To scatter or to cluster?

Nathan Reading
NC State University

Seminario di Algebra e Geometria
Dipartimento di Matematica Guido Castelnuovo
Università di Roma "La Sapienza"
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Scattering diagrams

Cluster algebras

Rank-2 affine scattering diagrams

Narayana numbers in affine rank 2

Overview

Scattering diagrams arose from mirror symmetry, Donaldson-Thomas theory (string theory), integrable systems, and I know almost nothing about any of that. Gross, Hacking, Keel, and Kontsevich recently applied scattering diagrams to prove longstanding conjectures about cluster algebras.

Today's goal: Introduce scattering diagrams and cluster algebras, make the connection between them, and point out some interesting combinatorics and discrete geometry.

Main points:

• Even in the two-dimensional case (e.g. affine type $\tilde{A}_1$), you have to work a bit to construct the cluster scattering diagram. I'll show how to do affine type $\tilde{A}_1$ using cluster algebras.

• I'll show how the generating function for alternating-signed Narayana numbers arises naturally.

Section 1: Scattering diagrams

Basic setup

Summary: skew-symmetric matrix, vector space and its dual, integer points $\leftrightarrow$ Laurent monomials.

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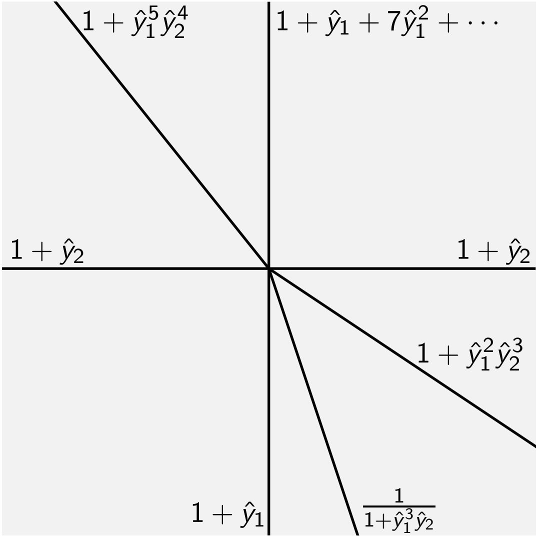
Details:

- B is $n \times n$ and skew-symmetric
- V real vector space, basis $\alpha_1, \ldots, \alpha_n$
- V^* its dual space, basis $\rho_1, \ldots, \rho_n$
- $\langle \rho_i, \alpha_j \rangle = \delta_{ij}$ (Kronecker delta)
- integer points in V^*: $\lambda = \sum_{i=1}^{n} c_i \rho_i \quad \leftrightarrow \quad x^\lambda = x_1^{c_1} \cdots x_n^{c_n}$
- integer points in V: $\beta = \sum_{i=1}^{n} d_i \alpha_i \quad \leftrightarrow \quad \hat{y}^\beta = \hat{y}_1^{d_1} \cdots \hat{y}_n^{d_n}$
- $\omega : V \times V \to \mathbb{R}$ skew-symmetric, bilinear. In the α_i basis, its matrix is B.

Scattering diagrams



A scattering diagram is a set of walls. Each wall is a codimension-1 cone in V^*, decorated with a scattering term—a formal power series in the $\hat{y}_i$.

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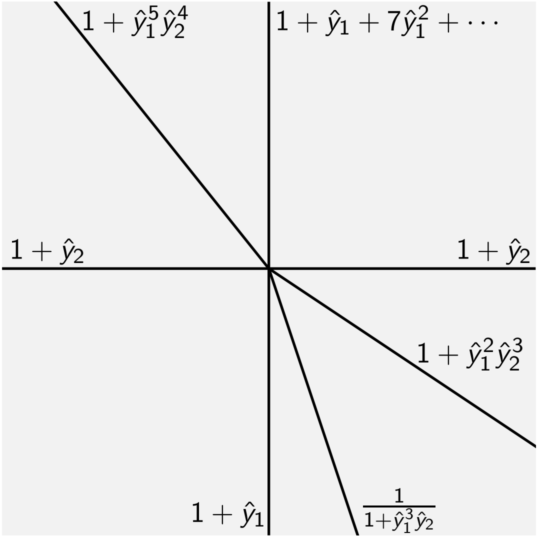

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- Each wall is normal to a primitive, positive integer vector β. (That is, $\beta = \sum c_i \alpha_i$ with $c_i \geq 0$, $\sum c_i > 0$, $\gcd(c_i) = 1$.)

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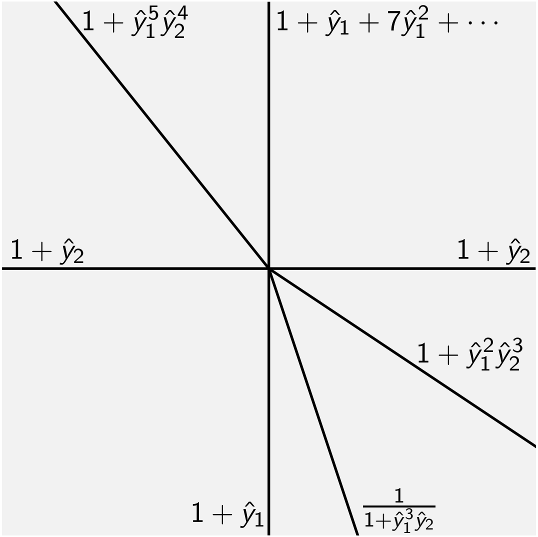

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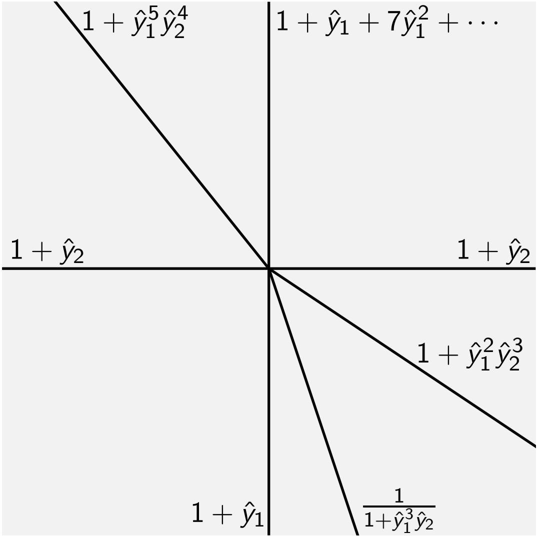

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- The scattering term is a univariate FPS in $\hat{y}^\beta$ with constant term 1.
- A finiteness condition

Wall-crossing homomorphisms and path-ordered products

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$ acts on polynomials (or FPS):

$$x^{\lambda} \mapsto x^{\lambda} f_{\mathfrak{d}}^{\langle \lambda, \pm\beta \rangle}$$
$$\hat{y}^{\phi} \mapsto \hat{y}^{\phi} f_{\mathfrak{d}}^{\omega(\pm\beta, \phi)}$$

Take "$-$" if crossing *with* β or "$+$" if crossing *against* β.

Path-ordered product $\mathfrak{p}_{\gamma}$: compose these along a path γ.

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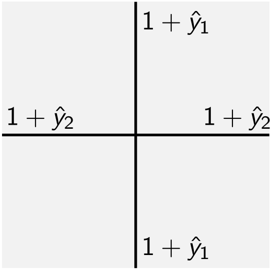

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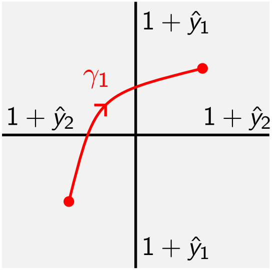

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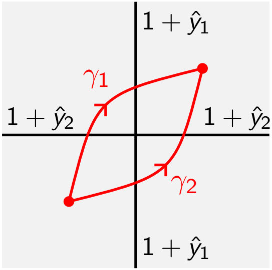

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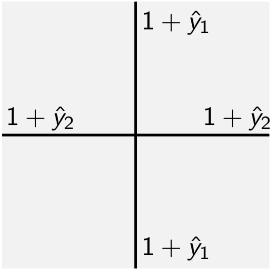

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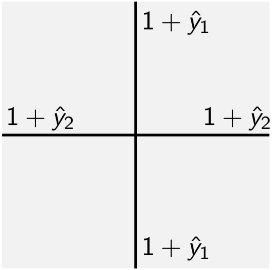

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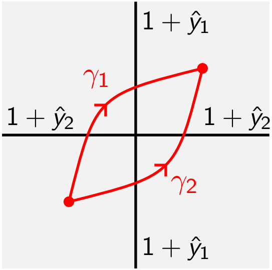

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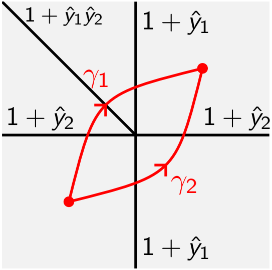

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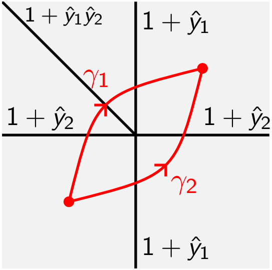

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$$\mathfrak{p}_{\gamma_1} : x_1^{-1} \mapsto x_1^{-1} \mapsto x_1^{-1}(1 + \hat{y}_1\hat{y}_2)$$
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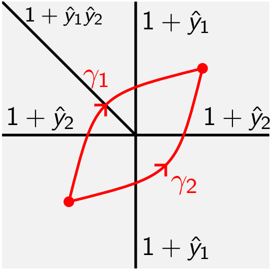

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$$\mathfrak{p}_{\gamma_2} : x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1) \mapsto x_1^{-1}(1+\hat{y}_1(1+\hat{y}_2))$$

Scattering fans

(Vaguely Stated) Theorem (R., 2017). A consistent scattering diagram cuts space into a complete fan.

A fan is a collection of convex cones, closed under passing to faces, with the property that, given any two cones in the collection, their intersection is a face of each.

If there is time, I'll give details on the construction and proof at the end. (But this seems unlikely.)

Cluster scattering diagrams

Theorem (Gross, Hacking, Keel, Kontsevich, 2014). Given a skew-symmetric integer matrix B, there is unique (up to equivalence) consistent scattering diagram $\mathfrak{D}$ such that

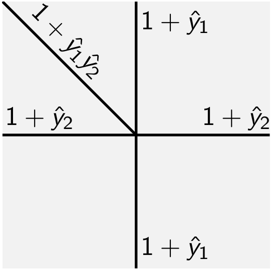
- $\mathfrak{D}$ contains the walls $(\alpha_i^\perp, 1 + \hat{y}_i)$.
- All other walls are outgoing.

A wall $(\mathfrak{d}, f_\mathfrak{d}(\hat{y}^\beta))$ is outgoing if it does not contain $\omega(\,\cdot\,, \beta)$.

This is the cluster scattering diagram $\mathrm{Scat}^T(B)$.

Example. The cluster scattering diagram for $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

One can check that the wall we added is outgoing.



Recap of Section 1: Scattering diagrams

A scattering diagram is a collection of walls. A wall is $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$

- $\mathfrak{d}$ is a codimension-1 cone.
- β is a positive integer normal vector.
- $f_{\mathfrak{d}}$ is the scattering term, a formal power series in $\hat{y}^{\beta}$.

Path-ordered product: at each wall crossing, replaces each monomial by itself times a power of the scattering term $f_{\mathfrak{d}}$.

Consistent scattering diagram: path-ordered products depend only on endpoints.

Cluster scattering diagram: Initial walls are coordinate hyperplanes, $\exists!$ way to add "outgoing" walls to get a consistent scattering diagram.

Questions?

Section 2: Cluster algebras

(Principal coefficients) Cluster algebras

Start with an initial seed consisting of initial cluster variables $x_1, \ldots x_n$ and a skew-symmetric integer matrix B.

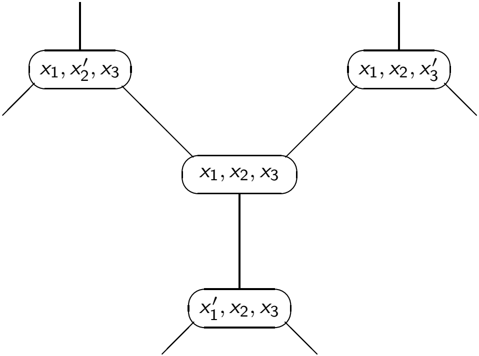
Mutation: an operation that takes a seed and gives a new seed.

- There are n "directions" for mutation.
- Mutation does two things:
  - switches out one cluster variable, replaces it with a new one;
  - changes B (and some extra rows) by matrix mutation.

  The result is a new seed.
- Mutation is involutive.

(Principal coefficients) Cluster algebras (continued)

Do all possible sequences of mutations, and collect all the cluster variables which appear.



The cluster algebra for the given initial seed is the subalgebra of $\mathcal{F}$ generated by all cluster variables.

Mutation

Write $[a]_+$ for $\max(a,0)$. The mutation of B in direction k is the matrix $B' = \mu_k(B)$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\}; \\ b_{ij} + \operatorname{sgn}(b_{kj})[b_{ik}b_{kj}]_+ & \text{otherwise.} \end{cases}$$

For principal coefficients, we replace B by $\left[\begin{smallmatrix} B \\ I \end{smallmatrix}\right]$ but we only mutate in directions $1,\ldots,n$.

We also introduce coefficients $y_1,\ldots,y_n$.

Mutating the cluster variables $x_1,\ldots,x_n$ in direction k means keeping x_i for $i \neq k$ and replacing x_k by x'_k according to the exchange relations

$$x_k x'_k = \prod_{i=1}^{n} x_i^{[b_{ik}]_+} y_i^{[b_{(n+i)k}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} y_i^{[-b_{(n+i)k}]_+}.$$

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Let's try it. . .

Mutation example

$$
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\xleftrightarrow{\mu_1}
\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}
\xleftrightarrow{\mu_2}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}
$$

$$\updownarrow \mu_2$$

$$
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$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\mu_1}{\longleftrightarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \overset{\mu_2}{\longleftrightarrow} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[x_1 \quad x_2]$$

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$$
\begin{array}{ccccc}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \overset{\mu_1}{\longleftrightarrow} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} & \overset{\mu_2}{\longleftrightarrow} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \\
\begin{bmatrix} x_1 & x_2 \end{bmatrix} & & \begin{bmatrix} \frac{x_2+y_1}{x_1} & x_2 \end{bmatrix} & & \\
\updownarrow \mu_2 & & & & \\
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$$\frac{y_1y_2+\frac{x_2+y_1}{x_1}}{x_2}$$

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$$\begin{bmatrix} x_1 & \frac{1+x_1y_2}{x_2} \end{bmatrix} \qquad \begin{bmatrix} \frac{x_1y_1y_2+x_2+y_1}{x_1x_2} & \frac{1+x_1y_2}{x_2} \end{bmatrix} \qquad \begin{bmatrix} \frac{x_1y_1y_2+x_2+y_1}{x_1x_2} & \frac{x_2+y_1}{x_1} \end{bmatrix}$$

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$$\updownarrow \mu_2$$

identify these

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The cluster variables

Continuing with $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, extended to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$:

We found that the cluster variables are

$$x_1, \quad x_2, \quad \frac{x_2 + y_1}{x_1}, \quad \frac{x_1 y_1 y_2 + x_2 + y_1}{x_1 x_2}, \quad \frac{1 + x_1 y_2}{x_2}$$

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$$x_1, \quad x_2, \quad \frac{x_2 + y_1}{x_1}, \quad \frac{x_1 y_1 y_2 + x_2 + y_1}{x_1 x_2}, \quad \frac{1 + x_1 y_2}{x_2}$$

Make a change of variables:

Set

The cluster variables

Continuing with $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, extended to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$:

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Make a change of variables:

Set $\hat{y}_1 = y_1 x_2^{-1}$

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Make a change of variables:

Set $\hat{y}_1 = y_1 x_2^{-1}$ and $\hat{y}_2 = y_2 x_1$.

The cluster variables

Continuing with $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, extended to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$:

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Make a change of variables:

Set $\hat{y}_1 = y_1 x_2^{-1}$ and $\hat{y}_2 = y_2 x_1$. The cluster variables become:

$$x_1, \;\; x_2, \;\; x_1^{-1} x_2 (1 + \hat{y}_1), \;\; x_1^{-1}(1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2), \;\; x_2^{-1}(1 + \hat{y}_2)$$

The cluster variables

Continuing with $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, extended to $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$:

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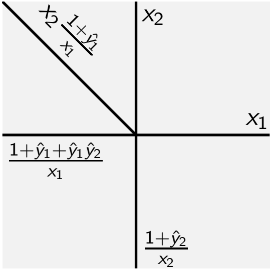
One of these should look familiar. It was the path-ordered product applied to x^{-1} in the cluster scattering diagram for $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Cluster algebras and scattering diagrams

Theorem (Fomin-Zelevinsky, 2007). Each cluster variable x is a Laurent monomial $x_1^{g_1} \cdots x_n^{g_n}$ times a polynomial in the $\hat{y}_i$. (The vector $\mathbf{g}(x) = (g_1, \ldots, g_n)$ is the **g**-vector of x.)



Theorem (GHKK, 2014). Part of the cluster scattering diagram cuts out the cluster fan, whose rays are spanned by the **g**-vectors of cluster variables and whose cones are spanned by the **g**-vectors of clusters. Scattering terms in these walls are $1 + \hat{y}^{\beta}$.

Theorem (morally GHKK, 2014, but R. had fun noticing it for himself, 2017). Let x be a cluster variable, let C be a cone in the cluster fan having a ray spanned by $\mathbf{g}(x)$, and let γ be a path from the interior of C to the interior of the positive cone. Then

$x = \mathfrak{p}_\gamma(x_1^{g_1} \cdots x_n^{g_n})$ (path-ordered product).

Recap of Section 2: Cluster algebras

Cluster algebras: Starting with B, a (very multi-directional) recurrence produces a lot of rational functions in the x_i and y_i. These generate the cluster algebra.

A global change of variables turns each cluster variable x into

$$x_1^{g_1} \cdots x_n^{g_n} \cdot (\text{a polynomial in the } \hat{y}_i).$$

The vector $\mathbf{g}(x) = (g_1, \ldots, g_n)$ is the $\mathbf{g}$-vector of x.

All of this fits into the cluster scattering diagram picture. Specifically:

Each cluster variable x is a path-ordered product for a path starting near $\mathbf{g}(x)$ and ending in the positive cone, applied to $x_1^{g_1} \cdots x_n^{g_n}$.

Questions?

Section 3: Rank-2 affine scattering diagrams

Rank-2 cluster scattering diagrams

$B = \left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right]$.

In every case, we know all the walls in the cluster fan using cluster-algebra ($\mathbf{g}$-vector) recurrences. Scattering terms for those walls are $1 + \hat{y}^{\beta}$.

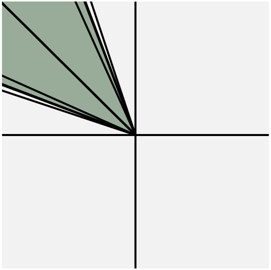
Finite case ($a < 2$): Well-understood. There are finitely many walls, all in the cluster fan.

Wild case ($a > 2$): Not well-understood. There is a region where we know essentially no scattering terms.

Affine case ($a = 2$): In between. Scattering term on one "limiting" wall is not obvious.

“Wild” rank-2 cluster scattering diagrams

$B = \left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right]$, $a > 2$.



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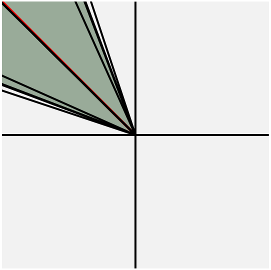
$B = \left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right]$, $a > 2$.



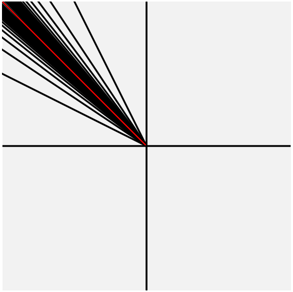
There is a formula for the scattering term on the red line (Reineke, 2011). No formulas are known for the other scattering terms in the gray region. As of [GHKK, 2014], it was not even known which rational rays have nontrivial scattering terms. [Bridgeland, 2017] says that all of them do.

Affine rank-2 cluster scattering diagrams

Scattering terms are $1 + \hat{y}^\beta$ except on the limiting ray.

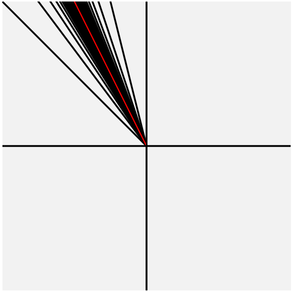
Theorem (Reineke, 2011 for $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, R. 2017 for $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$?).
The scattering term on the limiting walls are:

$$\frac{1}{(1-\hat{y}_1\hat{y}_2)^2} = 1 + 2\hat{y}_1^1\hat{y}_2^1 + 3\hat{y}_1^2\hat{y}_2^2 + \cdots$$



$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$$\frac{1+\hat{y}_1\hat{y}_2^2}{(1-\hat{y}_1\hat{y}_2^2)^2} = 1 + 3\hat{y}_1\hat{y}_2^2 + 5\hat{y}_1^2\hat{y}_2^4 + \cdots$$



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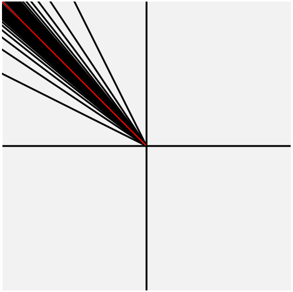

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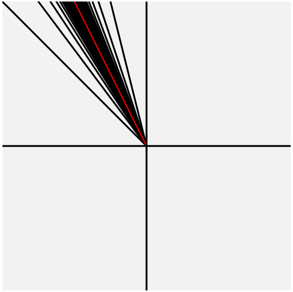
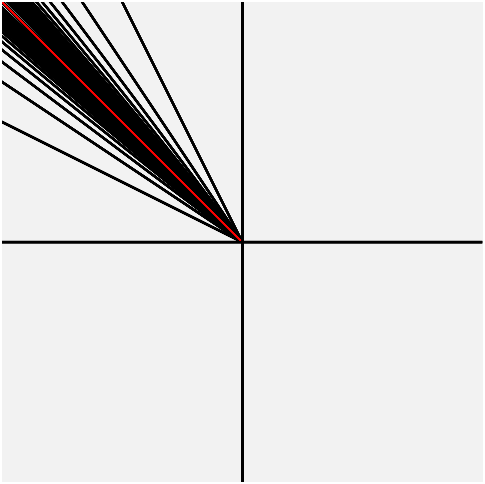


$\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

We'll spend some time on the proof of the case $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Why this is nontrivial (despite a simple answer)



The proof for $\left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$

We want to find the scattering term on the red ray. Call it f.

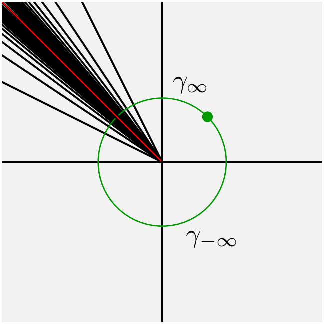
Let γ_∞ be a path starting (in the limit) from the red ray and going clockwise to the positive cone. Let $\gamma_{-\infty}$ be the same, but counterclockwise. Write $\mathfrak{p}_\infty$ for $\mathfrak{p}_{\gamma_\infty}$ and similarly $\mathfrak{p}_{-\infty}$ for $\mathfrak{p}_{\gamma_{-\infty}}$.

Crossing the red wall (moving Northeast) sends the monomial x_1x_2 to $x_1x_2f^{-2}$.

So consistency says:

$$\mathfrak{p}_{-\infty}(x_1x_2) = \mathfrak{p}_\infty(x_1x_2f^{-2}).$$

We will, in essence, solve this for f.



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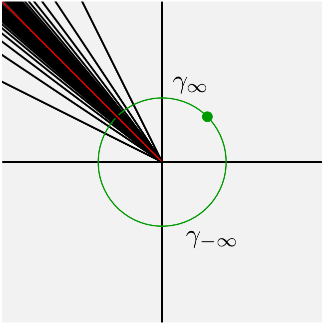

Key observations:

- all terms $\hat{y}_1^i\hat{y}_2^j$ of $x_1^{-1}x_2^{-1}\mathfrak{p}_\infty(x_1x_2f^{-2})$ have $i \geq j$.
- The diagonal terms (terms with $i = j$) are exactly f^{-2}.

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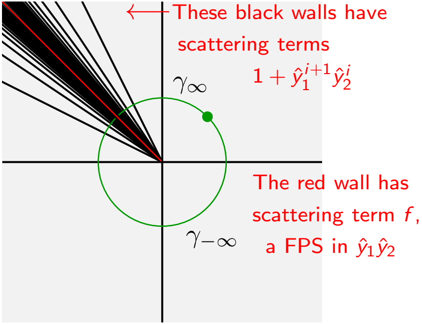

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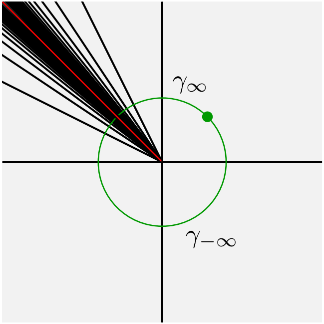
The proof for $\left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$ (continued)

The diagonal terms of $x_1^{-1}x_2^{-1}\mathfrak{p}_\infty(x_1x_2f^{-2})$ are exactly f^{-2}.

So the diagonal terms of $x_1^{-1}x_2^{-1}\mathfrak{p}_{-\infty}(x_1x_2)$ are exactly f^{-2}.

That is, $f =$ diagonal terms of $\sqrt{\frac{x_1x_2}{\mathfrak{p}_{-\infty}(x_1x_2)}}$



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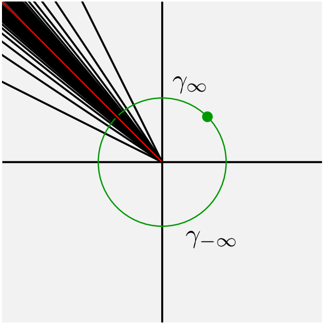

What is $\sqrt{\frac{x_1x_2}{\mathfrak{p}_{-\infty}(x_1x_2)}}$?

The proof for $\left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$ (continued)

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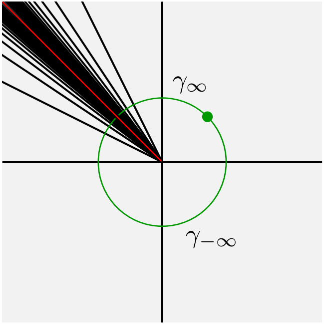
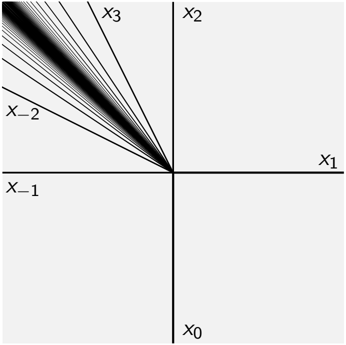


What is $\sqrt{\frac{x_1x_2}{\mathfrak{p}_{-\infty}(x_1x_2)}}$? A limit involving cluster variables!

How to understand $\sqrt{\frac{x_1 x_2}{\mathfrak{p}_{-\infty}(x_1 x_2)}}$

- Number the cluster variables.
- For $i \leq -1$, x_i is $x_1^i x_2^{-i-1} \cdot F_i$.



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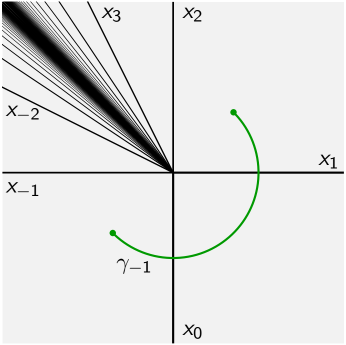

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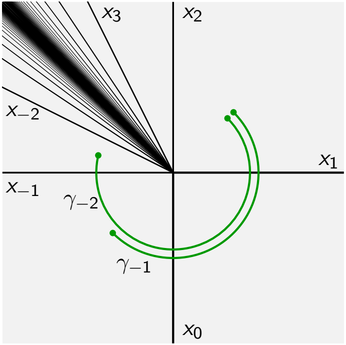

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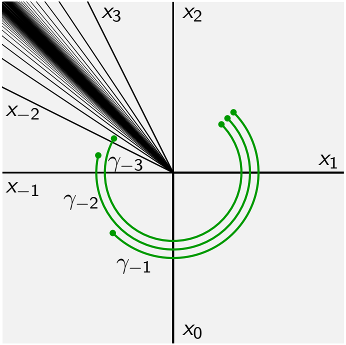

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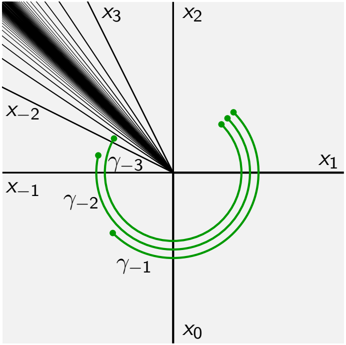
- Define paths γ_i for $i \leq -1$.
- Recall the connection between cluster variables and path-ordered products:

$$\begin{aligned}\mathfrak{p}_{\gamma_i}(x_1^i x_2^{-i-1}) &= x_1^i x_2^{-i-1} \cdot F_i \\ \mathfrak{p}_{\gamma_i}(x_1^{i+1} x_2^{-i-2}) &= x_1^{i+1} x_2^{-i-2} \cdot F_{i+1}\end{aligned}$$



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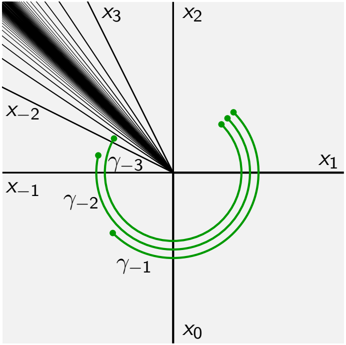
- Conclude:

$$\sqrt{\frac{x_1 x_2}{\mathfrak{p}_{-\infty}(x_1 x_2)}} = \sqrt{\lim_{i \to \infty} x_1 x_2 \cdot F_i^{2i+3} \cdot F_{i+1}^{-2i-1}}$$



Finally, we find the scattering term

We want the diagonal terms of $\sqrt{\lim_{i\to\infty} x_1 x_2 \cdot F_i^{2i+3} \cdot F_{i+1}^{-2i-1}}$, where each cluster variable x_i is $x_1^i x_2^{-i-1} \cdot F_i$.

Using a simple recursion for the F_i (coming from the exchange relations that define cluster variables), we prove just enough about the F_i. For example, the terms of F_{-4} are:

$$
\begin{array}{lllll}
1 & & & & \\
+3\hat{y}_1 & +2\hat{y}_1\hat{y}_2 & & & \\
+3\hat{y}_1^2 & +6\hat{y}_1^2\hat{y}_2 & +3\hat{y}_1^2\hat{y}_2^2 & & \\
+\hat{y}_1^3 & +4\hat{y}_1^3\hat{y}_2 & +6\hat{y}_1^3\hat{y}_2^2 & +4\hat{y}_1^3\hat{y}_2^3 & +\hat{y}_1^3\hat{y}_2^4
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\end{array}
$$

The diagonal terms of $x_1x_2 \cdot F_i^{2i+3} \cdot F_{i+1}^{-2i-1}$ limit to

$$(1 + 2\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2^2 + 4\hat{y}_1^3\hat{y}_2^3 + \cdots)^2 = (1 - \hat{y}_1\hat{y}_2)^{-4}$$

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$$(1 + 2\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2^2 + 4\hat{y}_1^3\hat{y}_2^3 + \cdots)^2 = (1 - \hat{y}_1\hat{y}_2)^{-4}$$

$$f = (1 - \hat{y}_1\hat{y}_2)^{-2}$$

Recap of Section 3: Rank-2 affine scattering diagrams

Rank-2 scattering diagrams are not understood in the "wild" case.

In the rank-2 affine case, one piece is not easy: the scattering term on the limiting ray. This term can be found using the connection to cluster algebras:

- Use consistency to reduce this to finding the diagonal terms in a path-ordered product evaluation.
- Rewrite the path-ordered product evaluation as a limit involving cluster variables (F-polynomials).
- Use the recursion on F-polynomials to find the diagonal terms of the limit.

Questions?

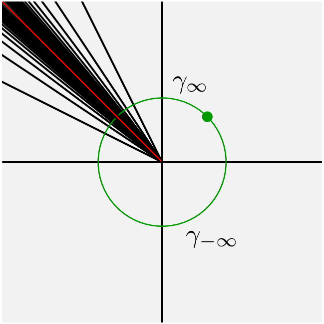
Section 4: Narayana numbers in affine rank 2

Something like a cluster variable

Recall that if x is a cluster variable with $\mathbf{g}(x) = (g_1, \ldots, g_n)$, then

$$x = \mathfrak{p}_\gamma(x_1^{g_1} \cdots x_n^{g_n}),$$

for any path γ starting near $\mathbf{g}(x)$ and ending in the positive cone.



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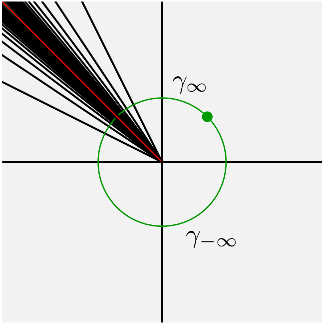
For $B = \left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$, there is a cluster variable for every ray except the limiting ray. What if we take the $\mathbf{g}$-vector of the limiting ray and a path starting near the limiting ray? In other words:



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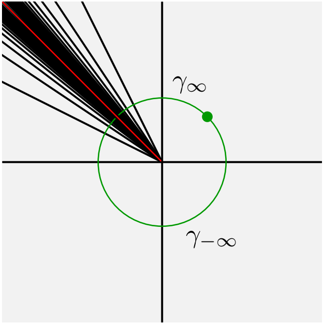
What is $\mathfrak{p}_\infty(x_1^{-1}x_2)$?



Narayana numbers (a refinement of the Catalan numbers)

By definition, $\mathfrak{p}_\infty(x_1^{-1}x_2) = x_1^{-1}x_2 \cdot \mathcal{N}(\hat{y}_1, \hat{y}_2)$ for some FPS $\mathcal{N}$.

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Theorem (R., 2017)

$$\mathcal{N}(\hat{y}_1, \hat{y}_2) = 1 + \hat{y}_1 \sum_{i,j\geq 0}(-1)^{i+j}\,\mathrm{Nar}(i,j)\hat{y}^i\hat{y}^j,$$

where the $\mathrm{Nar}(i,j)$ are the Narayana numbers:

$$\mathrm{Nar}(i,j) = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } ij = 0 \text{ otherwise, or} \\ \frac{1}{i}\binom{i}{j}\binom{i}{j-1} & \text{if } i,j \geq 1. \end{cases}$$

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$$\begin{array}{lrrrrrrr}
1 \\
+\hat{y}_1 \\
 & +\hat{y}_1^2\hat{y}_2 \\
 & -\hat{y}_1^3\hat{y}_2 & +\hat{y}_1^3\hat{y}_2^2 \\
 & +\hat{y}_1^4\hat{y}_2 & -3\hat{y}_1^4\hat{y}_2^2 & +\hat{y}_1^4\hat{y}_2^3 \\
 & -\hat{y}_1^5\hat{y}_2 & +6\hat{y}_1^5\hat{y}_2^2 & -6\hat{y}_1^5\hat{y}_2^3 & +\hat{y}_1^5\hat{y}_2^4 \\
 & +\hat{y}_1^6\hat{y}_2 & -10\hat{y}_1^6\hat{y}_2^2 & +20\hat{y}_1^6\hat{y}_2^3 & -10\hat{y}_1^6\hat{y}_2^4 & +\hat{y}_1^6\hat{y}_2^5 \\
 & -\hat{y}_1^7\hat{y}_2 & +15\hat{y}_1^7\hat{y}_2^2 & -50\hat{y}_1^7\hat{y}_2^3 & +50\hat{y}_1^7\hat{y}_2^4 & -15\hat{y}_1^7\hat{y}_2^5 & +\hat{y}_1^7\hat{y}_2^6 \\
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 & +\cdots
\end{array}$$

In fact, it's even nicer

Theorem (R., 2017)

$$\mathcal{N}(\hat{y}_1, \hat{y}_2) = \lim_{i \to -\infty} \frac{F_i}{F_{i+1}} = \lim_{i \to \infty} \frac{F_{i+1}}{F_i}$$

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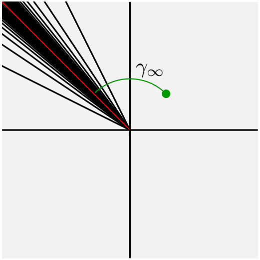

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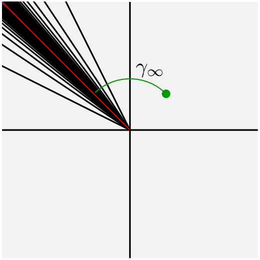

Proof ideas:

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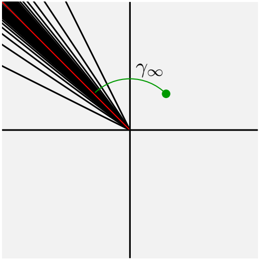

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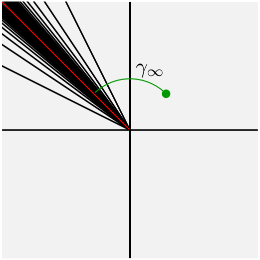

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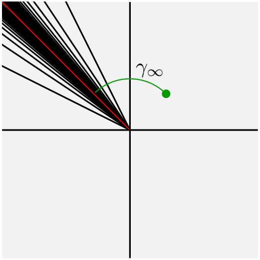

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- Use the expression for $\mathcal{N}$ as a limit of ratios of cluster variables to establish a functional equation:

$$(1 + \hat{y}_2) \cdot \mathcal{N}(\hat{y}_1(1 + \hat{y}_2)^{-2}, \hat{y}_2) = (1 + \hat{y}_1) \cdot \mathcal{N}(\hat{y}_2(1 + \hat{y}_1)^{-2}, \hat{y}_1).$$

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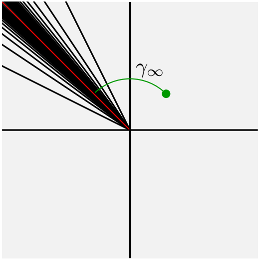

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- Verify, using the observation, that the given formula satisfies the functional equation.

Some details I won't have time for

To prove the functional equation:

- Define $\tilde{x}_i = x_{1-i}$ for all i.
- Know: $x_1^{-1}x_2 \cdot \mathcal{N}(\hat{y}_1, \hat{y}_2) = \lim_{i\to\infty} \frac{x_{i+1}}{x_i} = \lim_{i\to-\infty} \frac{\tilde{x}_i}{\tilde{x}_{i+1}}$
- RHS is basically $\tilde{x}_1^{-1}\tilde{x}_2 \cdot \mathcal{N}(\hat{y}_1, \hat{y}_2)$, but there is a technical issue (coefficients are not principal at the seed $\{\tilde{x}_1, \tilde{x}_2\}$). But we set $y_1 = y_2 = 1$ and remember $\hat{y}_1 = y_1 x_2^{-2}$ and $\hat{y}_2 = y_2 x_1^2$ and use exchange relations to write:
$$\tilde{x}_1 = x_0 = \frac{1+x_1^2}{x_2} \qquad \text{and} \qquad \tilde{x}_2 = x_{-1} = \frac{1+(1+x_1^2)^2 x_2^{-2}}{x_1}.$$
- Substitute:
$$x_1^{-1}x_2 \cdot \mathcal{N}(x_2^{-2}, x_1^2) = \mathcal{N}\left(\frac{x_1^2}{(1+(1+x_1^2)^2 x_2^{-2})^2}, \frac{(1+x_1^2)^2}{x_2^2}\right) \cdot \frac{x_2}{1+x_1^2} \cdot \frac{1+(1+x_1^2)^2 x_2^{-2}}{x_1}$$
- Replacing x_2^{-2} by $\hat{y}_1$ and x_1^2 by $\hat{y}_2$ and rearranging, we get
$$(1+\hat{y}_2) \cdot \mathcal{N}(\hat{y}_1(1+\hat{y}_2)^{-2}, \hat{y}_2) = (1+\hat{y}_1) \cdot \mathcal{N}(\hat{y}_2(1+\hat{y}_1)^{-2}, \hat{y}_1).$$

Some more details I won't have time for

To complete the proof:

- We can extract coefficients from the functional equation

$$(1+\hat{y}_2)\cdot\mathcal{N}(\hat{y}_1(1+\hat{y}_2)^{-2},\hat{y}_2) = (1+\hat{y}_1)\cdot\mathcal{N}(\hat{y}_2(1+\hat{y}_1)^{-2},\hat{y}_1).$$

  to get a relationship among the coefficients of $\mathcal{N}(\hat{y}_1,\hat{y}_2)$.
- This is not enough information to determine the coefficients uniquely.
- But, with the observation that

$$\mathcal{N}(\hat{y}_1,\hat{y}_2) = 1 + \text{terms } \hat{y}_1^i\hat{y}_2^j \text{ with } j < i,$$

  we have enough information to prove the theorem.
- The verification uses the Saalschutz ${}_3F_2$ evaluation.

Recap of Section 4: Narayana numbers in affine rank 2

Using the connection to cluster algebras, we find the Narayana numbers hiding in the scattering diagram for $B = \left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$.

Questions?

Thank you for listening.

Scattering diagrams and scattering fans. arXiv:1712.06968