Bases for the Flag f-Vectors of Eulerian Posets

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Abstract. We investigate various bases for the flag f-vectors of Eulerian posets. Many of the change-of-basis formulas are seen to be triangular. One change-of-basis formula implies the following: If the Charney-Davis Conjecture is true for order complexes, then certain sums of cd-coefficients are non-negative in all Gorenstein* posets. In particular, cd-coefficients with no adjacent c's are non-negative. A convolution formula for cd-coefficients, together with the proof by M. Davis and B. Okun of the Charney-Davis Conjecture in dimension 3, imply that certain additional cd-coefficients are non-negative for all Gorenstein* posets. In particular we verify, up to rank 6, Stanley's conjecture that the coefficients in the cd-index of a Gorenstein* ranked poset are non-negative.

1. Introduction

Much of the enumerative information about a graded poset is contained in its *flag f-vector*, which counts chains of elements according to the ranks they visit. Many naturally arising graded posets are Eulerian, that is, their Möbius function has the simple formula $\mu(x,y) = (-1)^{\rho(y)-\rho(x)}$, where ρ is the rank function (see [25, Sections 3.8, 3.14]). In this paper we study various bases for the flag f-vectors of Eulerian posets. M. Bayer and L. Billera [3] proved a set of linear relations on the flag f-vector of an Eulerian poset, now commonly called the Bayer-Billera relations. They also proved that the Bayer-Billera relations and the relation $f_\emptyset = 1$ are the complete set of affine relations satisfied by the flag f-vectors of all Eulerian posets. They exhibited a basis for the linear span $\mathcal{E}_n$ of flag f-vectors of Eulerian posets of rank $n+1$, and thereby showed the dimension of $\mathcal{E}_n$ to be F_n, where F_n is the Fibonacci number with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Another important basis for the space of Eulerian flag f-vectors is the cd-index Φ, introduced by M. Bayer and A. Klapper [5], following a suggestion by J. Fine. The cd-index is a polynomial in non-commuting variables c and d. Some of the results in this paper are motivated by a conjecture of R. Stanley [26] that the coefficients of the cd-index are non-negative whenever P is Gorenstein*, that is, whenever the order complex of $P - \{\hat{0}, \hat{1}\}$ triangulates a homology sphere. The coefficient of a cd-word w in the cd-index of P is denoted $\langle w|\Phi_P\rangle$.

An Eulerian poset P has the *Charney-Davis property* if $\langle d^k|\Phi_{[x,y]}\rangle \geq 0$ for any interval $[x,y] \subseteq P$ of rank $2k+1$, for any k. The name of this property refers to the Charney-Davis conjecture about Gorenstein* flag simplicial complexes (see [13, 24]

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for details). In the special case of order complexes, the conjecture is that any Gorenstein* poset has the Charney-Davis property. (This connection between the Charney-Davis conjecture and the cd-index was observed by E. Babson [13]). The conjecture is trivial for Gorenstein* flag simplicial complexes of dimension ≤ 2, and M. Davis and B. Okun [14] proved it for Gorenstein* flag complexes of dimension 3 and thus, trivially, for dimension 4. This implies in particular that Gorenstein* posets of rank ≤ 6 (corresponding to order complexes of dimension ≤ 4) have the Charney-Davis property.

The main results of this paper are the following: A common indexing set $\mathcal{F}_n$ is given for bases of $\mathcal{E}_n$. The Fibonacci partial order and three stronger orders are defined on $\mathcal{F}_n$. Each of these orders can be extended to a lexicographic order. Many of the basis-changes are shown to be upper- or lower-triangular in the Fibonacci order, or in one of its strengthenings. Most of the remaining basis-changes are not triangular.

We define the Charney-Davis index Γ, a non-commutative polynomial which serves as a basis for $\mathcal{E}_n$, and has non-negative coefficients when P has the Charney-Davis property. The equation relating the Charney-Davis index to the cd-index (Equation (29)), leads directly to the following:

Theorem 1. *Let P be an Eulerian poset with the Charney-Davis property. Then for any cd-word w:*

$$\sum_{v \geq w} 2^{l(v)} \langle v | \Phi_P \rangle \geq 0.$$

Here, "$\leq$" is the Fibonacci order. If w has no two adjacent c's, it is maximal in the Fibonacci order. Thus Theorem 1 has the following consequence:

Corollary 2. *Let P be an Eulerian poset with the Charney-Davis property. If w is a cd-word with no two adjacent c's, then $\langle w | \Phi_P \rangle \geq 0$.*

Other closely related changes-of-basis (Equations (27) and (28)) imply a convolution formula (Proposition 21) for cd-coefficients which is used to prove the following theorem, which also appears in [2]:

Theorem 3. *Let P be an Eulerian poset of rank $n+1$. The coefficients of dc^{n-2}, $c^{n-2}d$, cdc^{n-3} and $c^{n-3}dc$ are all non-negative. Also, let v be a cd-monomial starting and ending in d and alternating $dcdc \cdots cd$ with at least one c, such that $c^k v c^m$ has degree n. Then $\langle c^k v c^m | \Phi_P \rangle \geq 0$.*

In [2], it is also shown that there are no other lower or upper bounds on cd-coefficients of Eulerian posets.

The convolution formula and the result of Davis and Okun give the following theorem, which supports Stanley's conjecture.

Theorem 4. *Let P be a Gorenstein* poset of rank $n+1$. Let w be a cd-monomial of degree n with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \in \{1, 2\}$ for each i. (If $e_1 = 2$, require that $k \in \{0, 1\}$; if $e_j = 2$, require that $m \in \{0, 1\}$).*

Then $\langle w | \Phi_P \rangle \geq 0$.

The same proof gives the following strengthening of Corollary 2.

Theorem 5. *Let P be a poset of rank $n+1$ with the Charney-Davis property. Let w be a cd-monomial of degree n with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \geq 1$ for each i. (If $e_1 > 1$, require that $k \in \{0, 1\}$; if $e_j > 1$, require that $m \in \{0, 1\}$).*

Then $\langle w | \Phi_P \rangle \geq 0$.

For $n \leq 5$, any cd-monomial w satisfies the hypotheses of Theorem 4. Therefore:

Corollary 6. *If P is a Gorenstein* poset with rank$(P) \leq 6$, then Φ_P has non-negative coefficients.*

For Gorenstein* posets of rank 7, the only coefficients of Φ not shown in this paper to be non-negative are the coefficients of *ccdd*, *dccd*, *ddcc* and *ddd*. The most that the present work can say about these coefficients is that $2\langle v|\Phi\rangle + \langle ddd|\Phi\rangle \geq 0$ for $v = ccdd$, *dccd* or *ddcc*. (See Proposition 23 or the proof of Theorem 1).

The paper is structured as follows: In Section 2, we review five bases for the flag f-vectors of graded, not-necessarily-Eulerian posets, and give several change-of-basis formulas. In Section 3, a common indexing set $\mathcal{F}_n$ is given for bases of $\mathcal{E}_n$, the linear span of flag f-vectors of rank $n+1$ Eulerian posets. Several partial orders are defined on $\mathcal{F}_n$. Section 4 contains a list of bases for $\mathcal{E}_n$, and a summary of relationships between the various bases. In Sections 5 and 6, we give several change-of-basis formulas in $\mathcal{E}_n$. Section 7 contains the definition of the Charney-Davis index, its relations to other bases, and the proof of Theorem 1. Section 8 contains the remaining changes of basis. Convolution formulas for Φ and the proof of Theorem 3 appear in Section 9, and Theorems 4 and 5 are proven in Section 10. In Section 11, we pose some open questions for future study.

2. Bases for the Flag f-Vectors of Graded Posets

Throughout this paper, P is a graded poset of rank $n+1$ with rank function ρ. For a chain c in $P - \{\hat{0}, \hat{1}\}$ define $\rho(c) = \{\rho(x) : x \in c\}$. For any $S \subseteq [n]$, let $C(S)$ be the set of chains c in $P - \{\hat{0}, \hat{1}\}$ such that $\rho(c) = S$. The *flag f-vector* is $f_S := |C(S)|$. The flag f-vector is written $f_S(P)$ when it is important to specify the poset P explicitly. It is useful to think of each entry $f_S(\cdot)$ of the flag f-vector as a linear functional on graded posets, as in [10]. The notation f_S here corresponds to f_S^{n+1} in [10] for $S \subseteq [n]$. When we speak of bases for flag f-vectors, we mean bases for the vector space of linear combinations of these functionals.

Later on, P will be assumed to be Eulerian. But first, consider bases for the flag f-vectors of graded posets P of rank $n+1$, not necessarily Eulerian. There is the flag h-vector h_S, related to the flag f-vector by the equations:

$$h_S := \sum_{T \subseteq S} (-1)^{|S-T|} f_T \tag{1}$$

$$f_S = \sum_{T \subseteq S} h_T. \tag{2}$$

The flag k-vector is defined in [8], where it is used to determine the integral span of the flag f-vectors of zonotopes. It is related to the flag f- and h-vectors by

$$k_S := \sum_{T \subseteq S} (-1)^{|S-T|} h_T \tag{3}$$

$$h_S = \sum_{T \subseteq S} k_T \tag{4}$$

$$k_S = \sum_{T \subseteq S} (-2)^{|S-T|} f_T \tag{5}$$

$$f_S = \sum_{T \subseteq S} 2^{|S-T|} k_T. \tag{6}$$

In [9], the flag l-vector is used to describe the cone generated by the flag f-vectors of graded posets. In what follows and throughout the paper, $S^c := [n] - S$.

$$l_S := (-1)^{n-|S|} \sum_{T \supseteq S^c} (-1)^{|T|} f_T \tag{7}$$

$$f_S = \sum_{T \subseteq S^c} l_T \tag{8}$$

$$l_S = (-1)^{|S|} \sum_{T \supseteq S} h_T \tag{9}$$

$$h_S = (-1)^{|S|} \sum_{T \supseteq S} l_T. \tag{10}$$

The flag L-vector is introduced in [4]. It is a variant on the flag l-vector, used to describe some of the facets of the closed cone generated by the flag f-vectors of Eulerian posets.

$$L_S := (-1)^{n-|S|} \sum_{T \supseteq S^c} (-2)^{-|T|} f_T \tag{11}$$

$$f_S = 2^{|S|} \sum_{T \subseteq S^c} L_T. \tag{12}$$

By combining formulas and reversing the order of summation, it is easy to see that:

$$L_S = (-1)^{|S|} \sum_{T \supseteq S} 2^{-|T|} k_T \tag{13}$$

$$k_S = (-2)^{|S|} \sum_{T \supseteq S} L_T. \tag{14}$$

3. Orders on Sets Counted by the Fibonacci Numbers

The dimension of $\mathcal{E}_n$ is F_n, the Fibonacci number with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Thus it is necessary to index basis elements by a set counted by F_n. A convenient indexing set is $\mathcal{F}_n$, the set of words in the set $\{1, 2\}$ with the sum of the letters equal to n. These words are referred to as "1-2-words." The sum of the letters in a 1-2-word w is called the *degree* $\deg(w)$ and the *length* $l(w)$ is the number of letters in w. Many of the change-of-basis formulas involve partial orders on $\mathcal{F}_n$.

The Fibonacci order "$\leq$," is a partial order on $\mathcal{F}_n$, generated by the following covering relations: $v \gtrdot w$ if w is obtained from v by replacing some 2 with 11. The word $11\cdots 1$ is minimal and any 1-2-word with no adjacent 1's is maximal. Intervals in the Fibonacci order are boolean algebras, so if $w \leq v$ then the Möbius function is $\mu(w, v) = (-1)^{l(v)-l(w)}$. In fact, the Fibonacci order is an order-ideal in a boolean algebra. Therefore it is a meet semi-lattice, and if two elements have an upper bound then they have a join. To avoid confusion, the phrase "Fibonacci order," the interval notation $[v, w]$ and the symbol "$\leq$" always mean the "ordinary" Fibonacci order. Relations and intervals in all other orders defined below are always marked with subscripts.

Most of the bases discussed here are built on subsets of $[n]$. There are several natural ways to choose F_n subsets of $[n]$. The *(left) sparse* subsets of $[n]$ are the subsets which contain no pair of adjacent elements and which do not contain n. If S^c is left sparse, then S is called *(left) dense*. There is a right-handed version of sparseness, where the requirement is that a right-sparse set not contain 1. So the bases built on sparse sets have right and left-handed versions. We will deal only with left sparse sets, and omit the adjective "left" from now on. Each 1-2-word w corresponds to a sparse set S_w by lining w up with the set $[n]$ such that each 2 covers two elements of $[n]$. Then for each 2 in w, the left element covered by the 2 is included in S_w and the right element covered by the 2 is excluded from S_w. Elements covered by 1's are also excluded. This construction also gives rise to a dense set $D_w := (S_w)^c$. So, for example, if $w = 12221212$, then $S_w = \{2, 4, 6, 9, 12\} \subset [13]$ and $D_w = \{1, 3, 5, 7, 8, 10, 11, 13\}$. The Fibonacci order on $\mathcal{F}_n$ is ordinary containment of sparse sets. (This is the easiest way to see that it is an order-ideal in a boolean algebra). On dense sets, the Fibonacci order is reverse containment. Also, $l(w) = |D_w| = n - |S_w|$.

Even and anti-even sets are also counted by F_n. A set $S \subseteq [n]$ is *even* if it is a disjoint union of intervals of even cardinality. A set is *anti-even* if its complement is even. Given a 1-2-word w, an even set E_w is obtained by lining up w with the set $[n]$ as before, and including all elements covered by 2's. The corresponding anti-even set is $A_w := (E_w)^c$. So if $w = 12221212$, then $E_w = \{2, 3, 4, 5, 6, 7, 9, 10, 12, 13\}$ and $A_w = \{1, 8, 11\}$. The Fibonacci order on $\mathcal{F}_n$ corresponds to *even containment* of even sets, as defined in [2]. We recall the definition here: If U and V are even sets we say V contains U evenly and write $U \subseteq_e V$ if $U \subseteq V$ and if $V - U$ is an even set. The Fibonacci order on anti-even sets is reverse even containment. The length function is $l(w) = n - \frac{1}{2}|E_w| = \frac{1}{2}(|A_w| + n)$.

A natural extension of the Fibonacci order is the *strong Fibonacci order*, "$\leq_s$." Strong Fibonacci order corresponds to ordinary containment of even sets. The strong Fibonacci order is defined by the covering relations: $v \gtrdot_s w$ if w is obtained from v by replacing some 2^k with $12^{k-1}1$ for any $k \geq 1$. The minimal element in strong Fibonacci order is $11 \cdots 1$, and w is maximal if it has one or zero 1's. The Möbius function of the strong Fibonacci order, denoted by μ_s, has the following useful, easily-proved property:

Proposition 7. *If $v \leq_s w = w_1 c w_2$, then $v = v_1 c v_2$ for $v_1 \leq_s w_1$ and $v_2 \leq_s w_2$, and $[v, w]_s \cong [v_1, w_1]_s \times [v_2, w_2]_s$. In particular,*

$$\mu_s(v, w) = \mu_s(v_1, w_1) \cdot \mu_s(v_2, w_2). \quad \square$$

Two extensions of the strong Fibonacci order are also required. The *right Fibonacci order* "$\leq_r$" is defined by covers: $v \gtrdot_r w$ if w is obtained from v by replacing some 12 with 21, or by replacing an initial 2 with 11. It is easy to see that the right Fibonacci order indeed extends the strong Fibonacci order. There is a unique minimal element 1^n and a unique maximal element 2^k if $n = 2k$ or 12^k if $n = 2k+1$. The *left Fibonacci order* "$\leq_l$" is defined by the covers: $v \gtrdot_l w$ if w is obtained from v by replacing some 21 with 12, or by replacing a final 2 with 11. The left Fibonacci order on $\mathcal{F}_n$ is isomorphic to the right Fibonacci order on $\mathcal{F}_n$, by reversing the order of words. Notice also that $\leq_r$ and $\leq_l$ are dual orders on constant-length subsets of $\mathcal{F}_n$. To remember the difference between the two orders, it is convenient to think: "In the right Fibonacci order, moving a 2 to the right gives a 'greater' word."

The right Fibonacci order is extended by a lexicographic order, reading from the right of the word, with $2 > 1$. That is because replacing 12 by 21 or replacing an initial 2 by 11 yields an earlier word in the lexicographic order. The same is true of the left Fibonacci order, reading from the left of the word. Also, both strong and ordinary Fibonacci order are extended by either lexicographic order. Thus all the triangular basis-changes given below are in particular triangular with respect to some lexicographic order.

Properties of the right Fibonacci order. The rest of the section is devoted to properties of the right Fibonacci order which are worth mentioning, although they will not be used in this paper. Proofs will be omitted or abbreviated. Similar statements hold for the left Fibonacci order.

Proposition 8. *The right Fibonacci order on constant-length subsets of $\mathcal{F}_n$ is isomorphic to the poset of integer partitions with k or fewer parts of size at most $n-2k$, ordered by inclusion of Ferrers diagrams. In particular it is a distributive lattice.* □

The right Fibonacci order on all of $\mathcal{F}_n$ is also a distributive lattice, and in fact it is the poset of order ideals in a distributive lattice, as we now explain. An element of L is called *join-irreducible* if it cannot be written as the join of two elements below it in L. Given a finite distributive lattice L, let $\mathrm{Irr}(L)$ be the subposet of join-irreducible elements of L. For any poset P, let $J(P)$ be the lattice of order ideals in P, ordered by inclusion. The fundamental theorem for finite distributive lattices [25, Section 3.4] states that L is isomorphic to $J(\mathrm{Irr}(L))$.

The notation $(\mathcal{F}_n, \leq_r)$ refers to the set $\mathcal{F}_n$ partially ordered by $\leq_r$.

Proposition 9.
(i) *The right Fibonacci order $(\mathcal{F}_n, \leq_r)$ is a sublattice of the componentwise partial order on $\mathbb{N}^n$. In particular, $(\mathcal{F}_n, \leq_r)$ is a distributive lattice.*
(ii) *Irr$(\mathcal{F}_n)$ is a sublattice of $\mathbb{N}^2$, and thus is also distributive.*

Sketch of proof. For $w \in \mathcal{F}_n$ and $i \in [n]$, let $e_i(w) = |E_w \cap [n-i+1, n]|$, and write $e(w) \in \mathbb{N}^n$ for the vector $(e_1(w), e_2(w), \ldots e_n(w))$. The map $e : \mathcal{F}_n \to \mathbb{N}^n$ is injective, and a vector $g \in \mathbb{N}^n$ has $g = e(w)$ for some $w \in \mathcal{F}_n$ if and only if:

(1) $0 \leq g_1 \leq 1$,
(2) $g_i \leq g_{i+1} \leq g_i + 1$ for $1 \leq i \leq n-1$, and
(3) whenever g_i is odd, $g_{i+1} = g_i + 1$.

These facts can be used to establish that the map e embeds $(\mathcal{F}_n, \leq_r)$ as a sublattice of $\mathbb{N}^n$.

It follows directly from the definition of $\leq_r$ that $\mathrm{Irr}(\mathcal{F}_n)$ is the subposet of $\mathcal{F}_n$ consisting of words with exactly one string of consecutive 2's. Given an element w of $\mathrm{Irr}(\mathcal{F}_n)$, define its *breadth* $b(w)$ to be the largest element of E_w and its *height* $h(w)$ to be the number of d's in w. It can be shown that the map $w \mapsto (b(w), h(w))$ embeds $\mathrm{Irr}(\mathcal{F}_n)$ as a sublattice of $\mathbb{N}^2$. Specifically, $\mathrm{Irr}(\mathcal{F}_n)$ is the sublattice of pairs (b, h) satisfying $b \leq n$ and $1 \leq h \leq \frac{b}{2}$. □

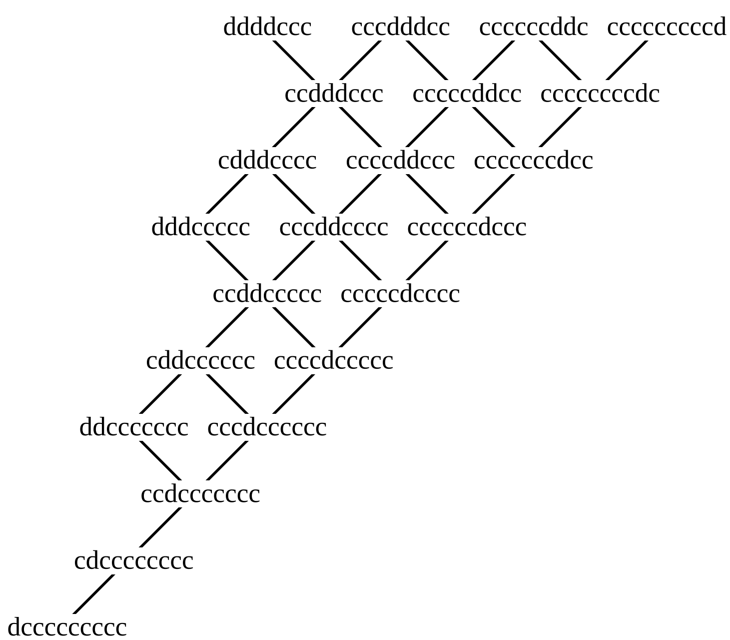
Remark 10. It is easy to see that $\mathrm{Irr}(\mathrm{Irr}(\mathcal{F}_n))$ is the subposet of $\mathbb{N}^2$ consisting of pairs (b, h) with $b \geq 3$ and $h = 1$ or $h = \frac{b}{2}$, which in general is not a distributive lattice.

Think for a moment of $\mathrm{Irr}(\mathcal{F}_n)$ as an abstract poset, forgetting that it is the subposet of join-irreducibles of $\mathcal{F}_n$. It is easy to see that the order ideals of $\mathrm{Irr}(\mathcal{F}_n)$ are counted by F_n. For example, the order ideals can be interpreted as lattice paths from $(n, 1)$ to the line $b = 2h$, which are well known to be counted by the Fibonacci numbers [20, 21]. Or, consider the element encoded by $(n, 1)$: Counting order ideals containing $(n, 1)$ and order ideals not containing $(n, 1)$ yields the Fibonacci recursion. A q-count, with the power of q counting the cardinality of the order ideal, shows that the rank-generating function $F_n(q)$ for $(\mathcal{F}_n, \leq_r)$ satisfies a q-Fibonacci recursion:

$$F_n(q) = F_{n-1}(q) + q^{n-1} F_{n-2}(q) \tag{15}$$

Order ideals in $\mathrm{Irr}(\mathcal{F}_n)$ can also be interpreted as partitions with part differences at least 2 and largest part at most $n-1$. (The cardinality of the order ideal is the sum of the parts of the partition, and the number of d's in an element of $\mathcal{F}_n$ is the number of parts.) Such partitions are well known to be counted by F_n [25, Exercise 14a, p. 46]. In the "limit" as $n \to \infty$, consider the distributive lattice $\mathcal{F}_\infty$, whose ground set is the infinite 1-2-words having a first letter but no last letter and containing only a finite number of d's. Then $\mathrm{Irr}(\mathcal{F}_\infty)$ is the sublattice of $\mathbb{N}^2$ with $1 \leq h \leq \frac{b}{2}$ (see Figure 11). The poset $\mathcal{F}_\infty$ can also be interpreted as *Rogers-Ramanujan partitions* [1], partitions with part differences at least 2, partially ordered by containment of Ferrers diagrams.

Figure 11. *Ranks 0 through 9 of the poset Irr($\mathcal{F}_\infty$). Elements of the poset are infinite words, but have been truncated to degree 11 for this figure.*



4. Bases for $\mathcal{E}_n$

The sparse flag f-vector is indexed by $w \in \mathcal{F}_n$, with $f^{sp}_w := f_{S_w}$. The dense flag f-vector is $f^d_w := f_{D_w}$. The even flag f-vector is $f^e_w := f_{E_w}$ and the anti-even flag f-vector is $f^a_w := f_{A_w}$. Similarly for h, k, l and L.

Some of the bases are not overtly related to subsets of $[n]$: in particular, the cd-index Φ and the ce-index Λ. The ce-index, defined by Stanley [26], is a polynomial in non-commuting variables c and e with $\deg(c) = \deg(e) = 1$, and e's occurring only in even-length strings. It is simple and useful to index these bases by $\mathcal{F}_n$ as well. A cd-monomial is obtained from $w \in \mathcal{F}_n$ by replacing 1 with c and 2 with d. To obtain a ce-monomial, replace 2 with ee instead. We abuse notation slightly by writing $\langle w|\Phi\rangle$ for $w \in \mathcal{F}_n$ when we mean the coefficient of the cd-monomial corresponding to w. Similarly for $\langle w|\Lambda\rangle$.

A consequence of the Bayer-Billera relations is that for all Eulerian posets, $h_S = h_{S^c}$. Therefore the sparse flag h-vectors are equivalent to the dense flag h-vectors, and the even and anti-even flag h-vectors are equivalent. The following turn out to be bases for $\mathcal{E}_n$ (We will define Γ later):

Bases for $\mathcal{E}_n$:

- f^{sp} The sparse flag f-vectors
- h^{sp} The sparse flag h-vectors, or equivalently the dense flag h-vectors h^{sp}
- k^{sp} The sparse flag k-vectors
- l^d The dense flag l-vectors
- f^a The anti-even flag f-vectors, or equivalently the even flag l-vectors l^e
- k^e The even flag k-vectors
- Φ The cd-index
- Λ The ce-index, or equivalently the even flag L-vectors L^e
- Γ The Charney-Davis index.

Given a partial order "$\leq$", we say a change-of-basis formula is triangular "in $\leq$" if it is triangular in any linear extension of $\leq$. Most of the relations between the above bases are either not triangular, or are triangular in one of the four partial orders defined in the previous section. Figure 12 is a summary of which bases are related triangularly, in which partial orders. The counterexamples to triangularity appear for $n = 3$ or 4. The basis-changes marked by question marks appear to be triangular for small examples.

Figure 12. Summary of bases and relations. *The symbols* $\diagdown$, ◺ *and* ◹ *indicate diagonal, lower- and upper-triangular relationships respectively. Subscripts indicate strong, left and right Fibonacci orders. Absence of a subscript indicates ordinary Fibonacci order. The symbol* □ *indicates that the relationship is not triangular in any order. The entry* ◺$_r$◹$_l$ *denotes that either* ◺$_r$ *or* ◹$_l$ *is appropriate.*

	f^{sp}	l^d	h^{sp}	k^{sp}	Φ	Γ	Λ	k^e	f^a
f^{sp}	id								
l^d	$\diagdown$	id							
h^{sp}	◺	◺	id						
k^{sp}	◺	◺	◺	id					
Φ	◺$_r$	◺$_r$	◺$_r$	◺$_r$◹$_l$	id				
Γ	?	?	□	◹$_l$	◹	id			
Λ	?	?	□	◹$_l$	◹	$\diagdown$	id		
k^e	□	□	□	◹$_l$	◹$_s$	◹$_s$	◹$_s$	id	
f^a	?	?	□	□	□	◺$_s$	◺$_s$	□	id

For the reader's convenience, we include the following index to the change-of-basis formulas.

Figure 13. Index to formulas. *For example, look in the k^e row, Γ column to find the equation number of the formula expressing k^e in terms of Γ. Asterisks indicate that the desired formula follows trivially from the numbered formula. An empty box indicates that we are not aware of a simple formula for the connection coefficients.*

	f^{sp}	l^d	h^{sp}	k^{sp}	Φ	Γ	Λ	k^e	f^a
f^{sp}	•	22	17	21					
l^d	22*	•	17*	21*					
h^{sp}	16	16*	•	19					
k^{sp}	20	20*	18	•	35				
Φ				36	•	30	27		
Γ					29	•	31		
Λ					28	32	•		
k^e						34	24	•	
f^a						33	23		•

Before we continue, a few words about what is not included in the list of bases. Below is a list of sets of flag vectors which one might guess are bases for $\mathcal{E}_n$, but which are not. Most of these are ruled out by results quoted or proved later. All of them can be seen not to be bases by considering $n = 2$ or 3.

Not bases for $\mathcal{E}_n$:

- f^d The dense flag f-vectors
- f^e The even flag f-vectors
- h^e The even flag h-vectors, or equivalently the anti-even flag h-vectors h^a
- k^d The dense flag k-vectors
- k^a The anti-even flag k-vectors
- l^{sp} The sparse flag l-vectors
- l^a The anti-even flag l-vectors
- L^{sp} The sparse flag L-vectors
- L^d The dense flag L-vectors
- L^a The anti-even flag L-vectors.

5. Sparse and Dense Bases

In this section, we discuss the bases f^{sp}, h^{sp}, k^{sp} and l^d. Bayer and Billera [3] showed that the sparse flag f-vectors are a basis for the linear span of flag f-vectors of Eulerian posets. Since the Fibonacci order is containment on sparse sets, Equations (1) through (8) imply that for all $w \in \mathcal{F}_n$:

$$h^{sp}_w = \sum_{v \le w} (-1)^{l(v)-l(w)} f^{sp}_v \tag{16}$$

$$f^{sp}_w = \sum_{v \le w} h^{sp}_v \tag{17}$$

$$k^{sp}_w = \sum_{v \le w} (-1)^{l(v)-l(w)} h^{sp}_v \tag{18}$$

$$h^{sp}_w = \sum_{v \le w} k^{sp}_v \tag{19}$$

$$k^{sp}_w = \sum_{v \le w} (-2)^{l(v)-l(w)} f^{sp}_v \tag{20}$$

$$f^{sp}_w = \sum_{v \le w} 2^{l(v)-l(w)} k^{sp}_v \tag{21}$$

$$f^{sp}_w = (-1)^{l(w)} l^d_w. \tag{22}$$

Equation (22) is verified by the following fact.

Proposition 14. *Let P be a graded poset of rank $n+1$ with $h_S = h_{S^c}$ for every $S \in [n]$ (e.g. if P is Eulerian). Then for any $S \in [n]$, $l_S = (-1)^{|S|} f_{S^c}$.*

Proof.

$$\begin{aligned} l_S &= (-1)^{|S|} \sum_{T \supseteq S} h_T \\ &= (-1)^{|S|} \sum_{T \subseteq S^c} h_T \\ &= (-1)^{|S|} f_{S^c}. \end{aligned}$$

□

When $h_S = h_{S^c}$, the flag f- and l vectors have interesting inclusion-exclusion properties, which are worth noting although they are not needed here.

Proposition 15. *Let P be a graded poset of rank $n+1$ with $h_S = h_{S^c}$ for every $S \in [n]$. Then for any $S \in [n]$,*

$$\begin{aligned} f_S &= \sum_{T \supseteq S} (-1)^{n-|T|} f_T \\ l_S &= (-1)^{|S|} \sum_{T \subseteq S} l_T. \end{aligned}$$

Proof.

$$\begin{aligned} (-1)^{|S|} \sum_{T \subseteq S} l_T &= (-1)^{|S|} \sum_{T \subseteq S} (-1)^{|T|} f_{T^c} \\ &= (-1)^{|S|} \sum_{T \supseteq S^c} (-1)^{n-|T|} f_T \\ &= l_S. \end{aligned}$$

The first formula follows by Proposition 14. □

6. Even and Anti-Even Bases

In this section we deal with the bases f^a, k^e and Λ. First, notice that the even flag l-vector is the same as the anti-even flag f-vector. By Proposition 14, for all w:

$$l^e_w = (-1)^{2n-2l(w)} f^a_w = f^a_w.$$

One can prove that the f^a span, using the Bayer-Billera relations [10, Corollary 3.7]. Alternately, one can appeal to a result about flag L-vectors. In [4], it is shown that in an Eulerian poset, $L_T = 0$ when T is not even, and $L_T = \langle w|\Lambda\rangle$ when $T = E_w$. Restricting Equation (12) to anti-even flag f-vectors yields:

$$f^a_w = 2^{2l(w)-n} \sum_{v \leq_s w} \langle v|\Lambda\rangle. \tag{23}$$

Since the coefficients of the ce-index span, so do the f^a_w.

Similarly, the even flag k-vector spans. Equation (14) yields

$$k^e_w = 4^{n-l(w)} \sum_{v \geq_s w} \langle v|\Lambda\rangle. \tag{24}$$

There is a simple convolution formula for f^a [10, Proposition 1.2]. If the degree of w_1 is $k-1$, then

$$f^a_{w_1 1 w_2} = \sum_{\substack{x \in P \\ \rho(x)=k}} f^a_{w_1}([\hat{0},x]) \cdot f^a_{w_2}([x,\hat{1}]). \tag{25}$$

Combined with Proposition 7, Equation (25) leads to a convolution formula for Λ.

Proposition 16. *If $w = w_1 c w_2$, with $\deg(w_1) = k-1$ then*

$$2\langle w|\Lambda_P\rangle = \sum_{\substack{x \in P \\ \rho(x)=k}} \langle w_1|\Lambda_{[\hat{0},x]}\rangle \cdot \langle w_2|\Lambda_{[x,\hat{1}]}\rangle.$$

Proof. Notice that $v \leq w$ if and only if $v = v_1 c v_2$ with $v_1 \leq w_1$ and $v_2 \leq w_2$. Therefore, by applying Möbius inversion to Equation (23),

$$\begin{aligned}
&\sum_{\substack{x \in P \\ \rho(x)=k}} \langle w_1|\Lambda_{[\hat{0},x]}\rangle \cdot \langle w_2|\Lambda_{[x,\hat{1}]}\rangle \\
&\qquad = \sum_{\substack{x \in P \\ \rho(x)=k}} 2^{n-1} \sum_{\substack{v_1 \leq w_1 \\ v_2 \leq w_2}} 4^{-l(v_1)-l(v_2)} \mu_s(v_1,w_1)\mu_s(v_2,w_2) f^a_{v_1}([\hat{0},x]) f^a_{v_2}([x,\hat{1}]) \\
&\qquad = 2^{n-1} \sum_{\substack{v_1 \leq w_1 \\ v_2 \leq w_2}} 4^{-l(v_1)-l(v_2)} \mu_s(v_1 c v_2, w_1 c w_2) f^a_{v_1 c v_2}(P) \\
&\qquad = 2^{n-1} \cdot 4 \sum_{v \leq w} 4^{-l(v)} \mu_s(v,w) f^a_v \\
&\qquad = 2\langle w|\Lambda_P\rangle.
\end{aligned}$$

□

For each $c \in C(S)$, write $c = (\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_{m+1} = \hat{1})$. Proposition 16 and induction imply the following:

Proposition 17.

$$2^{2l(w)-n}\langle w|\Lambda_P\rangle = \sum_{C(A_w)}\prod_{i=0}^{m}\langle e^{\rho(x_{i+1})-\rho(x_i)-1}|\Lambda_{[x_i,x_{i+1}]}\rangle.$$

□

7. The cd-Index and the Charney-Davis Index

The ce-index Λ is related to the cd-index Φ by the formula

$$\Lambda(c,e) = \Phi(c,(c^2-e^2)/2). \tag{26}$$

As observed in [2, Equation 1] and as a consequence of Equation (26), above:

$$\langle w|\Phi\rangle = (-2)^{n-l(w)}\sum_{v\geq w}\langle v|\Lambda\rangle \tag{27}$$

$$\langle w|\Lambda\rangle = (-2)^{-n}(-1)^{l(w)}\sum_{v\geq w}2^{l(v)}\langle v|\Phi\rangle. \tag{28}$$

Equations (24) and (28) can be combined to relate Φ to k^e upper-triangularly in the strong Fibonacci order.

For an Eulerian poset Q of rank $2k+1$, let $\kappa(Q) := \langle d^k|\Phi_Q\rangle$. Proposition 17 and Equation (28) imply:

Proposition 18.

$$\sum_{v\geq w}2^{l(v)}\langle v|\Phi_P\rangle = 2^{n-l(w)}\sum_{C(A_w)}\prod_{i=0}^{k}\kappa([x_i,x_{i+1}]). \quad \square$$

The form of Proposition 18 motivates the definition of the *Charney-Davis index* Γ, a polynomial in non-commuting variables γ and δ, degrees 1 and 2 respectively. Elements of $\mathcal{F}_n$ correspond to $\gamma\delta$-monomials w of degree n, and Γ is defined coefficient-wise:

$$\langle w|\Gamma\rangle := \sum_{C(A_w)}\prod_{i=0}^{k}\kappa([x_i,x_{i+1}]).$$

Proposition 18 and Möbius inversion yield the following formulas:

$$\langle w|\Gamma\rangle = 2^{l(w)-n}\sum_{v\geq w}2^{l(v)}\langle v|\Phi\rangle \tag{29}$$

$$\langle w|\Phi\rangle = 2^n(-2)^{-l(w)}\sum_{v\geq w}(-2)^{-l(v)}\langle v|\Gamma\rangle \tag{30}$$

$$\langle w|\Gamma\rangle = (-1)^n(-2)^{l(w)}\langle w|\Lambda\rangle \tag{31}$$

$$\langle w|\Lambda\rangle = (-1)^n(-2)^{-l(w)}\langle w|\Gamma\rangle \tag{32}$$

$$f^a_w = (-2)^{-n}4^{l(w)}\sum_{v\leq_s w}(-2)^{-l(v)}\langle v|\Gamma\rangle. \tag{33}$$

$$k^e_w = (-4)^n4^{l(w)}\sum_{v\geq_s w}(-2)^{-l(v)}\langle v|\Gamma\rangle. \tag{34}$$

Equation (29) and the definition of Γ are the proof of Theorem 1. If P has the Charney-Davis property, then the coefficients of Γ are non-negative, and therefore by Equation (29),

$$\sum_{v \geq w} 2^{l(v)} \langle v | \Phi \rangle = 2^{n-l(w)} \langle w | \Gamma \rangle \geq 0.$$

R. Ehrenborg and M. Readdy [19] suggested the following alternate proof of Equations (29) and (30). The proof uses coproduct techniques. For those familiar with coproducts, we sketch the proof, using notation from [18]. Let D be the linear functional on cd-polynomials which sets $c = 0$ and $d = \delta$. Then the Charney-Davis index can be written as follows. The first sum is over all chains $\hat{0} < x_1 < x_2 < \ldots < x_m < \hat{1}$.

$$\begin{aligned}\Gamma_P &= \sum D(\Phi_{[\hat{0},x_1]}) \cdot \gamma \cdot D(\Phi_{[x_1,x_2]}) \cdot \gamma \cdots \gamma \cdot D(\Phi_{[x_m,\hat{1}]}) \\ &= D(\Phi_P) + \sum_{\hat{0}<x<\hat{1}} D(\Phi_{[\hat{0},x]}) \cdot \gamma \cdot \Gamma_{[x,\hat{1}]}.\end{aligned}$$

The second term is a coproduct. Since the cd-index is a coalgebra homomorphism, by induction $\Gamma_P = g(\Phi_P)$ where g is some linear map on cd-polynomials, and g satisfies the functional equation:

$$g(w) = D(w) + \sum_w D(w_{(1)}) \cdot \gamma \cdot g(w_{(2)}).$$

Using the functional equation, it is easy to calculate $g(cw) = 2\gamma g(w)$ and $g(dw) = (\delta + 2\gamma^2)g(w)$. Thus by induction $\Gamma_P = \Phi_P(2\gamma, \delta + 2\gamma^2)$. Equations (29) and (30) follow.

Remark 19. There is the following curious formula for Γ:

$$\begin{aligned}\langle w | \Gamma \rangle &= \sum_{A_w \subseteq T \subseteq D_w} (-2)^{|D_w - T|} f_T \\ &= \sum_{T \subseteq D_w} (-1)^{|D_w - (A_w \cup T)|} h_T.\end{aligned}$$

Notice that these are ordinary flag f- or h-vectors, not sparse, dense, etc. The two expressions on the right side are easily seen to be equal using the relation between h_S and f_S and the binomial theorem. To see why they hold, consider that a "right sparse" version of Equation (35) below, implies that for an Eulerian poset Q of rank $2k+1$,

$$\kappa(Q) = \sum_{T \subseteq D_{(d^k)}} (-1)^{|D_{(d^k)} - (A_{(d^k)} \cup T)|} h_T = \sum_{A_{(d^k)} \subseteq T \subseteq D_{(d^k)}} (-2)^{|D_{(d^k)} - T|} f_T.$$

Now, apply the definition of Γ.

8. Other Changes of Basis

In this section, we give the remaining triangular basis-change formulas. Each remaining triangular relation is a consequence of the change-of-basis formula relating Φ to k^{sp}. In [8], k^{sp} is defined as a sum of cd-coefficients, and is shown to satisfy Equation (19). In [7], the cd-coefficients are given as a signed sum of sparse k-vectors. Formulas relating the cd-index to the full flag k-vector can be found in [15].

To relate Φ to k^{sp} the following two operations on 1-2-words are useful. The *left 2-shift* of w is ${}^2w$, obtained by removing the rightmost 2 from w and placing it at the beginning of the word. The *right 1-shift* of w is w^1, obtained by removing the leftmost 1 from w and placing it at the end of the word. The inversion number $i(w)$ is the number of instances of a 2 appearing before a 1 in w. For example, if $w = 12221212$, then ${}^2w = 21222121$, $w^1 = 22212121$ and $i(w) = 7$. The following are respectively [8, Definition 6.5] and [7, Proposition 7.1], translated into the language of the right Fibonacci order.

$$k^{sp}_w = \sum_{w^1 \leq_r v \leq_r w} \langle v|\Phi\rangle \tag{35}$$

$$\langle w|\Phi\rangle = \sum_{{}^2w \leq_r v \leq_r w} (-1)^{i(v)-i(w)} k^{sp}_v. \tag{36}$$

For readers familiar with the encoding of the flag f- or h-vector in an ab-polynomial, the following remark may provide insight into Equation (35): If we encode the flag k-vector as an ab-polynomial in the same way, then this "k-index" is obtained from Φ by setting $c = a$ and $d = ab + ba - 2bb$.

Remark 20. It is easy to see that Equations (35) and (36) are equivalent using a sign-reversing involution, which we will not give here. To construct the involution, it is useful to interpret 1-2-words as integer partitions as in Section 3. The left 2-shift and right 1-shift correspond to natural operations on Ferrers diagrams: Deleting the first row of the Ferrers diagram for w gives the Ferrers diagram for ${}^2w$. Deleting the first column of the Ferrers diagram for w gives the Ferrers diagram for w^1.

We can now explain more of the entries in the table in Figure 12. Recall that on constant-length subsets of $\mathcal{F}_n$, the left and right Fibonacci orders are dual. The intervals $[w^1, w]_r$ and $[{}^2w, w]_r$ are constant-length subsets. Thus Φ is related to k^{sp} lower-triangularly in the right order, or upper-triangularly in the left order. Because the right Fibonacci order extends the ordinary and strong Fibonacci orders, Equations (35) and (36) combine with Equations (16) through (22), relating Φ to h^{sp}, f^{sp} and l^d lower-triangularly in the right Fibonacci order. Similarly, Equations (35) and (36) combine with Equations (24) through (30), relating k^{sp} to Λ, Γ and k^e upper-triangularly in the left Fibonacci order.

9. Convolution Formulas for cd-Coefficients

The change-of-basis formulas can be applied to give convolution formulas for some of the bases. This was already done in Proposition 16, where Equation (25), a trivial convolution formula for f^a, became a less trivial convolution for Λ. In the same way, the convolution formula for Λ yields a convolution formula for Φ, which has important consequences for non-negativity of cd-coefficients.

Proposition 21. *If* $w = w_1 dcd w_2$ *with* $\deg(w_1 d) = k - 1$ *then*

$$2\langle w|\Phi_P\rangle = \sum_{\substack{x \in P \\ \rho(x) = k}} \langle w_1 d|\Phi_{[\hat{0},x]}\rangle \cdot \langle dw_2|\Phi_{[x,\hat{1}]}\rangle.$$

If $w = cdw_2$, *then*

$$2\langle w|\Phi_P\rangle = \sum_{\substack{x \in P \\ \rho(x) = 1}} \langle dw_2|\Phi_{[x,\hat{1}]}\rangle.$$

Proof. If $w = w_1dcdw_2$, then $v \geq w$ if and only if $v = v_1cv_2$ for $v_1 \geq w_1d$ and $v_2 \geq dw_2$. Thus by Proposition 16 and Equations (27) and (28),

$$\begin{aligned}
&\sum_{\substack{x\in P\\ \rho(x)=k}} \langle w_1d|\Phi_{[\hat{0},x]}\rangle \cdot \langle dw_2|\Phi_{[x,\hat{1}]}\rangle \\
&\qquad = \sum_{\substack{x\in P\\ \rho(x)=k}} (-2)^{(n-1)-(l(w)-1)} \sum_{\substack{v_1\geq w_1d\\ v_2\geq dw_2}} \langle v_1|\Lambda_{[\hat{0},x]}\rangle \cdot \langle v_2|\Lambda_{[x,\hat{1}]}\rangle \\
&\qquad = (-2)^{n-l(w)} \sum_{v_1cv_2\geq w} \sum_{\substack{x\in P\\ \rho(x)=k}} \langle v_1|\Lambda_{[\hat{0},x]}\rangle \cdot \langle v_2|\Lambda_{[x,\hat{1}]}\rangle \\
&\qquad = 2(-2)^{n-l(w)} \sum_{v\geq w} \langle v|\Lambda_P\rangle \\
&\qquad = 2\langle w|\Phi_P\rangle
\end{aligned}$$

A similar calculation can be made in the case where $w = cdw_2$. □

Proposition 21 leads to the proof of Theorem 3 from the Introduction.

Theorem 3. *Let P be an Eulerian poset of rank $n+1$. The coefficients of dc^{n-2}, $c^{n-2}d$, cdc^{n-3} and $c^{n-3}dc$ are all non-negative. Also, let v be a cd-monomial starting and ending in d and alternating $dcdc\cdots cd$ with at least one c, such that c^kvc^m has degree n. Then $\langle c^kvc^m|\Phi_P\rangle \geq 0$.*

Proof. For any Eulerian poset P of rank $n+1$:

$$f_{\{i\}} = \begin{cases} 2+\langle dc^{n-2}|\Phi_P\rangle & \text{if } i=1\\ 2+\langle c^{i-1}dc^{n-i-1}|\Phi_P\rangle + \langle c^{i-2}dc^{n-i}|\Phi_P\rangle & \text{if } 2\leq i\leq n\\ 2+\langle c^{n-2}d|\Phi_P\rangle & \text{if } i=n \end{cases} \tag{37}$$

Because $f_{\{1\}} \geq 2$ and $f_{\{n\}} \geq 2$, $\langle dc^{n-2}|\Phi_P\rangle$ and $\langle c^{n-2}d|\Phi_P\rangle$ are non-negative. The remaining coefficients are non-negative by Proposition 21, and induction. □

Proposition 21 has several generalizations, which we state without proof.

Proposition 22. *If $w = w_1dc^mdw_2$ with $\deg(w_1d) = k-1$, then*

$$\sum_{v\geq c^m} 2^{l(v)}\langle w_1dvdw_2|\Phi_P\rangle = \sum_{(x_1\lessdot\cdots\lessdot x_m)\in C([k,k+m-1])} \langle w_1d|\Phi_{[\hat{0},x_1]}\rangle \cdot \langle dw_2|\Phi_{[x_m,\hat{1}]}\rangle.$$

If $w = c^mdw_2$, then

$$\sum_{v\geq c^m} 2^{l(v)}\langle vdw_2|\Phi_P\rangle = \sum_{(x_1\lessdot\cdots\lessdot x_m)\in C([m])} \langle dw_2|\Phi_{[x_m,\hat{1}]}\rangle.$$

□

Besides the obvious generalization of replacing c by c^m, there is another way of generalizing Proposition 21. In the proof of Proposition 21, we found a c in w and wrote down a convolution formula by splitting w at that c. The formula came out nicely because the c was between two d's. There are other formulas when the c is not between two d's.

Proposition 23. *If* $w = w_1 dccw_2$ *with* $\deg(w_1 d) = k-1$, *then*

$$2\langle w|\Phi_P\rangle + \langle w_1 ddw_2|\Phi_P\rangle = \sum_{x\in P:\rho(x)=k} \langle w_1 d|\Phi_{[\hat{0},x]}\rangle \cdot \langle cw_2|\Phi_{[x,\hat{1}]}\rangle.$$

If $w = ccw_2$ *with* $\deg(w_1 d) = k-1$, *then*

$$2\langle w|\Phi_P\rangle + \langle dw_2|\Phi_P\rangle = \sum_{x\in P:\rho(x)=1} \langle cw_2|\Phi_{[x,\hat{1}]}\rangle.$$

□

Proposition 24. *If* $w = w_1 cccw_2$ *with* $\deg(w_1 c) = k-1$, *then*

$$2\langle w|\Phi_P\rangle + \langle w_1 cdw_2|\Phi_P\rangle + \langle w_1 dcw_2|\Phi_P\rangle = \sum_{x\in P:\rho(x)=k} \langle w_1 c|\Phi_{[\hat{0},x]}\rangle \cdot \langle cw_2|\Phi_{[x,\hat{1}]}\rangle.$$

□

Propositions 21 through 24 also have coproduct proofs, suggested by R. Ehrenborg and M. Readdy [19]. For example, the right side of Proposition 21 can be rewritten as

$$\sum_{\hat{0}<x<\hat{1}} \delta_{w_1 d}(\Phi_{[\hat{0},x]}) \cdot \delta_{dw_2}(\Phi_{[x,\hat{1}]}),$$

where δ_w is a linear functional on cd-polynomials which returns the coefficient of w. Now the proposition follows, after some calculation, from the fact that Φ is a coalgebra homomorphism.

10. Non-Negative cd-Index Coefficients of Gorenstein* Posets

In this section, we prove Theorem 4. The proof of Theorem 5 is nearly identical. In what follows, topological statements about a poset P apply to the geometric realization of the order complex of $P - \{\hat{0}, \hat{1}\}$ [25, Section 3.8]. The m-skeleton of a graded poset is the subposet $Sk_m(P) = \{p \in P : \mathrm{rank}(p) \leq m+1\} \cup \{\hat{1}\}$. If P is the face poset of a CW-complex [11], this corresponds to the usual notion of skeleton.

Theorem 4. *Let* P *be a Gorenstein* poset of rank* $n+1$. *Let* w *be a cd-monomial of degree* n *with* $w = c^k d^{e_1} cd^{e_2} c \cdots cd^{e_j} c^m$ *for* $j \geq 1$ *and* $e_i \in \{1,2\}$ *for each* i. *(If* $e_1 = 2$, *require that* $k \in \{0,1\}$; *if* $e_j = 2$, *require that* $m \in \{0,1\}$*).*

Then $\langle w|\Phi_P\rangle \geq 0$.

Proof. Davis and Okun [14] proved that $\langle d^2|\Phi\rangle$ is non-negative for a Gorenstein* poset of rank 5. We use their result, and induction on j. The case where $j = 1$ and $e_1 = 1$ is Proposition 25, below. If $j = 1$ and $e_1 = 2$, then the Davis-Okun result and Proposition 21 give non-negativity. For $j > 1$, the result follows by Proposition 21 and induction. □

Proposition 25. *If* P *is an Eulerian poset of rank* $n+1$ *then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = (-1)^m \tilde{\chi}\left(Sk_m(P)\right) - 1.$$

If P *is Gorenstein* then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = \mathit{rank}\left[H_m\left(Sk_m(P)\right)\right] - 1 \geq 0.$$

Proof.

$$(-1)^m \tilde{\chi}\left(Sk_m(P)\right) = f_{\{m+1\}} - f_{\{m\}} + \cdots + (-1)^m f_{\{1\}} + (-1)^{m+1}.$$

By Equation (37), this is $\langle c^m dc^{n-m-2} | \Phi_P\rangle + 1$. Also,

$$(-1)^m \tilde{\chi}\left(Sk_m(P)\right) = \sum_i (-1)^{m-i} \mathrm{rank}\left[H_i\left(Sk_m(P)\right)\right].$$

If P is a homology sphere then by the Rank-Selection Theorem for Cohen-Macaulay [12] posets, $H_i\left(Sk_m(P)\right) = 0$ for $i < m$. To show that $H_m\left(Sk_m(P)\right) \neq 0$, consider any element $p \in P$ of rank $m+2$. The interval $[\hat{0}, p]$ is a Gorenstein* poset, i.e. the order complex of $(\hat{0}, p)$ is a homology m-sphere. The orientation class of that homology sphere is a non-zero element of $H_m\left(Sk_m(P)\right)$. □

The first assertion of Proposition 25 was also observed by I. Novik [22, Section 2].

11. Comments and Further Questions

Non-negativity of the cd-index. A possible next step towards resolving the non-negativity of the cd-index is to examine the coefficients of *ddcc* and *dccd* in Gorenstein* posets. The coefficient of *dd* can be interpreted as the Charney-Davis quantity. Is there a way to interpret the coefficient of *ddcc* which also takes advantage of the work being done on the Charney-Davis Conjecture? If so, the interpretation can probably be generalized to other coefficients.

Can the Charney-Davis conjecture be proven in the special case of order complexes? If so, Corollary 2 would imply the non-negativity of cd-coefficients with no adjacent c's, for any Gorenstein* poset. Theorem 5 would imply the non-negativity of additional coefficients.

Relation to P-partitions. L. Billera and N. Liu [10] defined a non-commutative Hopf algebra of flag-enumeration functionals on graded posets. The algebra restricts nicely to Eulerian posets, so the bases discussed here are graded bases for the algebra of flag-enumeration functionals on Eulerian posets. N. Bergeron, S. Mykytiuk, F. Sottile and S. Van Willigenburg [6] showed that the algebra of flag-enumeration functionals on Eulerian posets is dual to the peak algebra of Stembridge, related to enriched P-partitions. It would be interesting to know how the bases relate. Do any of the bases discussed here correspond to interesting new bases for the Stembridge peak algebra? Do any known bases of the peak algebra correspond to new bases for the flag f-vectors of Eulerian posets?

Which bases are "good?" The cd-index is a "good" basis for several reasons. Ehrenborg and Readdy [19] suggested that it would be interesting to know if any other bases are good in the same way. The cd-index is compatible with various geometric operations. Several authors [18, 16, 17] have studied polytope operations like pyramid, prism and cutting and the corresponding operations on posets, and have determined the effect of these operations on the full flag f- and h-vectors and on the cd-index. Are there similarly simple ways of describing the effect of these operations on any other bases besides the cd-index? The cd-index has good geometric properties because it inherits a coproduct structure from the full flag f-vector. Do any of the other bases have a coproduct structure?

The cd-index also has a product structure: it is a non-commutative polynomial in c and d. The operation of multiplication corresponds to the *join* of posets [25, Lemma 1.1]. Any of the other bases can be given given the structure of a non-commutative polynomial, say in γ and δ. For example, to obtain a "$\gamma\delta$-index" F^{sp} for the sparse flag f-vector, make a $\gamma\delta$-word in the obvious way from each 1-2-word w and set $\langle w|F^{sp}\rangle = f^{sp}_w$. Does multiplication of these non-commutative polynomials have any meaning in terms of the posets?

Other bases. This paper followed Bayer and Billera [3] in using what is here called the "left" sparse f-vectors as a basis. The relation of the right sparse bases to the other bases can be worked out easily by analogy to the left sparse case. Still unanswered is the question of how the left sparse basis is related to the right sparse basis. There are F_{n-1} out of the F_n formulas which are easy: If w is a 1-2-word of degree $n-1$ then $f^{sp}_{1w} = f^{rsp}_{w1}$. Here f^{rsp} is the right sparse flag f-vector, but better notation will be needed if this vector is studied further.

There are additional ways of choosing F_n subsets of $[n]$. For example, call a subset *2-sparse* if it omits the element 2 and contains no adjacent elements, considering 1 and 3 to be adjacent. Similarly, for any $j \in [n]$, there is a collection of j-sparse subsets. Thus n-sparse means left sparse and 1-sparse means right sparse. Is the j-sparse flag f-vector a basis, for the other j as well? What is the relationship between the j-sparse and j'-sparse bases for $j \neq j'$?

More generally, which sets of flag f-vectors are bases? In other words, it would be interesting to study the matroid of flag f-vectors of Eulerian posets. Since this matroid arises from a collection of vectors in $\mathbb{R}^{F_n}$, there is also an associated hyperplane arrangement.

Basis-change formulas. There are several question marks in Figure 12 and many blank spaces in the table in Figure 13. Are there good formulas for connection coefficients in any of the other basis-change formulas?

The partial orders. The Fibonacci order and the strong Fibonacci are not new. The Fibonacci order is the face poset of the matching complex of a path, and has been studied by V. Reiner and V. Welker [23]. They showed that its homotopy type is that of a point if $n \equiv 2 (\mathrm{mod}\ 3)$ and $\mathbb{S}^{\lfloor \frac{n}{3} \rfloor - 1}$ if $n \equiv 0, 1 (\mathrm{mod}\ 3)$. A partial order dual to the strong Fibonacci interval $[c^{2k}, d^k]_s$ occurs in Exercise 52, Chapter 3 of [25], attributed there to K. Baclawski and P. Edelman. Translated into the language of this paper, the exercise is to prove that $\mu_s(c^{2k}, d^k) = (-1)^k C_k$ (Catalan number) and that $[v, d^k]_s \cong [c^{2j}, d^j]_s$, where $j = l(v) - k$. Together with Proposition 7, this is enough to determine the Möbius function for any interval in the strong Fibonacci order.

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