Cambrian lattices and generalized associahedra

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Cambrian Lattices

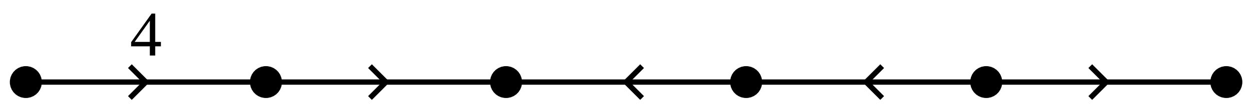
For any finite Coxeter group W, we define a family of *Cambrian lattices*.

- Defined as certain lattice quotients of the weak order.

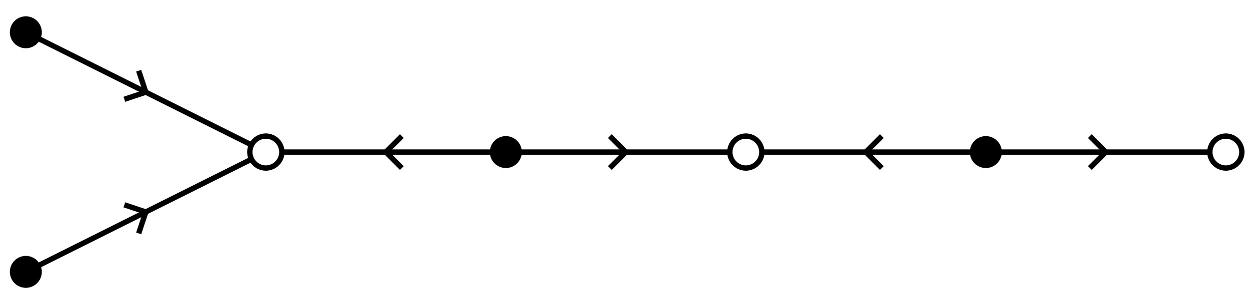
- Each orientation of the diagram for W gives a Cambrian lattice and a related complete *Cambrian fan*.

Orientations

An orientation of the B_6 diagram.



A *bipartite* orientation of the D_7 diagram. The white vertices are sinks and the black vertices are sources.



Conjectures

(Theorems for A_n and B_n):

- When W is crystallographic, each Cambrian fan is combinatorially isomorphic to the normal fan of Fomin and Zelevinsky's *generalized associahedron* for W.

- When the orientation is bipartite:

  The Cambrian fan is linearly isomorphic to this normal fan.

  The Cambrian lattice is isomorphic to a certain natural partial order on the *clusters* (the vertices of the generalized associahedron).

Proofs in types A and B follow from a(n) (equivariant) fiber-polytope construction.

Other results

- One of the type-A Cambrian lattices is the Tamari lattice.

- We identify two Cambrian lattices for B_n as "type-B Tamari lattices" and characterize them by signed pattern avoidance.

- Intervals in Cambrian lattices are either contractible or homotopy-equivalent to spheres.

- In types A and B, and conjecturally in general, Cambrian lattices are sublattices of weak order. (Tamari lattice case: Björner and Wachs).

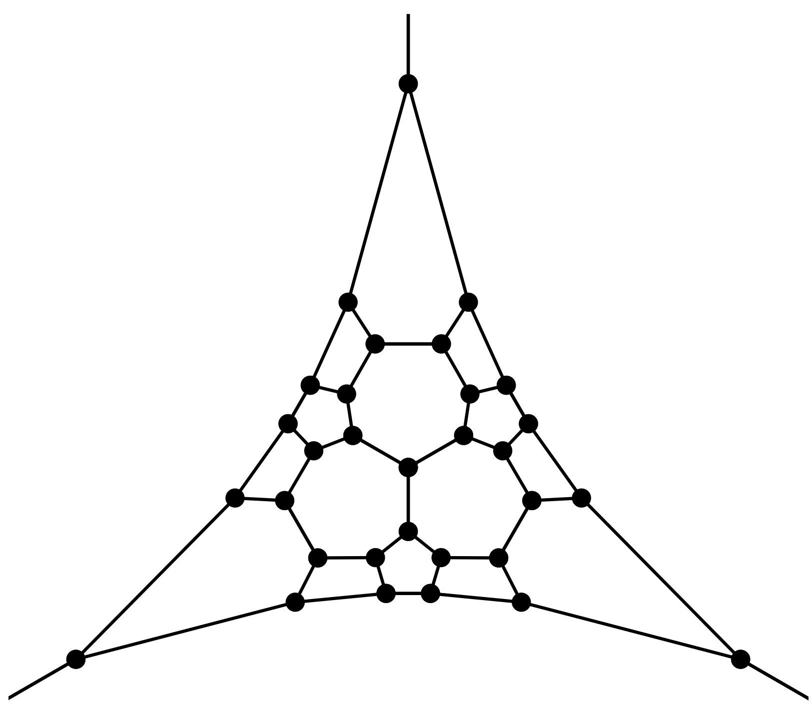
Non-crystallographic types

- Cambrian fans suggest a definition of generalized associahedra applicable to all (not necessarily crystallographic) types.

- The $I_2(m)$-associahedron is an $(m+2)$-gon.

  The 1-skeleton of the H_3-associahedron is shown below (one vertex is at ∞).

  These have the f-vectors one would expect.



Associahedra

- **Catalan numbers:**

  $C_{n+1} = \#\{\text{triangulations of an } (n+3)\text{-gon}\}$

- **Associahedron:** a simple n-polytope.

  Vertices: triangulations of an $(n+3)$-gon.

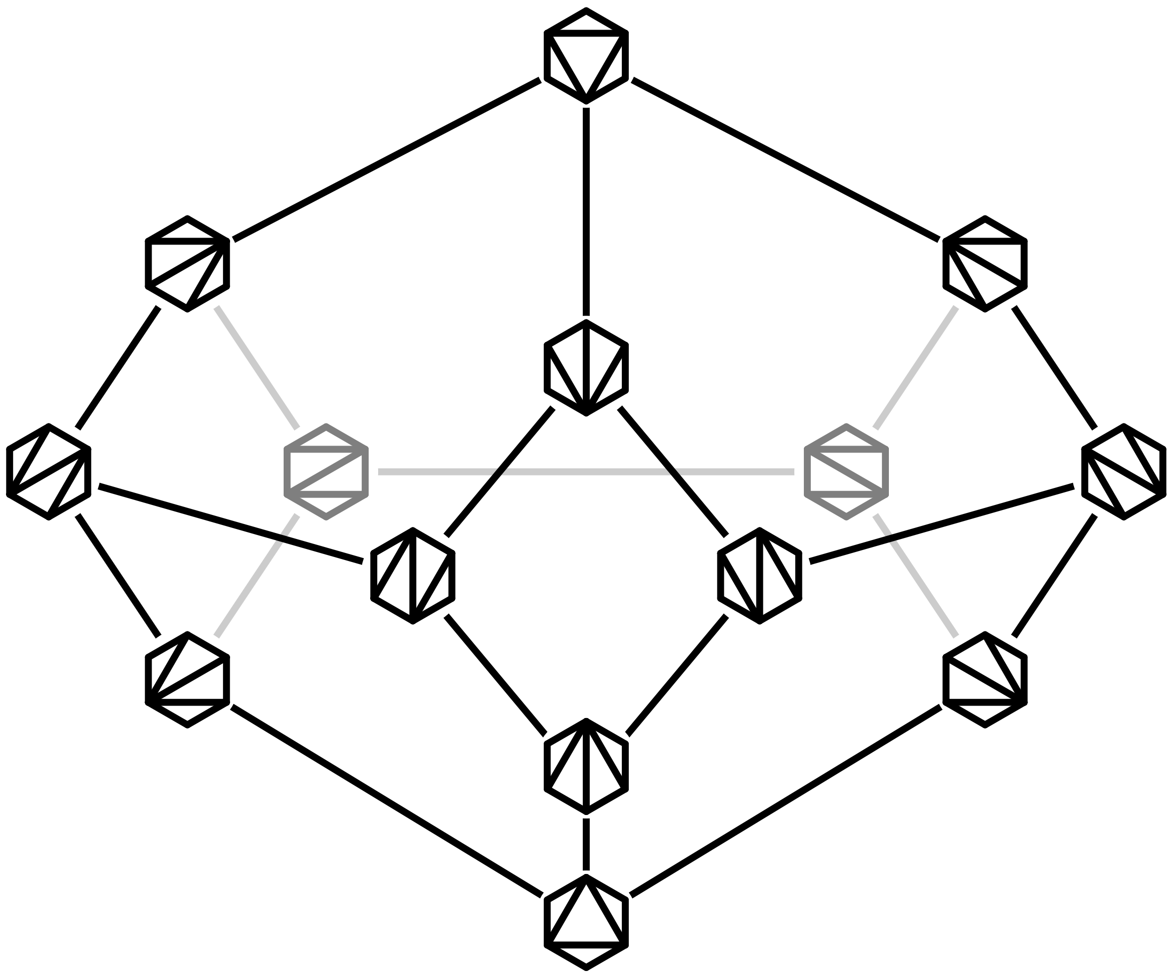
  Edges: "diagonal flips."

- **Tamari lattice:** a partial order on triangulations.

  Hasse diagram isomorphic to the 1-skeleton of the associahedron.

  Isomorphic to weak order restricted to 312-avoiding permutations. (Björner and Wachs).

The 3-dimensional associahedron



Coxeter groups

- **The symmetric group S_n:**

  Generators: $s_i := (i\ \ i{+}1)$.

  Relations: $s_i^2 = 1$, $(s_i s_{i+1})^3 = 1$ and
  $(s_i s_j)^2 = 1$ for $|i - j| > 1$.

- **Coxeter group** W:

  Generators: a set S of involutions called *simple generators*.

  Relations: $(st)^m = 1$ for $s, t \in S$.

Coxeter groups (continued)

- **Coxeter diagram** on vertex set S.

  Edges: s — t for $(st)^m = 1$ and $m \geq 3$.

  Labeled with m if $m > 3$.

- **Classification** of finite Coxeter groups.

  Infinite families include $A_n = S_{n+1}$:

  and the group B_n of signed permutations (the "hyperoctahedral group"):

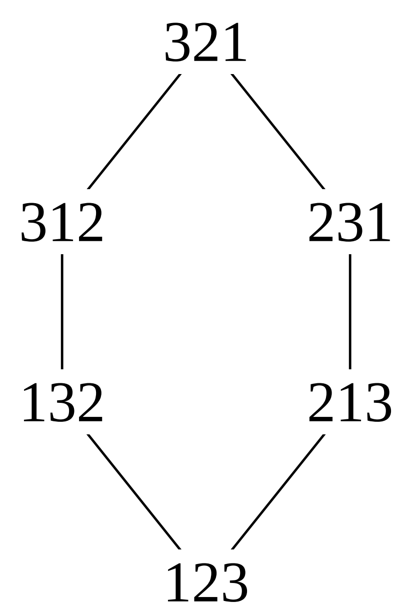
The (right) weak order

- **In general:** The covers are $w \lessdot ws$ for every $w \in W$ and $s \in S$ with $l(w) < l(ws)$ (using the usual length function).

- **In the symmetric group S_n:** Covers are transpositions of adjacent entries.

  Going "up" means putting the entries out of numerical order.

  The weak order on S_3:



The (usual) Catalan numbers are "type A."

- They are known to count, for $W = A_n$, various objects which can be defined for general (crystallographic) W, including

  antichains in the root poset,

  positive regions of the Shi arrangement,

  non-crossing partitions,

  conjugacy classes of elements of finite order in the associated compact Lie group.

- There is a well-known map from permutations to triangulations with nice properties.

Generalized Associahedra

- W-**Catalan numbers**

  A general-type formula involving the exponents and Coxeter number of W.

- B_n-**associahedron or cyclohedron** (Bott and Taubes, Simion). A simple n-polytope.

  Vertices: centrally symmetric triangulations of a $(2n+2)$-gon.

  Edges: diameter flips or symmetric pairs of diagonal flips.

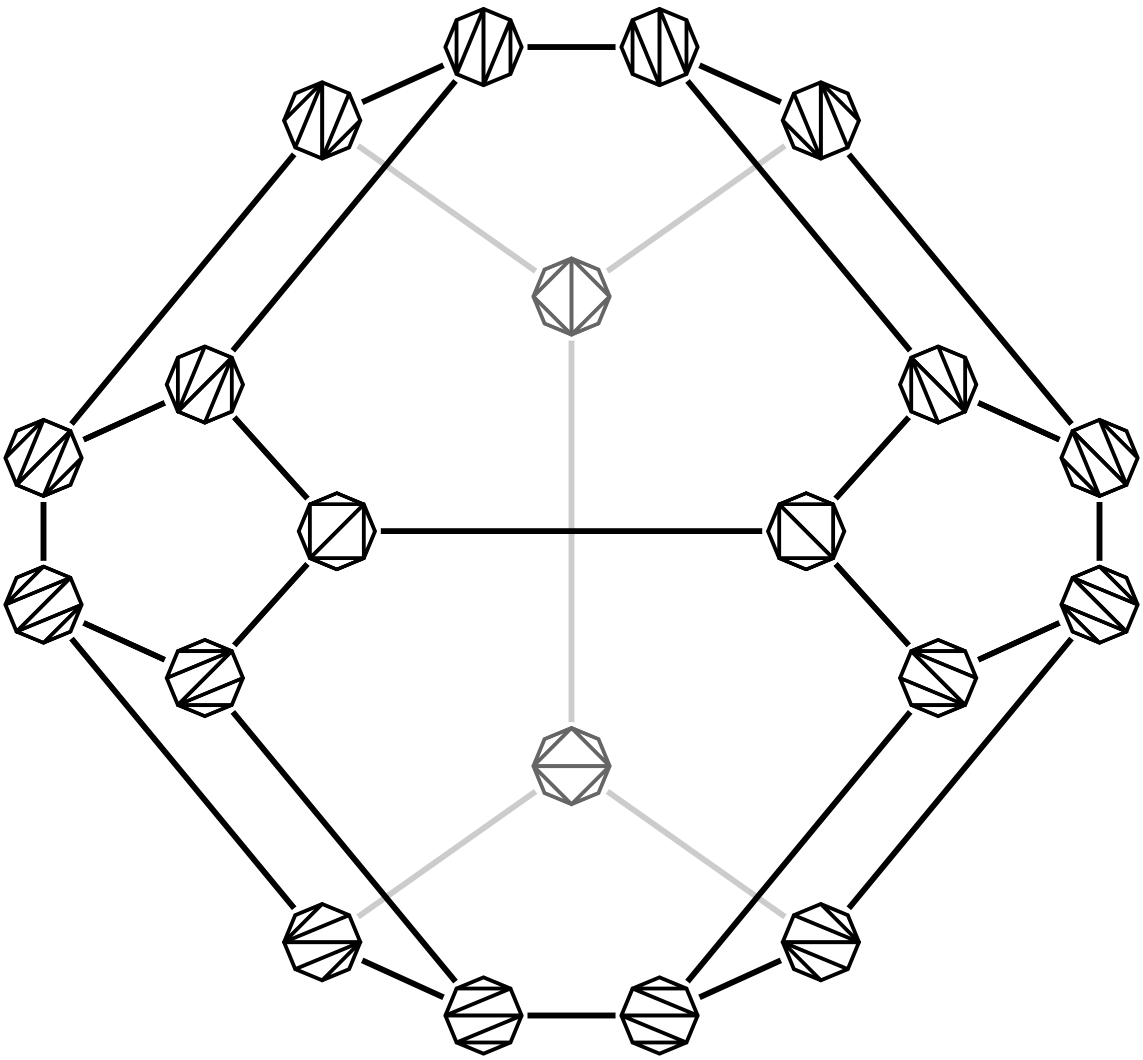
- W-**associahedron** (for crystallographic W)

  A simple polytope defined by Fomin and Zelevinsky (and Chapoton).

  Dimension: rank of W (i.e. $|S|$)

  Vertices: counted by W-Catalan numbers

The B_3-associahedron



Clusters

Fomin and Zelevinsky constructed complete simplicial fans and conjectured they were polytopal. Later, with Chapoton, they proved this conjecture.

- **Description:**

  Rays: positive roots and negative simple roots.

  Cones: sets of "compatible" roots.

  Maximal sets of compatible roots are called *clusters* (cf. cluster algebras).

- **Tools:**

  A bipartition of the diagram of W.

  Piecewise-linear maps τ_+ and τ_- which generate a finite dihedral group of combinatorial symmetries of the associahedron.

A type-B Tamari lattice?

Simion: asked for a "B_n Tamari lattice."

Reiner: defined maps from B_n to vertices of the B_n-associahedron, and asked if the maps define a partial order.

Our approach was guided by this observation:

The map from weak order on S_n to the Tamari lattice is a lattice homomorphism.

Surprisingly: Each of Reiner's maps is a lattice homomorphism and defines a "Tamari-like" lattice. A similar family of Tamari-like lattices exists in type A. This led to the general-type definition of Cambrian lattices.

(Hugh Thomas constructed a type-B Tamari lattice at about the same time, using a different approach.)

Lattice congruences

- **Definition:** equivalence relations respecting meet and join. (Analogous to congruences on rings).

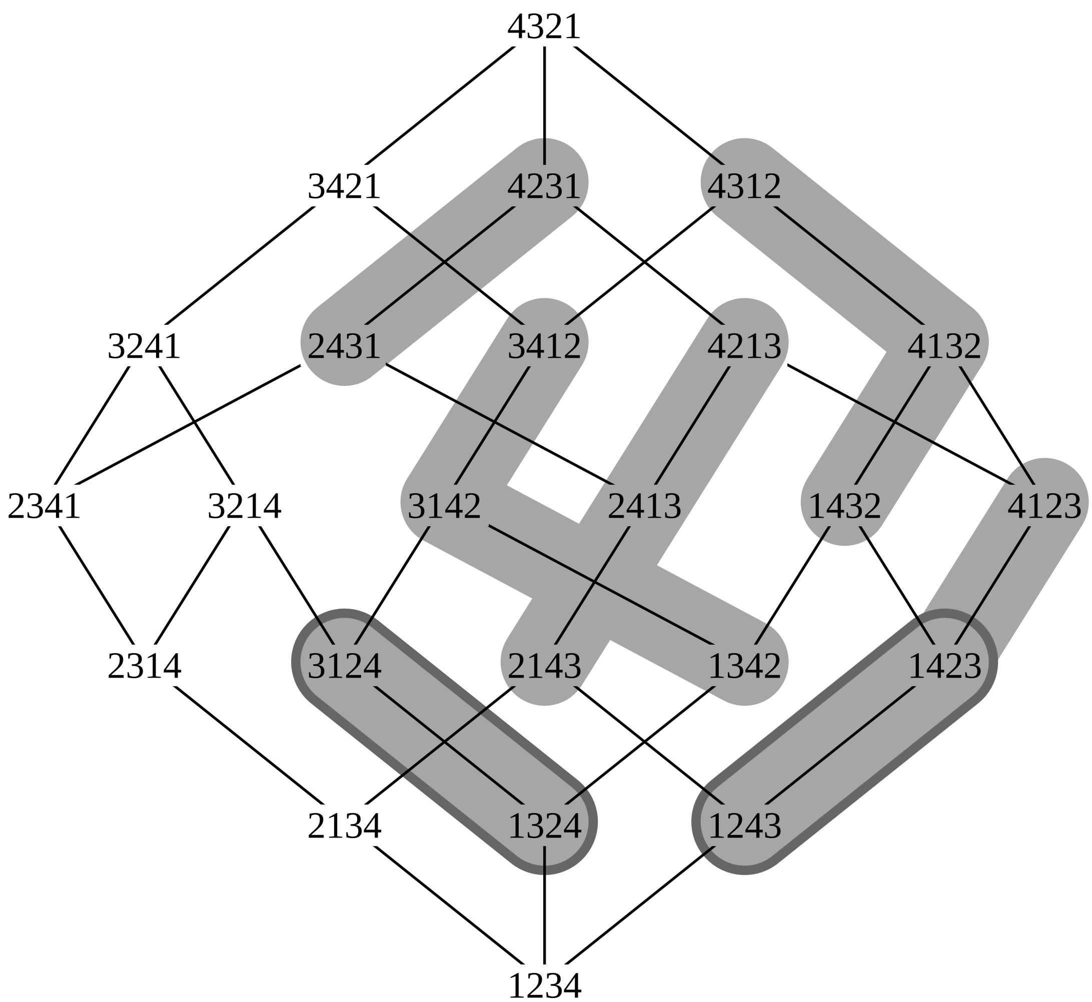
- They arise as the fibers of lattice homomorphisms.

- A congruence *contracts* an edge $x \lessdot y$ if $x \equiv y$.

- Given a collection of edges, there is a well-defined smallest lattice congruence contracting those edges.

Example

The smallest congruence of the weak order with $1324 \equiv 3124$ and $1243 \equiv 1423$.



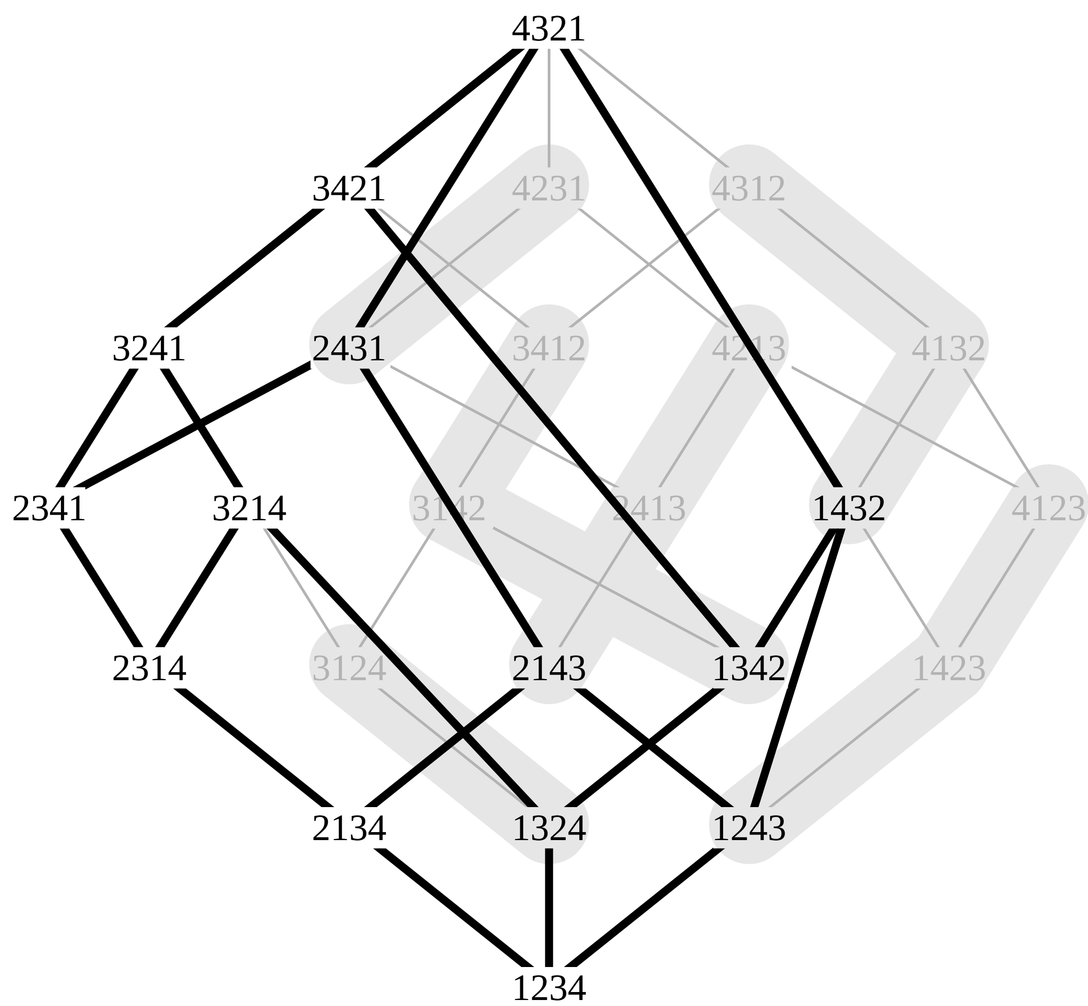
Congruence classes

- Congruence classes in a finite lattice are always intervals.

- The quotient mod the congruence is isomorphic to the subposet induced by the bottoms of intervals.

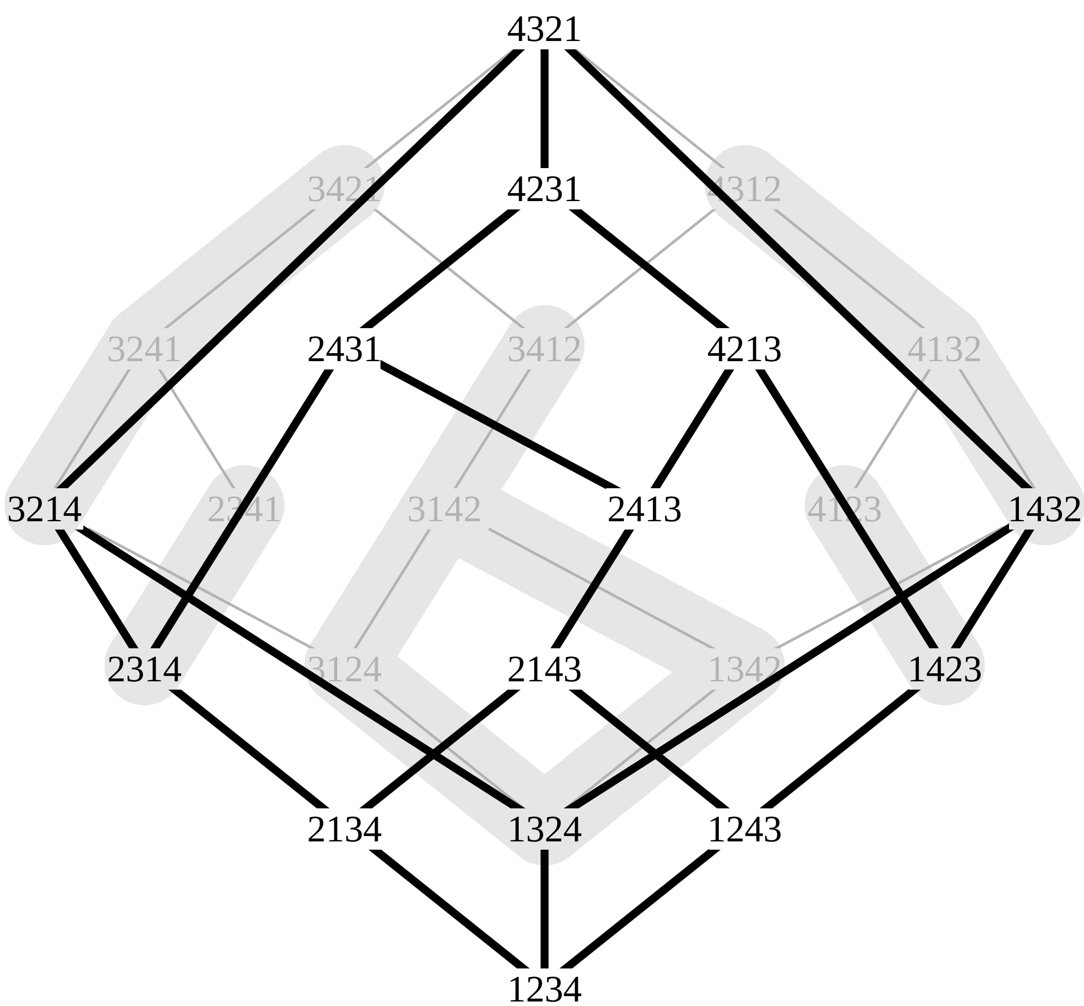
Example (continued)

In this example, the bottoms of congruence classes are exactly the 312-avoiding permutations. Thus the quotient is the Tamari lattice.



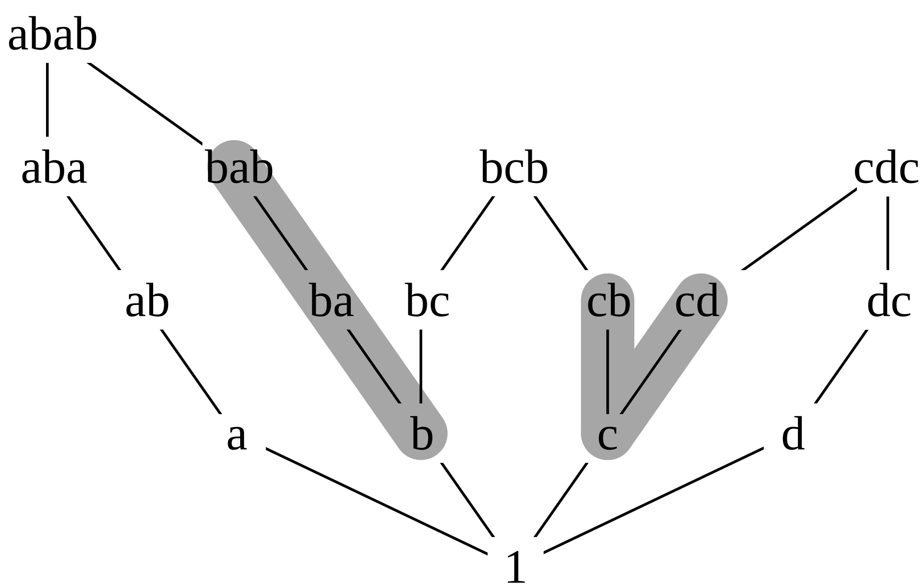
Example

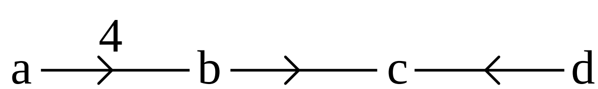
The quotient of the weak order mod the smallest congruence with $1324 \equiv 3124$ and $1324 \equiv 1342$. This is not the Tamari lattice, but its Hasse diagram is still the 1-skeleton of the associahedron. This is a Cambrian lattice.



Cambrian congruences

An orientation of the diagram for W specifies certain edges in the weak order which are to be contracted.



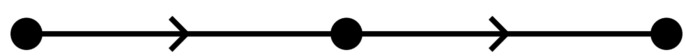


The *Cambrian congruence* for this orientation is the smallest congruence contracting the specified edges.

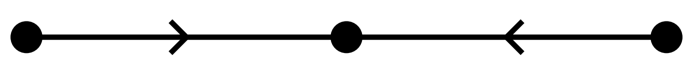
Cambrian lattices

- The Cambrian lattice for an orientation is the weak order on W mod the Cambrian congruence for the orientation.

- We have seen the Cambrian lattices for the orientations



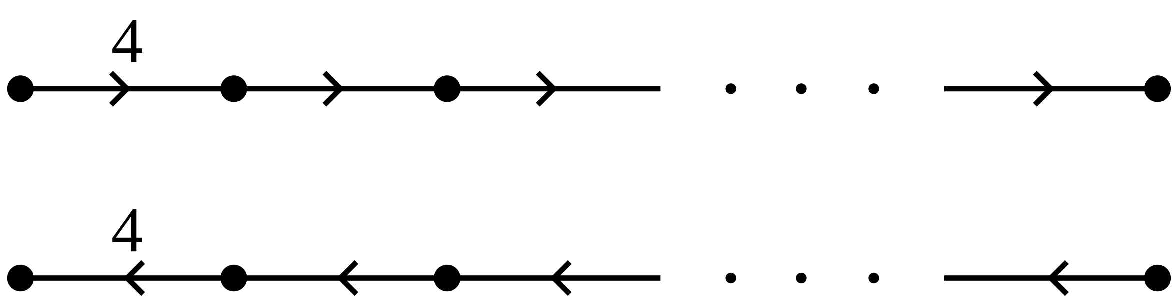
and



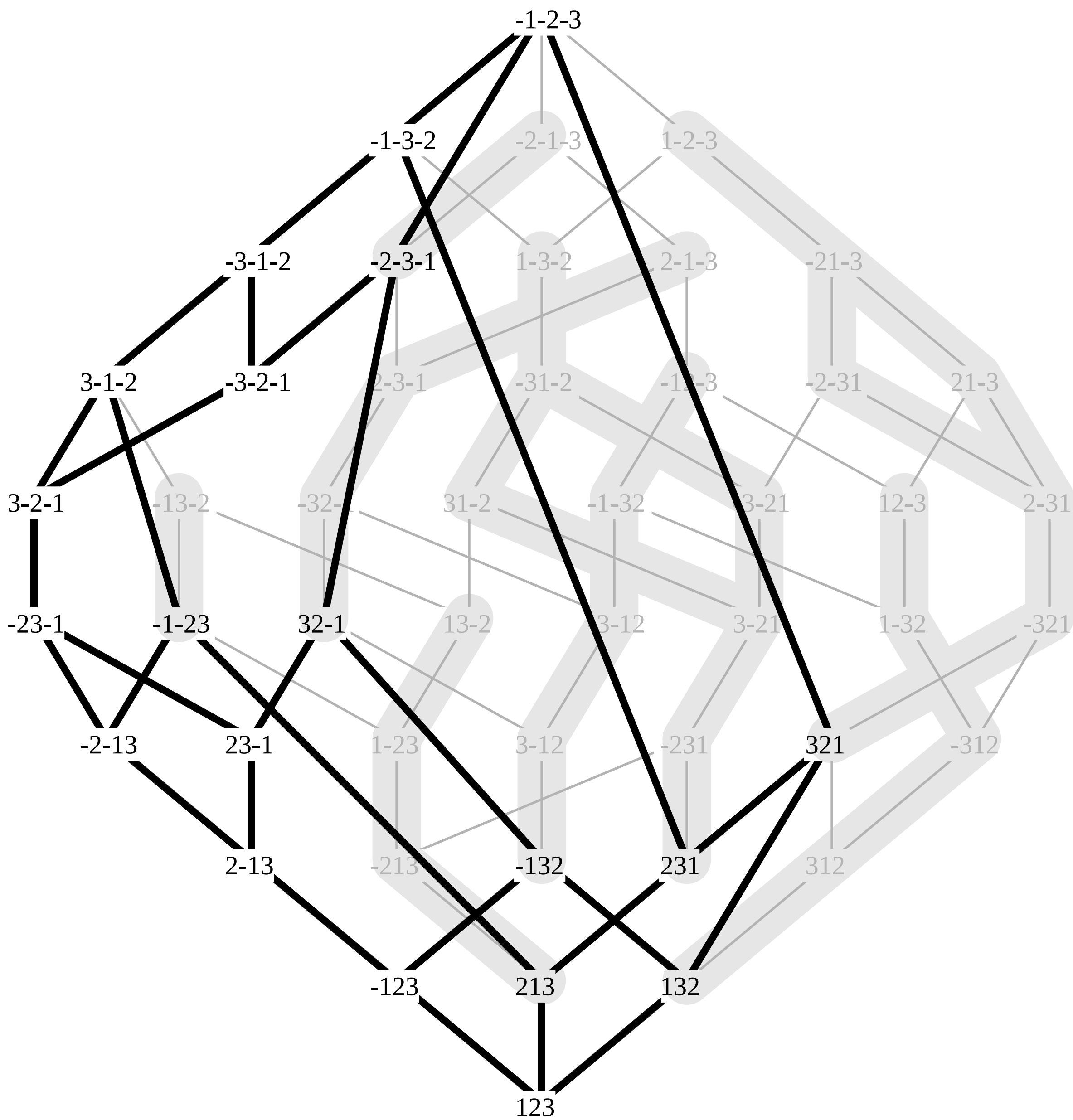
Tamari lattices

The (usual, type A) Tamari lattice is the Cambrian lattice for the orientation

In type B, there are two anti-isomorphic analogs. Each can be characterized by signed pattern avoidance.



The B_3 Tamari lattice

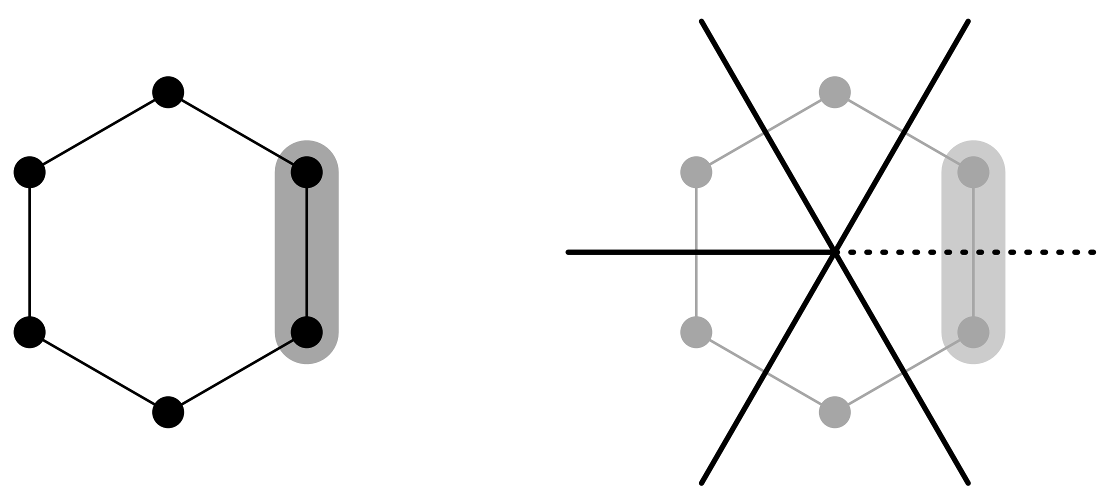


Cambrian fans

Any congruence of the weak order defines a complete fan.

One takes the complete fan arising from the Coxeter arrangement for W and glues together its maximal faces according to the congruence.

For example, in A_2:



The fan arising from the Cambrian congruence is the *Cambrian fan*.

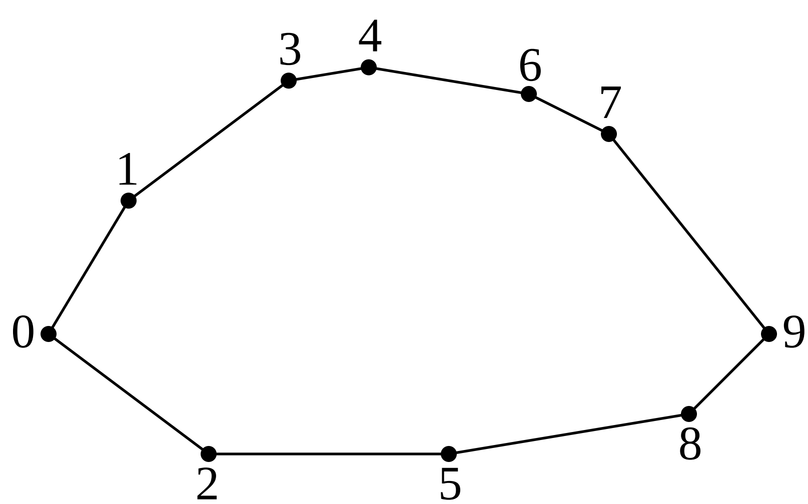
Type A_{n-1}

Define a convex polygon Q:

Vertices $0, 1, \ldots, n+1$, with 0 and $n+1$ on a horizontal line.

Make horizontal positions of the other vertices consistent with numerical order.

For example, with $n = 8$:



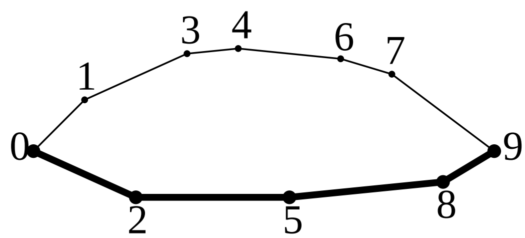
Permutations to Triangulations

From an iterated fiber polytope construction (Billera, Sturmfels). Given a permutation $\pi \in S_n$, define a triangulations of Q as follows:

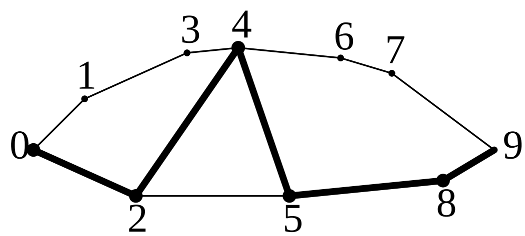
- Start with a path along the bottom of Q.
- Read a permutation π from left to right.
- For each symbol encountered, add/delete the corresponding vertex to/from the path.
- The union of the paths is a triangulation.

Example: $\pi = 42783165$

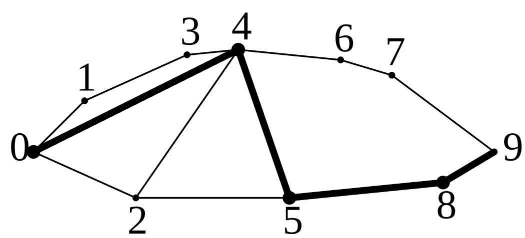
Start:



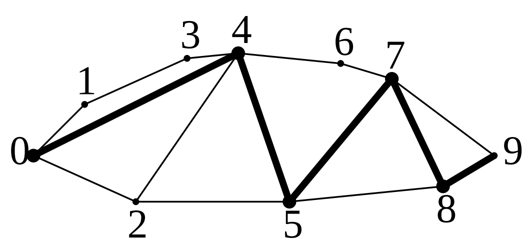
Add 4:



Delete 2:

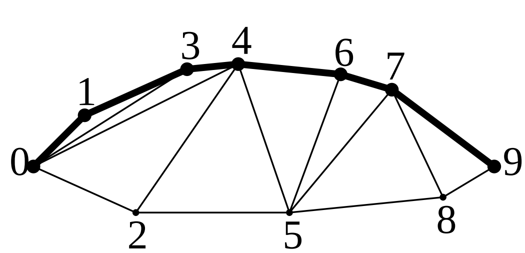


Add 7:



⋮

End:

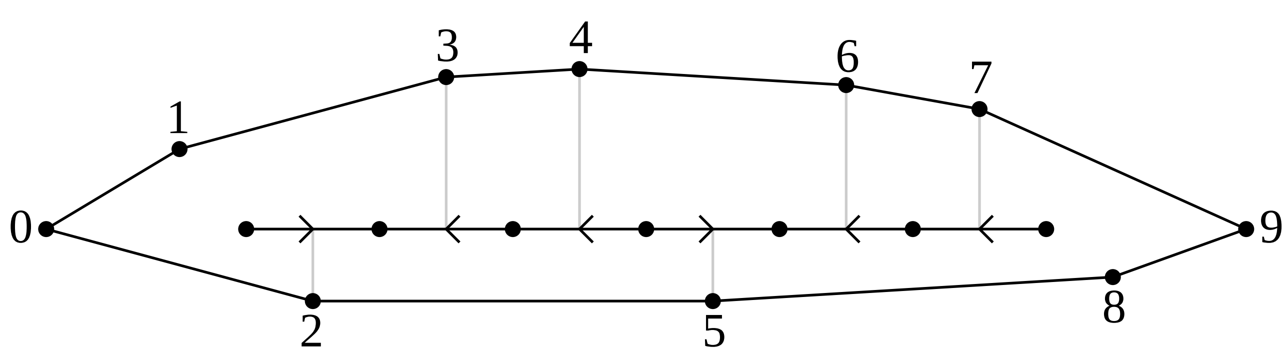


Orientations to polygons

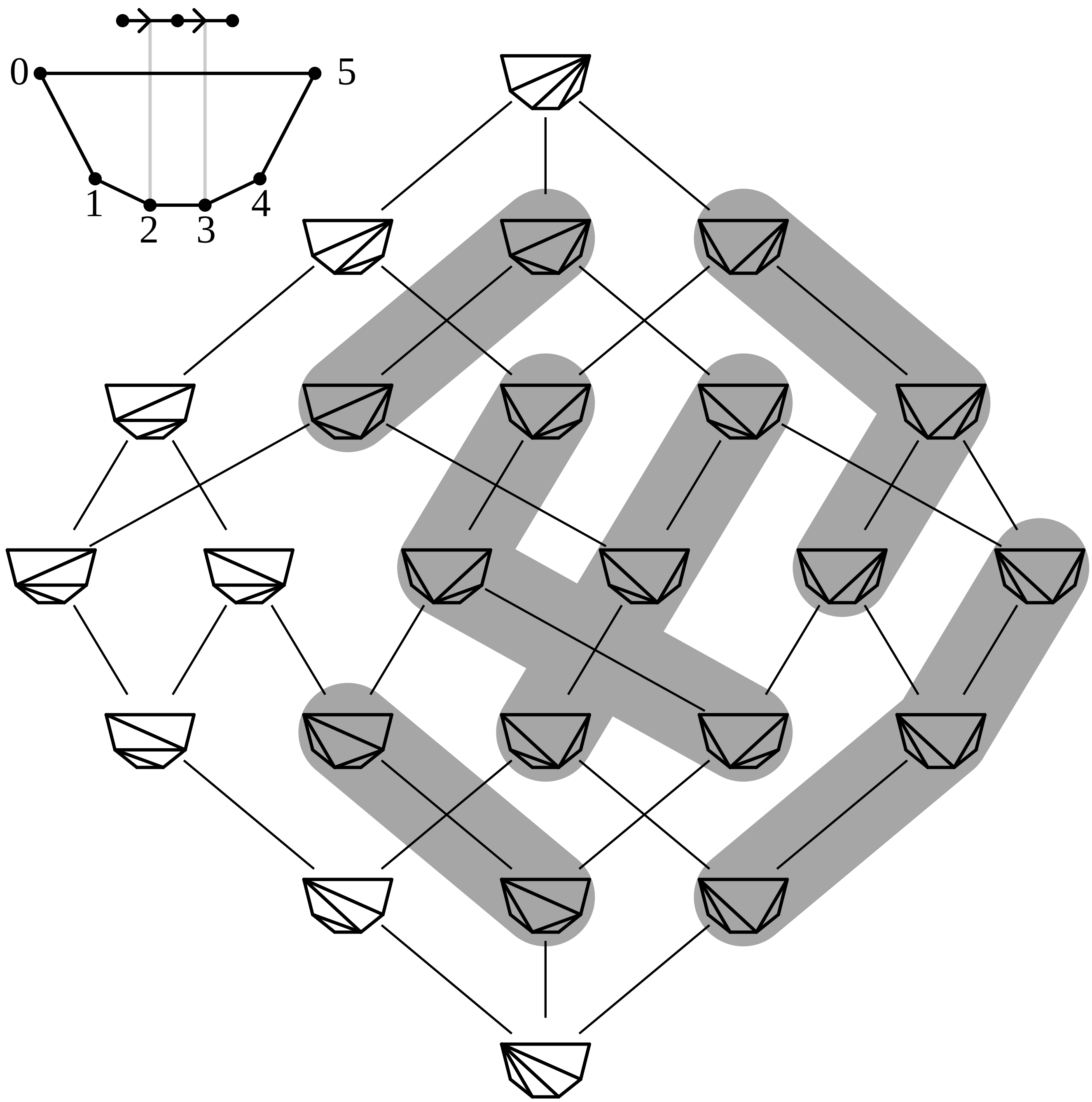
Each choice of direction of an edge of the diagram corresponds to choosing a vertex to be on top of Q or on bottom.

Only the vertices 2 through $n-1$ really matter.

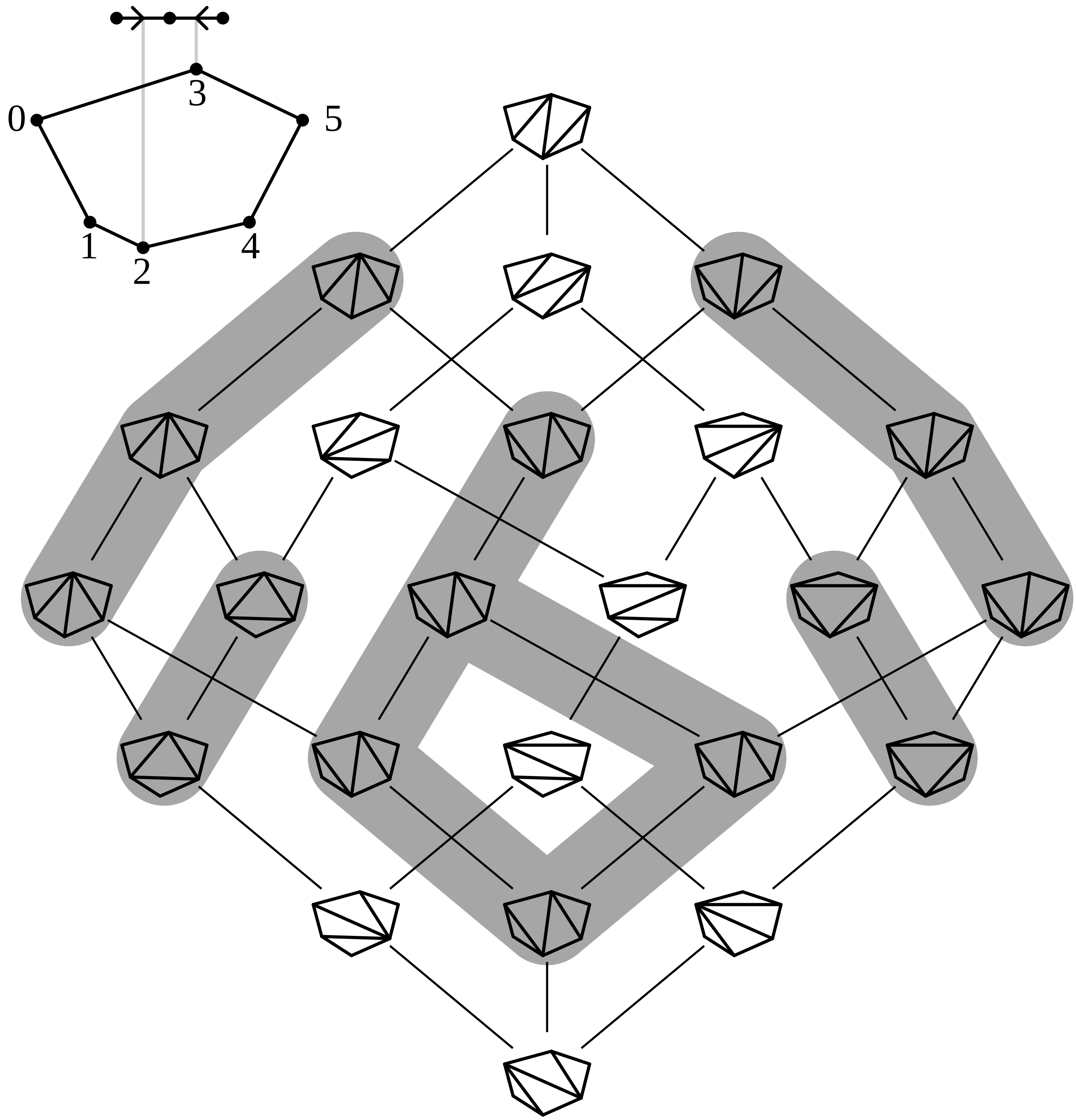
For example, the Q of the previous diagram corresponds to this orientation of the diagram of A_7:



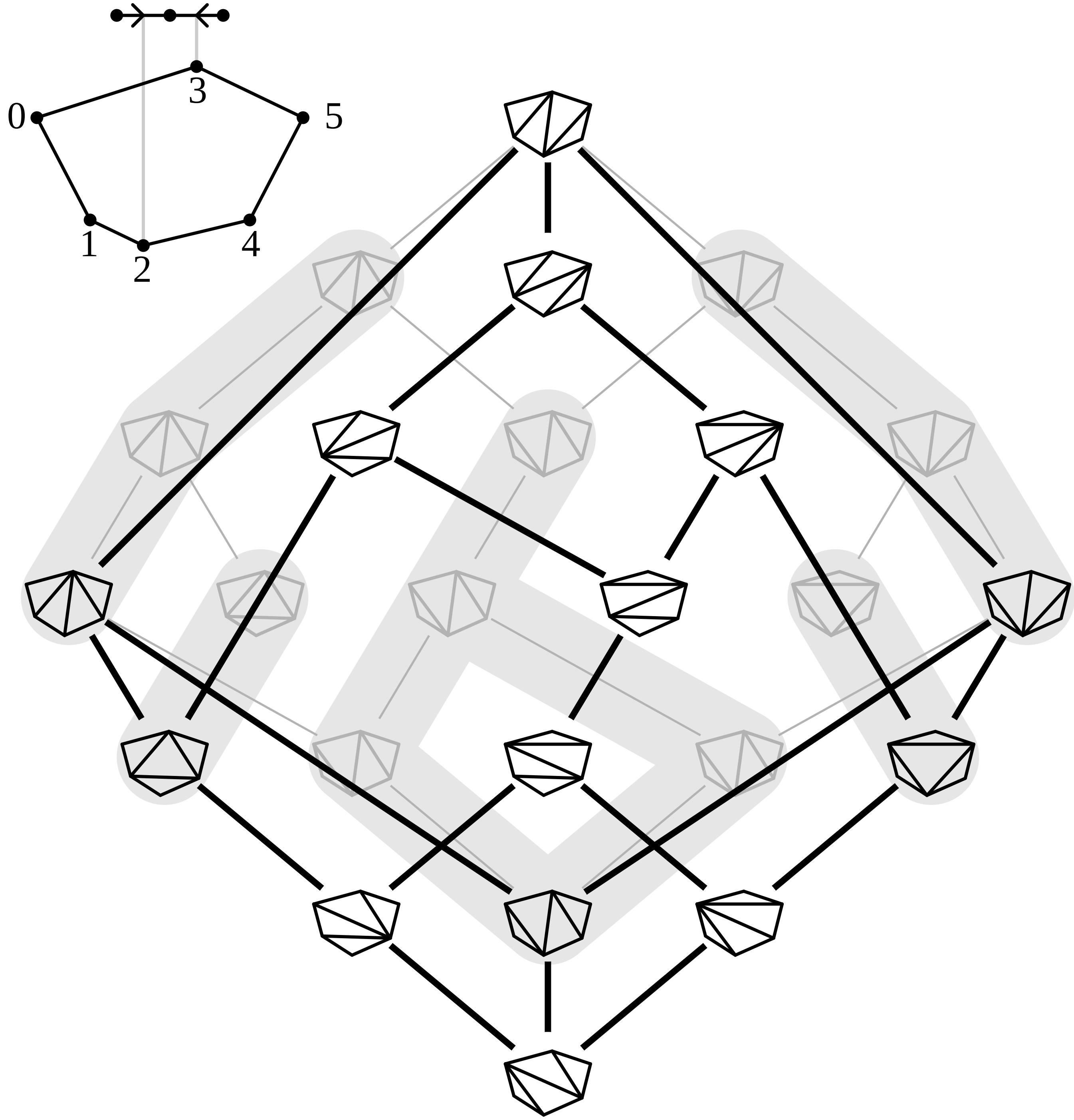
Example



Example



Example (continued)



Flips and Slopes

In both of these examples, and in general, the Cambrian lattice can be realized as follows:

- Fix a polygon Q for the orientation as described previously.

- Covers are diagonal flips.

- Going "up" means increasing the slope of the diagonal.

The type-B Cambrian lattices have a similar realization using centrally-symmetric polygons. (Uses an equivariant fiber polytope contruction due to Reiner).

Pattern avoidance

The Cambrian lattices of type A can also be realized in terms of pattern avoidance.

- One “colors” the symbols in $[n]$ using two colors “up” and “down.”

- The Cambrian lattice is the subposet consisting of permutations avoiding certain “colored patterns.”

- For the Tamari lattice, every symbol is the same color, so we get ordinary pattern avoidance.

A similar characterization exists for type B.

Preprints available on the arXiv.

Thank you

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