NON-NEGATIVE CD-COEFFICIENTS OF GORENSTEIN* POSETS

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Abstract. We give a convolution formula for cd-index coefficients. The convolution formula, together with the proof by Davis and Okun of the Charney-Davis Conjecture in dimension 3, imply that certain cd-coefficients are non-negative for all Gorenstein* posets. Additional coefficients are shown to be non-negative by interpreting them in terms of the top homology of certain Cohen-Macaulay complexes. In particular we verify, up to rank 6, Stanley's conjecture that the coefficients in the cd-index of a Gorenstein* ranked poset are non-negative.

1. Introduction

We begin by summarizing the main results, putting off most definitions until Section 2. Throughout this paper, P is an Eulerian poset with rank function ρ and cd-index Φ_P, and $\langle w|\Phi_P\rangle$ is the coefficient of a cd-word w in Φ_P. We prove the following convolution formula for coefficients of the cd-index:

Theorem 1. *If $w = w_1dcdw_2$ with $\deg(w_1d) = k-1$ then*

$$2\langle w|\Phi_P\rangle = \sum_{\substack{x\in P\\ \rho(x)=k}} \langle w_1d|\Phi_{[\hat{0},x]}\rangle \cdot \langle dw_2|\Phi_{[x,\hat{1}]}\rangle.$$

A similar formula holds when w_1d or dw_2 is replaced by an empty cd-word. Theorem 1 has been discovered independently by Mahajan [10] and Stenson [15]. The idea of using convolution formulas to generate new inequalities for flag numbers was originated by G. Kalai in [9].

We establish directly the non-negativity of the coefficient of any cd-word having at most one d. If P is a graded poset with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$, we denote by $\widetilde{\chi}(P)$ the reduced Euler characteristic of the poset $P-\{\hat{0},\hat{1}\}$ and by $H_m(P)$ the m^{th} simplicial homology of the order complex of $P-\{\hat{0},\hat{1}\}$. For a graded poset P, the subposet $Sk_m(P) := \{x\in P : \rho(x)\leq m+1\}\cup\{\hat{1}\}$ is called the *m-skeleton* of P. If P is the face poset of a CW-complex, this corresponds to the usual notion of skeleton.

Theorem 2. *If P is an Eulerian poset of rank $n+1$ then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = (-1)^m\widetilde{\chi}\left(Sk_m(P)\right) - 1.$$

If P is Gorenstein then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = rank\left[H_m\left(Sk_m(P)\right)\right] - 1 \geq 0.$$

1991 *Mathematics Subject Classification.* Primary 06A07; Secondary 05E99, 52B05.

Key words and phrases. cd-index, Charney-Davis Conjecture, Eulerian poset, flag f-vector, Gorenstein.

Theorem 2 is similar to a result of Novik [11] for Eulerian Buchsbaum simplicial complexes. In particular, Novik's result implies Theorem 2 in the case when P is a simplicial poset, i.e. when every lower interval of P is a Boolean lattice.

An Eulerian poset P has the *Charney-Davis property* if $\langle d^k|\Phi_{[x,y]}\rangle \geq 0$ for any interval $[x,y] \subseteq P$ of rank $2k+1$, for any k. The name of this property refers to the Charney-Davis conjecture about Gorenstein* flag simplicial complexes (see [5, 12] for details). In the special case of order complexes, the conjecture is that any Gorenstein* poset has the Charney-Davis property. This connection between the Charney-Davis conjecture and the cd-index was observed by Babson (see [5]). Stanley [14] conjectured that the coefficients of the cd-index are non-negative whenever P is Gorenstein*. Davis and Okun [6] proved the Charney-Davis conjecture for Gorenstein* flag complexes of dimension 3. Their result, combined with Theorems 1 and 2, yields the following theorem, which supports Stanley's conjecture. The case where $j = 1$ and $e_1 = 1$ is Theorem 2. If $j = 1$ and $e_1 = 2$, then the Davis-Okun result and Theorem 1 give non-negativity. For $j > 1$, the result follows by Theorem 1 and induction.

Theorem 3. *Let P be a Gorenstein* poset of rank $n+1$. Let w be a cd-monomial of degree n with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \in \{1,2\}$ for each i. If $e_1 \neq 1$, require that $k \in \{0,1\}$; if $e_j \neq 1$, require that $m \in \{0,1\}$. Then $\langle w|\Phi_P\rangle \geq 0$.* □

As far as we are aware, Theorem 3 gives the complete list of cd-coefficients which are known to be non-negative for all Gorenstein* posets. If the Charney-Davis conjecture is true for order complexes of all dimensions, then the restriction $e_i \in \{1,2\}$ can be replaced by $e_i \geq 1$.

For any Eulerian poset P, the coefficient of c^n is 1, and for $n \leq 5$, any other cd-monomial w satisfies the hypotheses of Theorem 3. Therefore:

Corollary 4. *If P is a Gorenstein* poset with rank$(P) \leq 6$, then Φ_P has non-negative coefficients.*

For Gorenstein* posets of rank 7, the only coefficients of Φ not shown in this paper to be non-negative are the coefficients of *ccdd*, *dccd*, *ddcc* and *ddd*. The most that the present work can say about these coefficients is that $2\langle v|\Phi\rangle + \langle ddd|\Phi\rangle \geq 0$ for $v =$ *ccdd*, *dccd* or *ddcc*. (See Proposition 9).

The paper is structured as follows: In Section 2, we give a few basic definitions and formulas. Theorem 1 and several generalizations are proven in Section 3, and Theorem 2 is proven in Section 4.

2. The cd-index

In this section, we define the flag f-vector and the flag L-vector, then use the flag L-vector to define the cd-index. It is more common to transform the flag f-vector to the flag h-vector, and use the flag h-vector to define the cd-index. For more details on flag f-vectors and the cd-index, including the usual definitions, the reader is referred to [14].

Let P be an Eulerian poset of rank $n+1$ with rank function ρ. A poset with $\hat{0}$ and $\hat{1}$ is Eulerian if its Möbius function has the simple formula [13, Sections 3.8, 3.14]

$$\mu(x,y) = (-1)^{\rho(y)-\rho(x)}.$$

For a chain C in $P - \{\hat{0}, \hat{1}\}$ define $\rho(C) = \{\rho(x) : x \in C\}$. For any $S \subseteq [n]$, let $\mathcal{C}(S)$ be the set of chains C in $P - \{\hat{0}, \hat{1}\}$ such that $\rho(C) = S$. The *flag f-vector* is $f_S := |\mathcal{C}(S)|$. The flag f-vector is written $f_S(P)$ when it is important to specify the poset P explicitly.

A set $S \subseteq [n]$ is *even* if it is a disjoint union of intervals of even cardinality. A set is *anti-even* if its complement is even. Bayer and Hetyei defined the flag L-vector

$$L_S \quad := \quad (-1)^{n-|S|} \sum_{T \supseteq S^c} (-2)^{-|T|} f_T. \tag{1}$$

This formula can be inverted, with $S^c := \{1, 2, \ldots, n\} - S$.

$$f_S \quad = \quad 2^{|S|} \sum_{T \subseteq S^c} L_T. \tag{2}$$

Bayer and Hetyei [2] noted that $L_S = 0$ when S is not an even set and that the L_S are actually the coefficients of the ce-index [14], which we will not define here.

Let c and d be non-commuting variables in c and d, and let the degree of c be 1 and the degree of d be 2. The length $l(w)$ of a cd-word is the number of letters in the word. To define the cd-index in terms of the flag L-vector, it is convenient to introduce two partial orders on cd-words. The Fibonacci order "$\le$," is a partial order on cd-words, generated by the following covering relations: $v \gtrdot w$ if w is obtained from v by replacing some d with cc. Intervals in the Fibonacci order are boolean algebras, so the Möbius function is $\mu(w, v) = (-1)^{l(v)-l(w)}$ for $w \le v$. The *strong Fibonacci order* is also defined by covering relations. Say that $v \gtrdot_s w$ if w is obtained from v by replacing some d^k with $cd^{k-1}c$ for any $k \ge 1$, and let "$\le_s$" be the transitive closure of these cover relations. The Möbius function of the strong Fibonacci order, denoted by μ_s, has the following useful, easily-proved property:

Proposition 5. *If $v \le_s w = w_1 c w_2$, then $v = v_1 c v_2$ for $v_1 \le_s w_1$ and $v_2 \le_s w_2$, and $[v, w]_s \cong [v_1, w_1]_s \times [v_2, w_2]_s$. In particular,*

$$\mu_s(v, w) = \mu_s(v_1, w_1) \cdot \mu_s(v_2, w_2). \quad \square$$

Given a cd-word w, an even set E_w is obtained by lining up w with the set $[n]$ such that each d covers two elements of $[n]$. Then for each d in w, both elements covered by the d are included in E_w and each element covered by a c is excluded from E_w. An anti-even set can be defined by $A_w := (E_w)^c$. So if $w = cdddcdcd$, then $E_w = \{2, 3, 4, 5, 6, 7, 9, 10, 12, 13\}$ and $A_w = \{1, 8, 11\}$. Because $L_S = 0$ when S is not even, we can index L by cd-words, with $L_w := L_{E_w}$. We wish to consider flag f-vectors in the case when S is anti-even, as was done in [3]. We denote the anti-even flag f-vectors by $f^a_w := f_{A_w}$.

Restricting Equation (2) to anti-even flag f-vectors and rewriting in terms of the strong Fibonacci order yields:

$$f^a_w = 2^{2l(w)-n} \sum_{v \le_s w} L_v. \tag{3}$$

Bayer [1] gave the following formula for the cd-index, which we take to be the definition. Specifically, define the cd-index Φ_P to be the homogenous polynomial of degree n in non-commuting c and d whose coefficients are given by:

$$\langle w | \Phi \rangle \quad = \quad (-2)^{n-l(w)} \sum_{v \ge w} L_v \tag{4}$$

The partial order here is the Fibonacci order.

3. Convolution Formulas for cd-Coefficients

In this section we prove Theorem 1 and several generalizations.

There is a simple convolution formula for f^a, also found in [3, Proposition 1.2]. If the degree of w_1 is $k-1$, then

$$f^a_{w_1cw_2} = \sum_{\substack{x\in P\\ \rho(x)=k}} f^a_{w_1}([\hat{0},x])\cdot f^a_{w_2}([x,\hat{1}]). \tag{5}$$

Combined with Proposition 5, Equation (5) leads to a convolution formula for the flag L-vector [2, Appendix B].

Proposition 6. *If $w = w_1cw_2$, with $\deg(w_1) = k-1$ then*

$$2L_w(P) = \sum_{\substack{x\in P\\ \rho(x)=k}} L_{w_1}([\hat{0},x])\cdot L_{w_2}([x,\hat{1}]).$$

Proof. Notice that $v \le w$ if and only if $v = v_1cv_2$ with $v_1 \le w_1$ and $v_2 \le w_2$. Therefore we can use the Möbius inversion of Equation (3) to rewrite the right side, then apply Proposition 5 and Equation (5) to simplify the products of Möbius functions and of anti-even flag f-vectors. We finish by applying the Möbius inversion of Equation (3) again. $\square$

Proposition 6 allows us to prove Theorem 1.

Proof of Theorem 1. If $w = w_1dcdw_2$, then $v \ge w$ if and only if $v = v_1cv_2$ for $v_1 \ge w_1d$ and $v_2 \ge dw_2$. Thus we can use Equation (4) to rewrite the right side in terms of flag L-vectors, then simplify using Proposition 6 and Equation (4). When dw_2 is the empty word, the intervals $[x,\hat{1}]$ on the right side are two-element chains, so $\Phi_{[x,\hat{1}]} = 1$, and the same proof goes through. $\square$

Theorem 1 implies the following, which is the easier direction of a theorem of Bayer [1]. Bayer's theorem also shows that there are no other lower or upper bounds on cd-coefficients of Eulerian posets.

Theorem 7. *Let P be an Eulerian poset of rank $n+1$. The coefficients of dc^{n-2}, $c^{n-2}d$, cdc^{n-3} and $c^{n-3}dc$ are all non-negative. Also, let v be a cd-monomial starting and ending in d and alternating $dcdc\cdots cd$ with at least one c, such that c^kvc^m has degree n. Then $\langle c^kvc^m|\Phi_P\rangle \ge 0$.*

Proof. The coefficients $\langle dc^{n-2}|\Phi_P\rangle$ and $\langle c^{n-2}d|\Phi_P\rangle$ are easily seen to be non-negative, and the other coefficients follow by Theorem 1. $\square$

Theorem 1 has several generalizations, which we state without proof.

Proposition 8. *If $w = w_1dc^mdw_2$ with $\deg(w_1d) = k-1$, then*

$$\sum_{v\ge c^m} 2^{l(v)}\langle w_1dvdw_2|\Phi_P\rangle = \sum_{(x_1\lessdot\cdots\lessdot x_m)\in C([k,k+m-1])} \langle w_1d|\Phi_{[\hat{0},x_1]}\rangle\cdot\langle dw_2|\Phi_{[x_m,\hat{1}]}\rangle.$$

$\square$

Besides replacing c by c^m, there is another way of generalizing Theorem 1. In the proof of Theorem 1, we found a c in w and wrote down a convolution formula by splitting w at that c. The formula came out nicely because the c was between two d's. There are other formulas when the c is not between two d's. These formulas were also discovered independently by Stenson [15].

Proposition 9. *If $w = w_1 dccw_2$ with $\deg(w_1 d) = k-1$, then*

$$2\langle w|\Phi_P\rangle + \langle w_1 ddw_2|\Phi_P\rangle = \sum_{x\in P:\rho(x)=k} \langle w_1 d|\Phi_{[\hat{0},x]}\rangle \cdot \langle cw_2|\Phi_{[x,\hat{1}]}\rangle.$$

□

Proposition 10. *If $w = w_1 cccw_2$ with $\deg(w_1 c) = k-1$, then*

$$2\langle w|\Phi_P\rangle + \langle w_1 cdw_2|\Phi_P\rangle + \langle w_1 dcw_2|\Phi_P\rangle = \sum_{x\in P:\rho(x)=k} \langle w_1 c|\Phi_{[\hat{0},x]}\rangle \cdot \langle cw_2|\Phi_{[x,\hat{1}]}\rangle.$$

□

Theorem 1 and Propositions 8 through 10 also have coproduct proofs, suggested by Ehrenborg and Readdy [8]. For example, the right side of Theorem 1 can be rewritten as

$$\sum_{\hat{0}<x<\hat{1}} \delta_{w_1 d}(\Phi_{[\hat{0},x]}) \cdot \delta_{dw_2}(\Phi_{[x,\hat{1}]}),$$

where δ_w is a linear functional on cd-polynomials which returns the coefficient of w. Now the proposition follows, after some calculation, from the fact that Φ is a coalgebra homomorphism. See [7] for more information on coproducts as they relate to the cd-index.

4. cd-Coefficients and Skeleta

We conclude by proving Theorem 2. In what follows, topological statements about a poset P apply to the geometric realization of the order complex of $P - \{\hat{0},\hat{1}\}$, as in [13, Section 3.8]. Recall from the introduction that the m-skeleton of a graded poset is the subposet $Sk_m(P) := \{x \in P : \rho(x) \le m+1\} \cup \{\hat{1}\}$.

Theorem 2. *If P is an Eulerian poset of rank $n+1$ then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = (-1)^m \widetilde{\chi}(Sk_m(P)) - 1.$$

If P is Gorenstein then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = \mathit{rank}\,[H_m(Sk_m(P))] - 1 \ge 0.$$

Proof.

$$(-1)^m \widetilde{\chi}(Sk_m(P)) = f_{\{m+1\}} - f_{\{m\}} + \cdots + (-1)^m f_{\{1\}} + (-1)^{m+1}.$$

This is easily seen to be equal to $\langle c^m dc^{n-m-2}|\Phi_P\rangle + 1$. Also,

$$(-1)^m \widetilde{\chi}(Sk_m(P)) = \sum_i (-1)^{m-i} \mathrm{rank}\,[H_i(Sk_m(P))]\,.$$

If P is a homology sphere then by the Rank-Selection Theorem for Cohen-Macaulay posets [4], the skeleton $Sk_m(P)$ is Cohen-Macaulay, so $H_i(Sk_m(P)) = 0$ for $i < m$. To show that $H_m(Sk_m(P)) > 0$, consider any element $p \in P$ of rank $m+2$. The interval $[\hat{0}, p]$ is a Gorenstein* poset, i.e. the order complex of $(\hat{0}, p)$ is a homology

m-sphere. The orientation class of that homology sphere is a non-zero element of $H_m\,(Sk_m(P))$. $\square$

5. Acknowledgments

The author wishes to thank his advisor Vic Reiner for helpful conversations, and in particular for suggesting that there might be a convolution formula for $\langle w_1 dcdw_2 | \Phi_P \rangle$. The author also wishes to thank Lou Billera, Richard Ehrenborg and Margaret Readdy for helpful conversations.

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