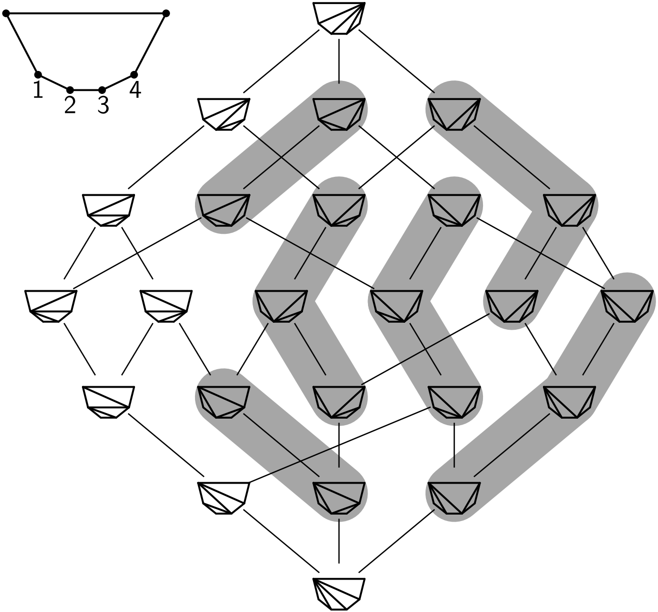
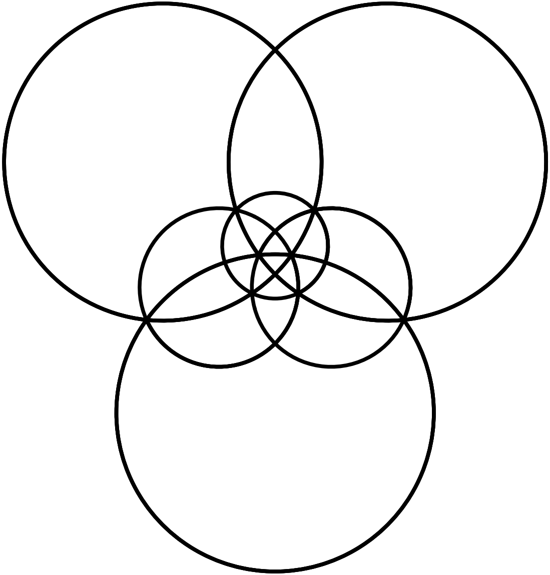


S_4 to triangulations, geometric view



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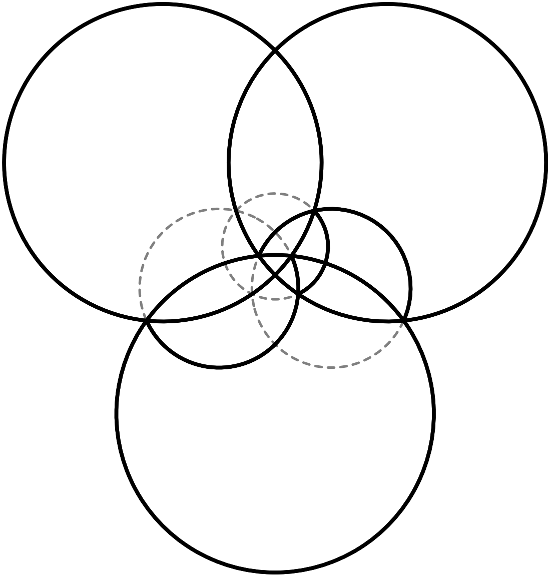

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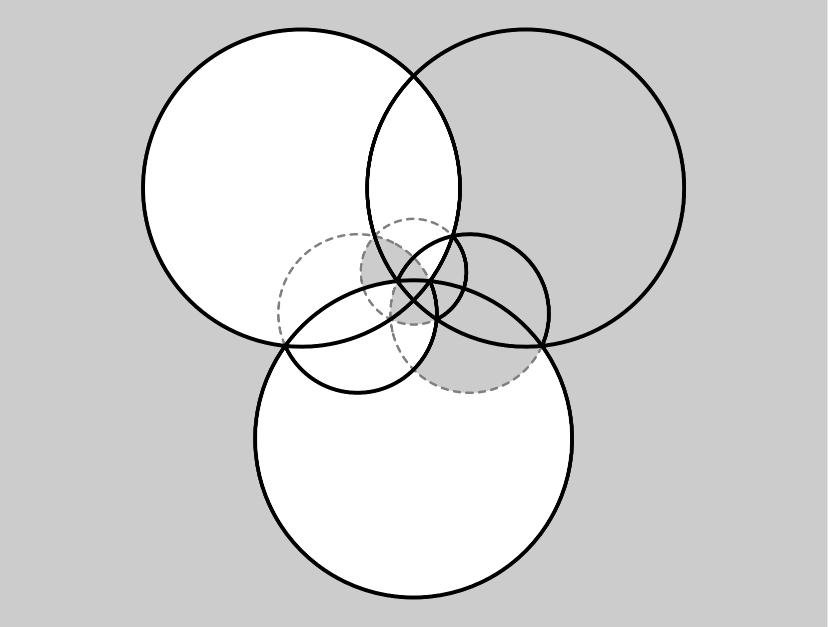

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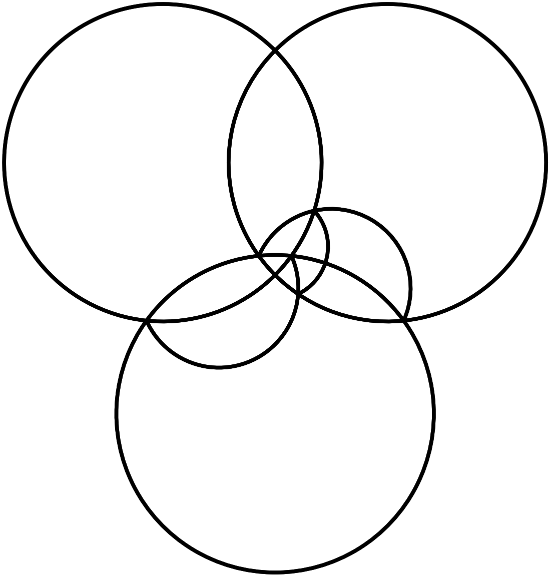

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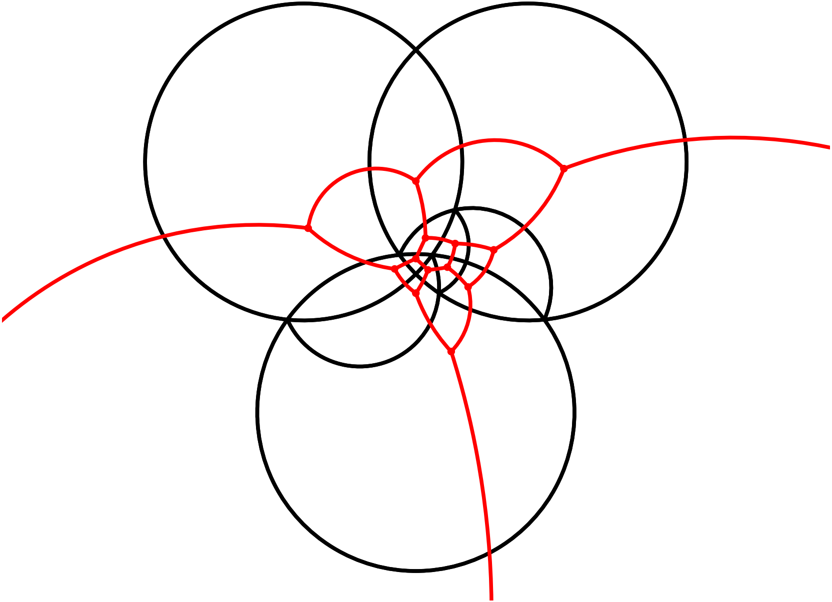

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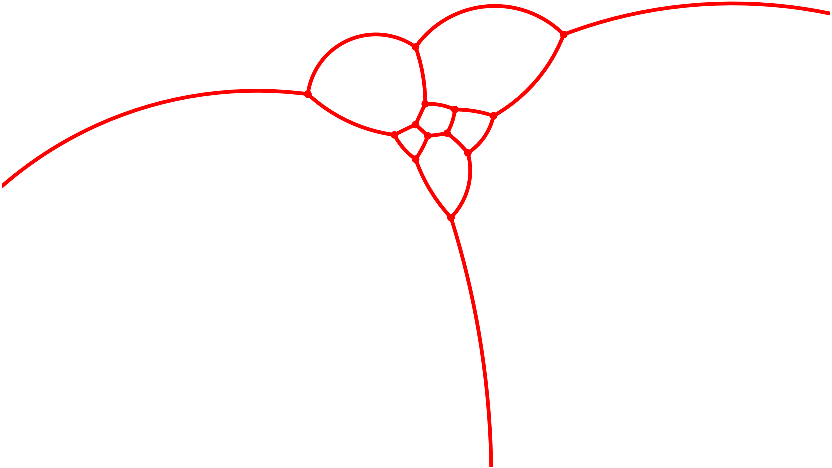
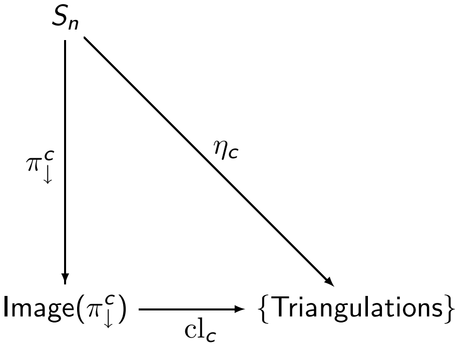


Summary of the S_n example



c: the choice of how to label the polygon.

η_c: Permutations to triangulations. Factors through $\pi_\downarrow^c$.

$\pi_\downarrow^c : x \longmapsto$ bottom element of $\eta_c^{-1}(\eta_c(x))$.

cl_c: a bijection from bottom elements to triangulations.

Key point: Combinatorics of associahedron encoded in fibers of $\pi_\downarrow^c$.

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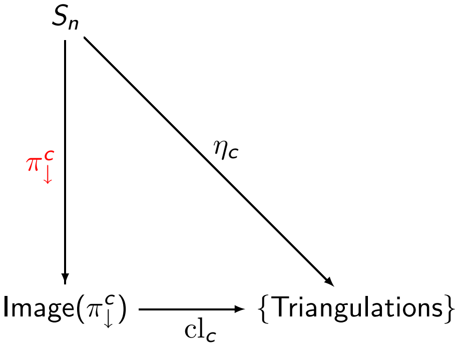

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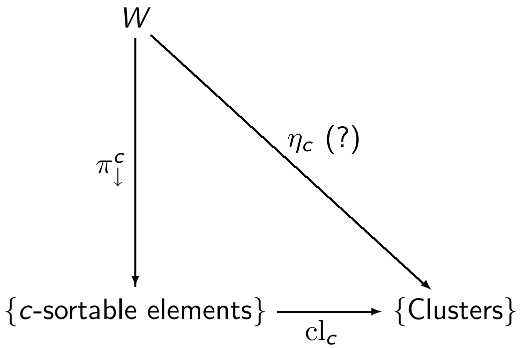
Third take-home lesson (for general finite W)



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c-sortables and $\pi_\downarrow^c$ defined using weak order and/or reduced words and/or geometry.

cl_c: still a bijection.

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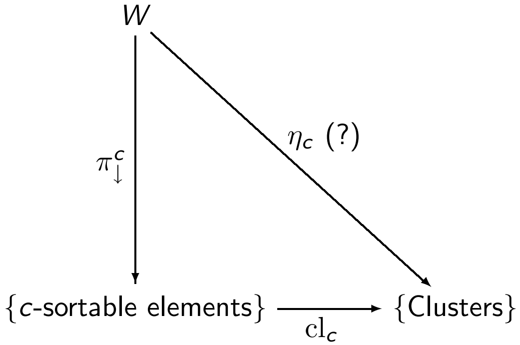

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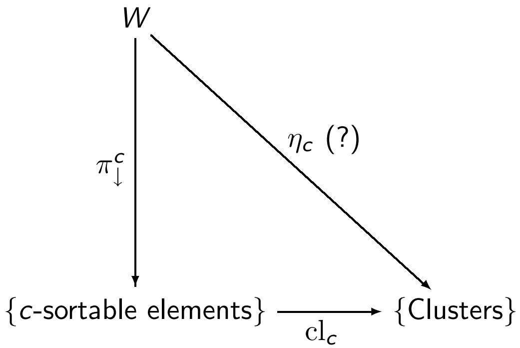

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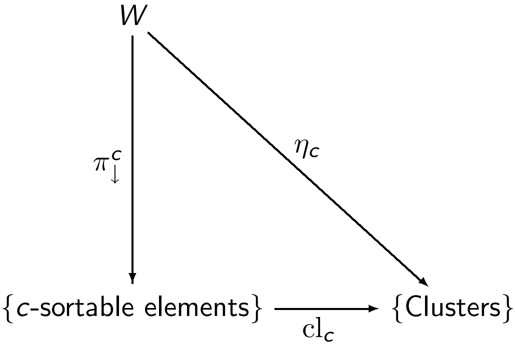
Key point: Fibers of $\pi_\downarrow^c$ define Cambrian fan, combinatorially isomorphic to cluster complex (coincides with "g-vector fan").

Infinite Coxeter groups/cluster algebras of infinite type



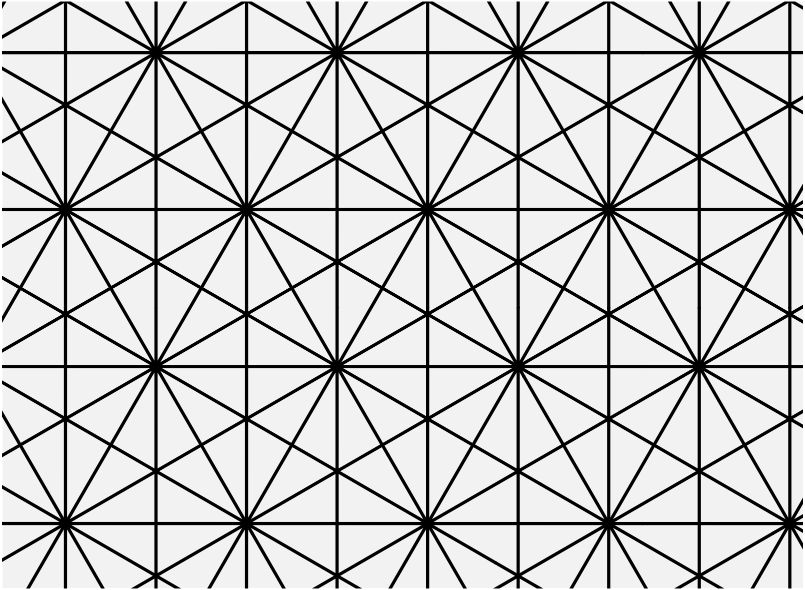
c-sortables and $\pi_\downarrow^c$: definitions (weak order, reduced words, geometry) hold verbatim.

cl_c: No suitable general definition of combinatorial clusters.
nc_c: No suitable general definition of noncrossing partitions.

Cambrian fan: Conjecturally, a subfan of the g-vector fan.

The issue: Cambrian fan doesn't reach outside the Tits cone.

An "affine" example (W of type $\widetilde{G}_2$)



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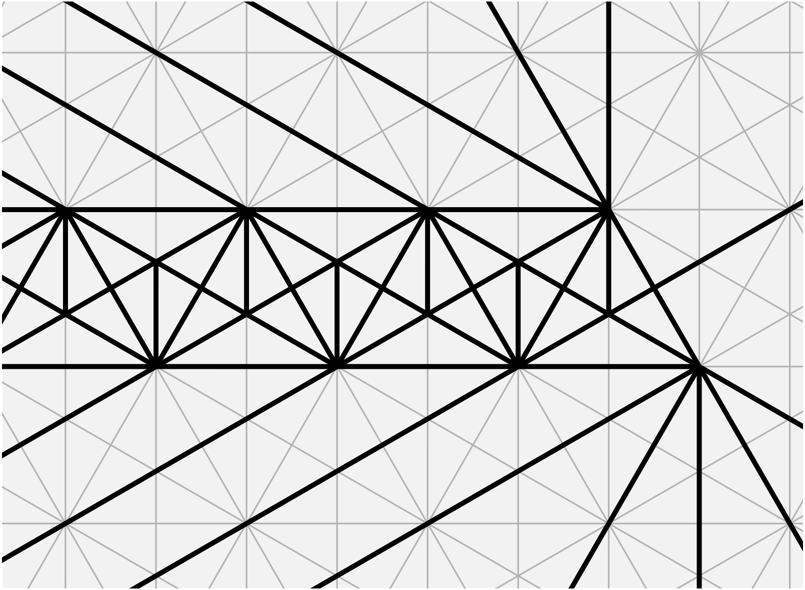

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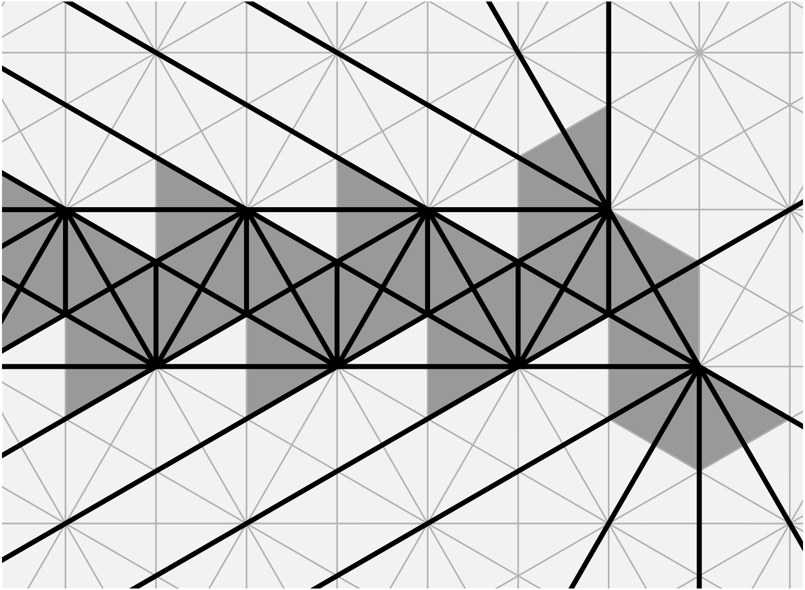

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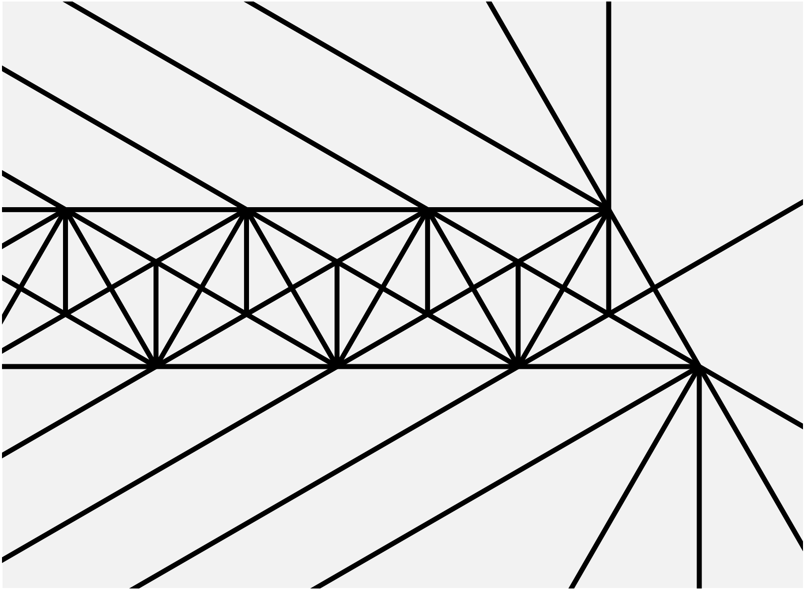
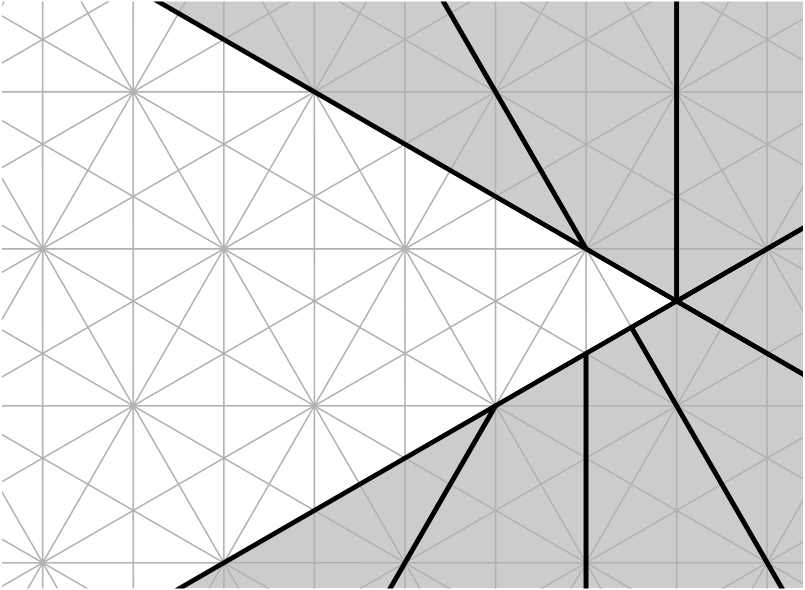
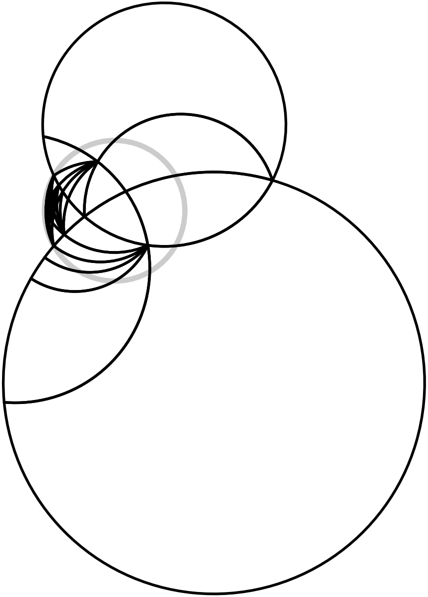


The same affine example (Negative of Tits cone)



The same affine example (stereographic view)



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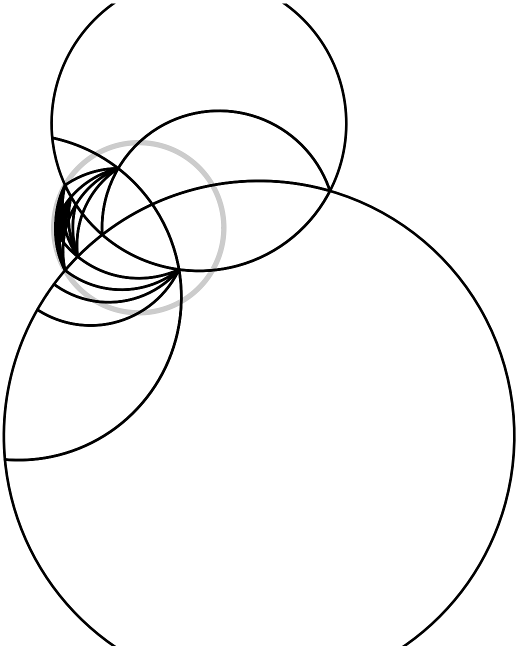

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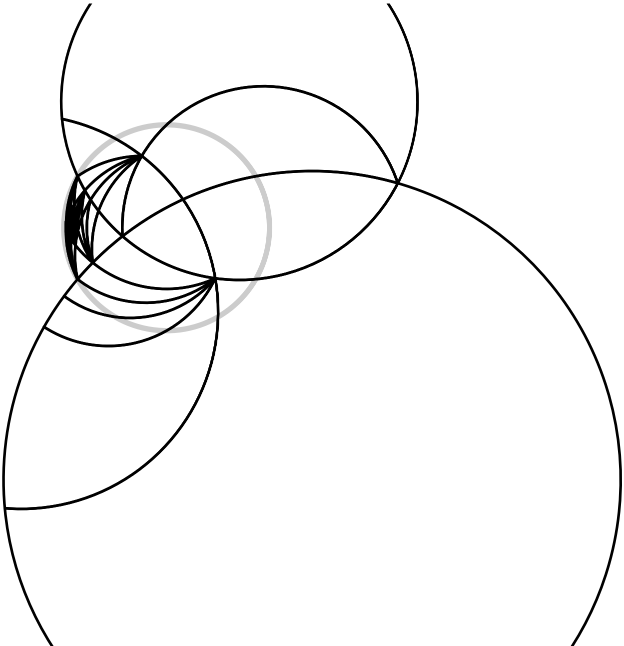

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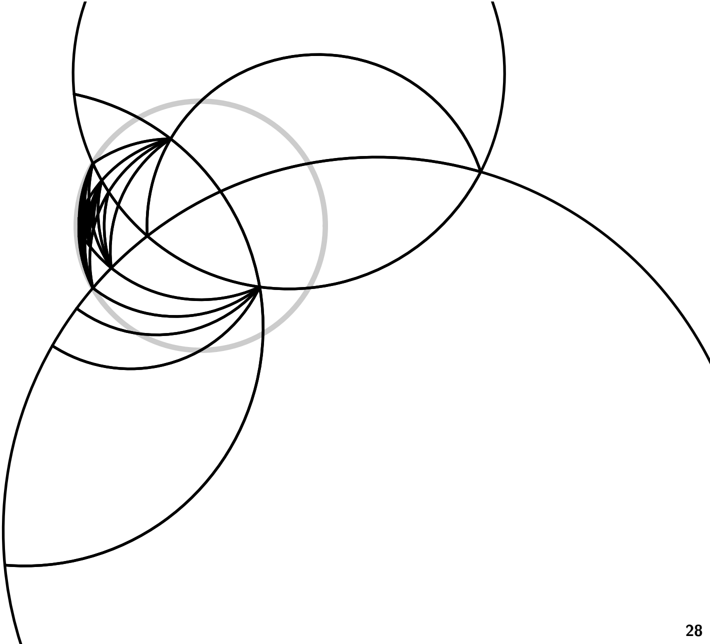

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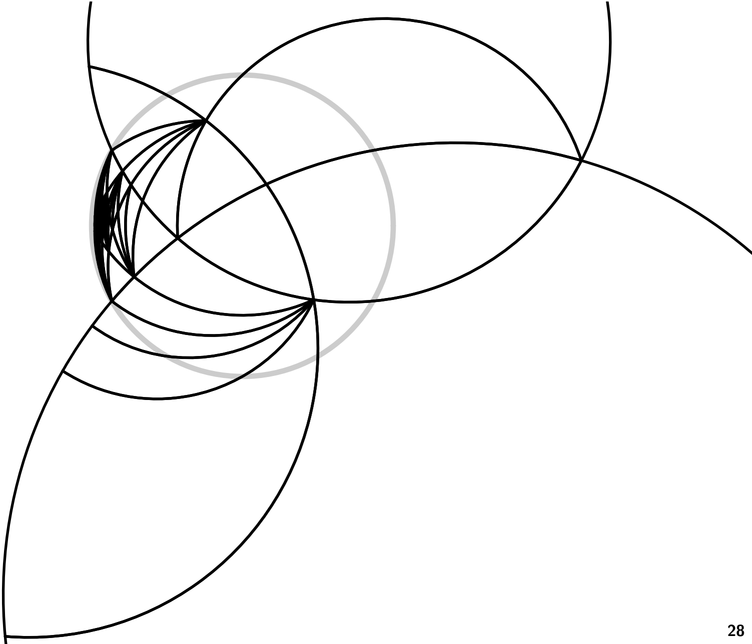

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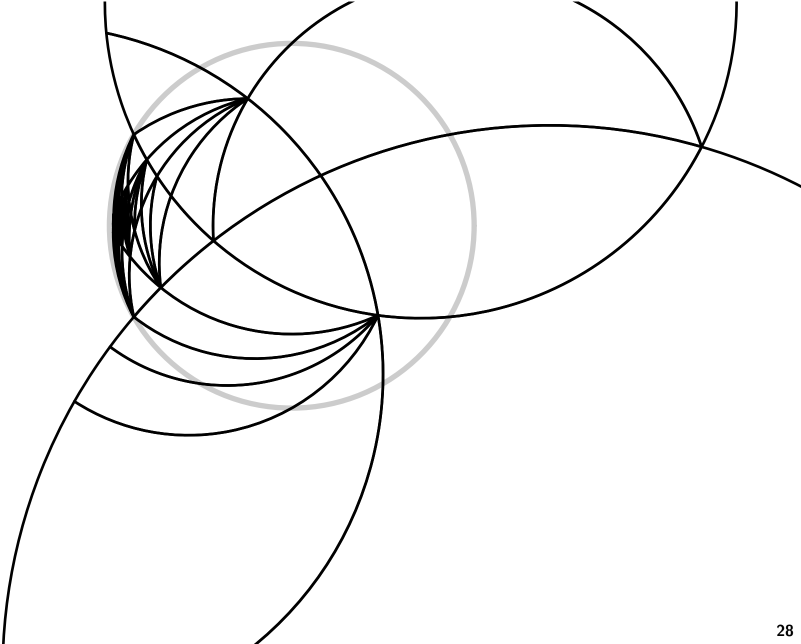


Analysis of the affine example

The Cambrian fan becomes periodic in one direction.

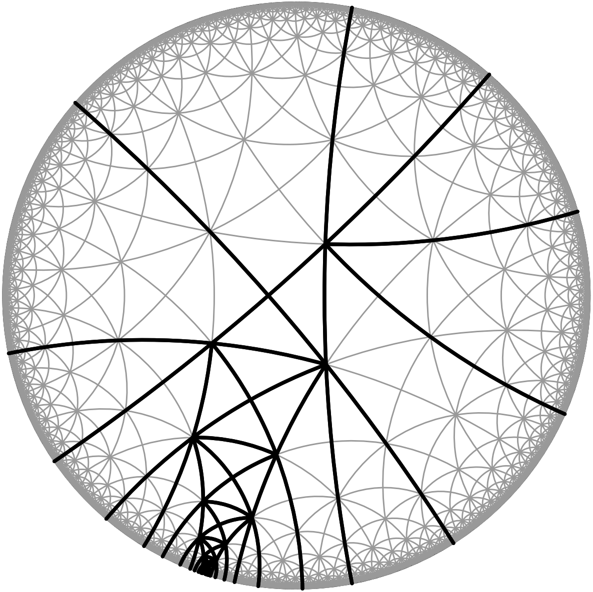
Cluster algebras tell us that the dual "affine associahedron" should be combinatorially periodic in both directions.

Part of the associahedron is outside of the Cambrian fan, because the Cambrian fan doesn't extend (very far) outside the Tits cone.

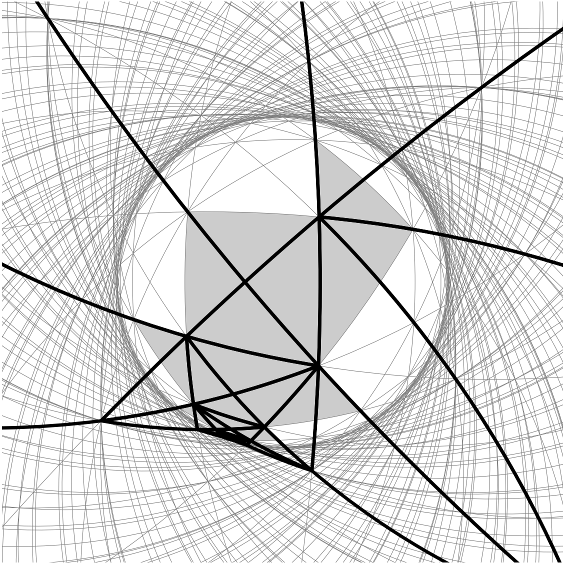
Speyer and I think we know how to deal with the affine case.
A key point: all of the combinatorics is already present in the (limiting) periodic part.

The non-affine cases are harder...

A hyperbolic example



The same hyperbolic example (stereographic view)



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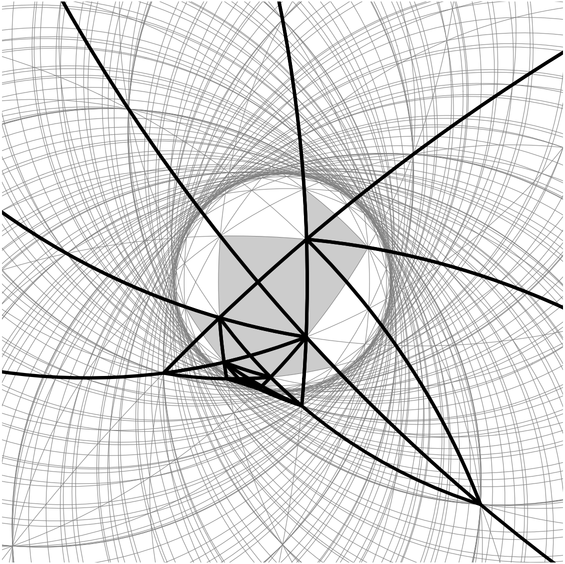

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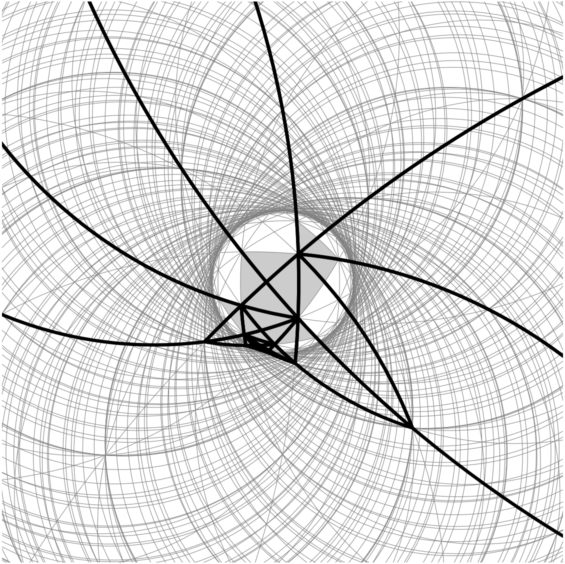

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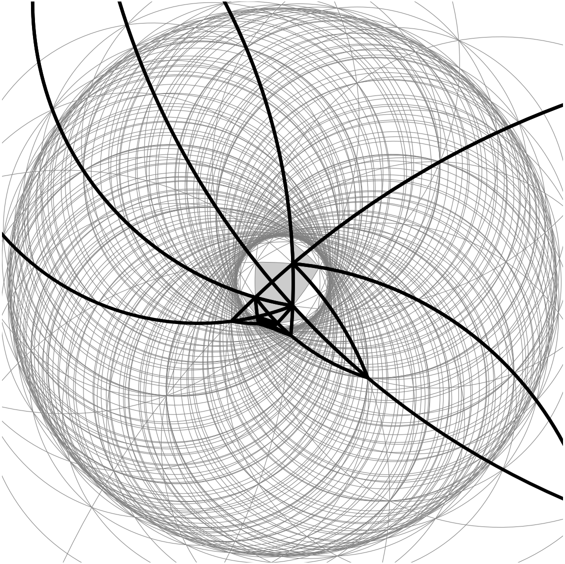

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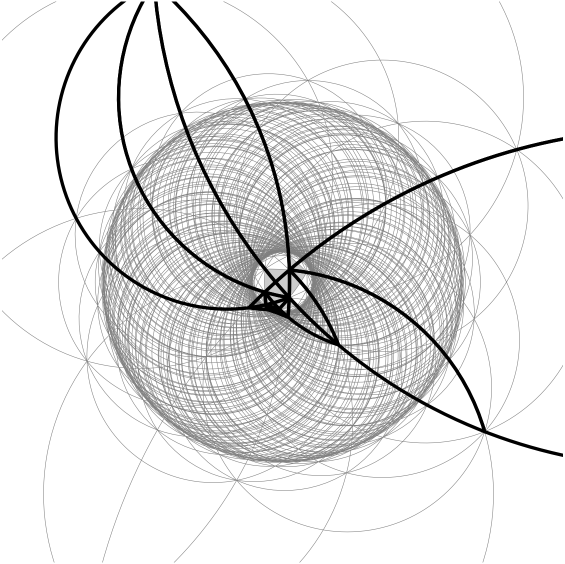

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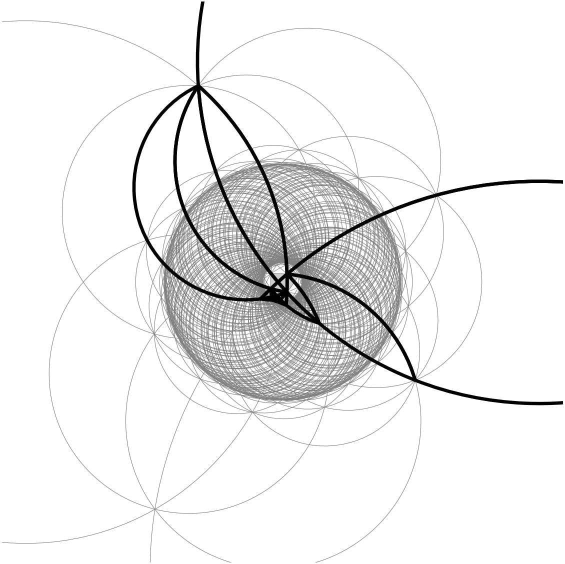

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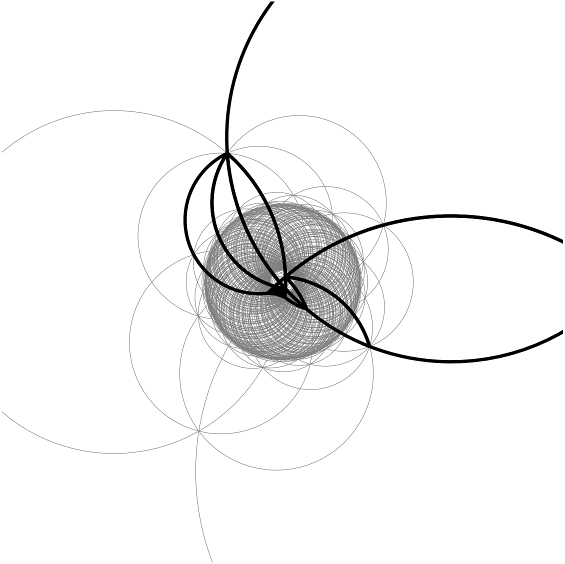

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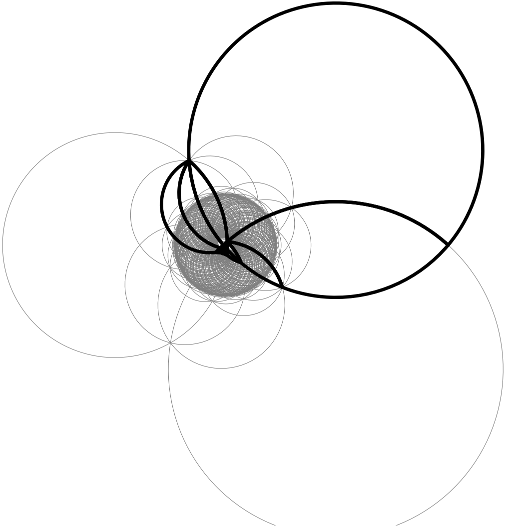
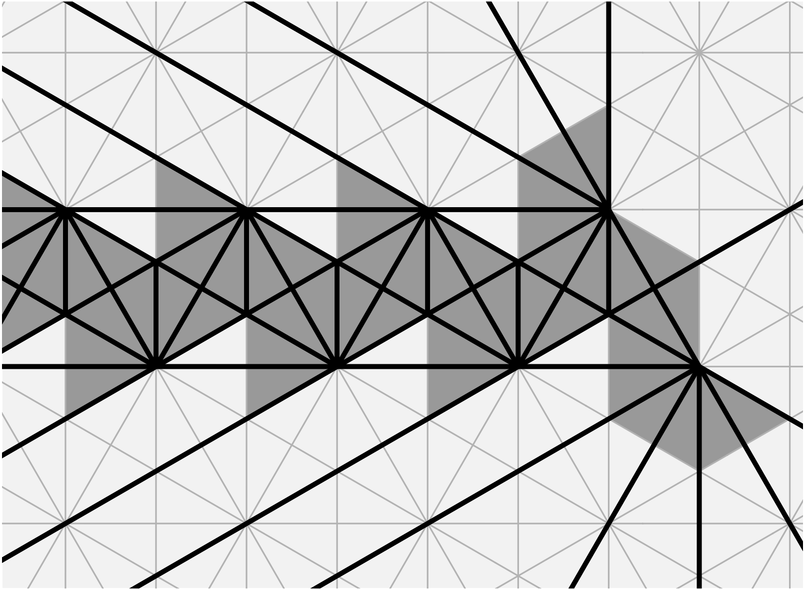


Analysis of the hyperbolic example

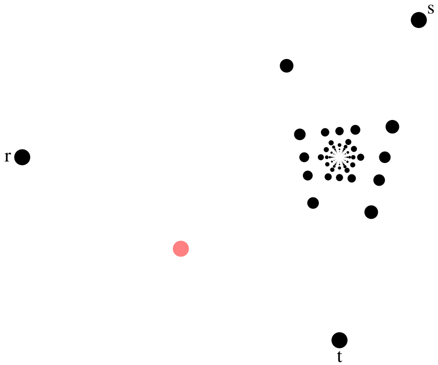
Speyer and I don't think we can deal with the non-affine infinite case with only sortable elements/Cambrian lattices as tools.

There are indications that some larger object (properly containing the Coxeter group W) exists. The key may be to generalize sortable elements and Cambrian lattices to this bigger setting.

The geometry of sortable elements



The geometry of sortable elements (continued)



The word-combinatorics of sortable elements

A Coxeter element is $c = s_1 \cdots s_n$ for $S = \{s_1, \ldots, s_n\}$

Fix (some reduced word for) a Coxeter element c.

Form an infinite word

$$c^\infty = c|c|c|c|\cdots$$

c-Sorting word for w is the the lexicographically leftmost subword of c^∞ which is a reduced word for w.

Example: $W = S_5, \quad c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

Step	c-Sorting word	Permutation
0		42351

Example: $W = S_5$, $c = s_1 s_2 s_3 s_4$

$$c^{\infty} = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

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Step	c-Sorting word	Permutation
0		42351
1	s_1	41352

Example: $W = S_5, \quad c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

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The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

Step	c-Sorting word	Permutation
0		42351
1	s_1	41352
2	$s_1 s_2$	41253

Example: $W = S_5$, $c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

Step	c-Sorting word	Permutation
0		42351
1	s_1	41352
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Example: $W = S_5$, $c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

Step	c-Sorting word	Permutation
0		42351
1	s_1	41352
2	$s_1 s_2$	41253
3	$s_1 s_2 s_3$	31254

Example: $W = S_5$, $c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

The c-sorting word for 42351 is $s_1 s_2 s_3 s_4 | s_2 | s_1$.

Step	c-Sorting word	Permutation
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1	s_1	41352
2	$s_1 s_2$	41253
3	$s_1 s_2 s_3$	31254

Example: $W = S_5$, $c = s_1 s_2 s_3 s_4$

$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

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3	$s_1 s_2 s_3$	31254
4	$s_1 s_2 s_3 s_4 \|$	31245

Example: $W = S_5, \quad c = s_1 s_2 s_3 s_4$

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3	$s_1 s_2 s_3$	31254
4	$s_1 s_2 s_3 s_4 |$	31245
5	$s_1 s_2 s_3 s_4 |$	31245

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4	$s_1 s_2 s_3 s_4 \vert$	31245
5	$s_1 s_2 s_3 s_4 \vert$	31245

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7	$s_1 s_2 s_3 s_4 \| s_2$	21345
8	$s_1 s_2 s_3 s_4 \| s_2$	21345
9	$s_1 s_2 s_3 s_4 \| s_2 \| s_1$	12345

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$$c^\infty = s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | s_1 s_2 s_3 s_4 | \cdots$$

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Sortable elements

A sorting word can be interpreted as a sequence of sets (letters between dividers "$|$"). If the sequence is nested then w is c-sortable.

Example: w with c-sorting word $s_1 s_2 s_3 s_4 | s_2 | s_1$ is not c-sortable because $\{s_1\} \not\subseteq \{s_2\}$.

Example: $W = S_3, \quad c = s_1 s_2$.

c-sortable:		not c-sortable:	
1	123	$s_2 \| s_1$	312
s_1	213		
$s_1 s_2$	231		
$s_1 s_2 \| s_1$	321		
s_2	132		

Example: $W = S_n$
For one choice of c, the c-sortable elements are the "231-avoiding" or "stack-sortable" permutations.
For another c, "c-sortable" = "312-avoiding".