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To dispense with the obligatory running jokes:

• I did an email search to remember why I wasn't at the INI CAR. Not very funny: "Omicron variant" was part of the reason.

• My talk is about posets, which can be thought of as quivers, so I am using slides rather than drawing any posets on the board.

• I'm embarrassed to say that the paper is already on the arXiv. (arXiv:2311.06033)

It's better to speak earlier in the week! I could have saved 1 minute of talk time, and maybe could have *started* the running jokes.

Posets for F-polynomials in marked surfaces

Nathan Preading
NC State University

Cluster Algebras and Its Applications
Oberwolfach, January 18, 2024

Reporting on joint work with Vincent Pilaud and Sibylle Pschroll

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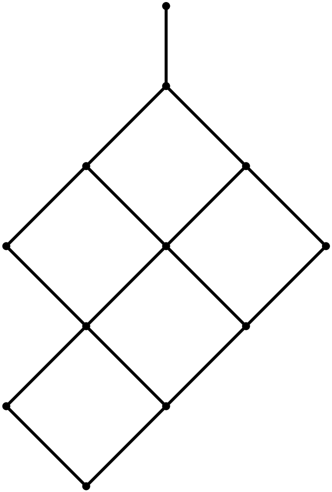
Section 1: Background

The Fundamental Theorem of Finite Distributive Lattices

Suppose L is a distributive lattice.

L



The Fundamental Theorem of Finite Distributive Lattices

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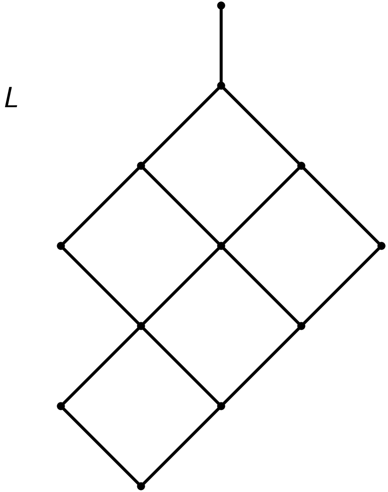
FTFDL says: There exists P such that L is the lattice of downsets of P.



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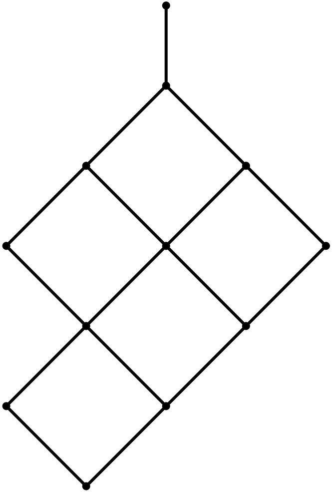
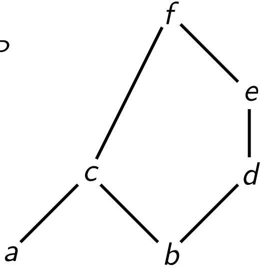


P



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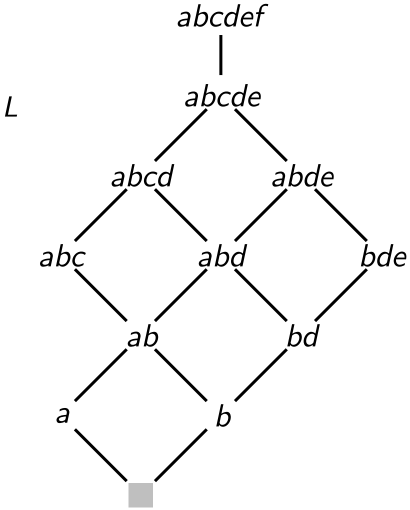

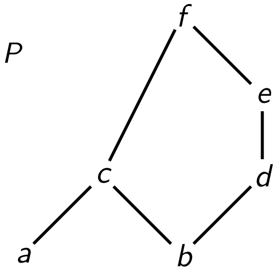

Philosophy

Philosophy

If you see a distributive lattice, you must use FTFDL.

Our cluster algebras conventions

In six words: Cluster algebras IV with principal coefficients.

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Specifically:

Initial cluster variables $x_1, \ldots, x_n$

Tropical variables/initial coefficients $y_1, \ldots, y_n$

Monomials $\hat{y}_1, \ldots, \hat{y}_n$ with $\hat{y}_j = y_j \prod_i x_i^{b_{ij}}$

g-vectors and F-polynomials: Each cluster variable is $x^{\mathbf{g}} \cdot F(\hat{y})$.

Main question

In a marked surface $(\mathbf{S}, \mathbf{M})$ and triangulation T, choose a lambda length x_γ for each arc γ in T.

There is a unique way to put a hyperbolic metric and horocycles on $\mathbf{S}$ so that each arc $\gamma \in T$ is a geodesic with lambda length x_γ.

Cluster variables are lambda lengths of tagged arcs.

Question: Find a formula for the lambda length of a tagged arc α in terms of the lambda lengths x_γ for $\gamma \in T$.

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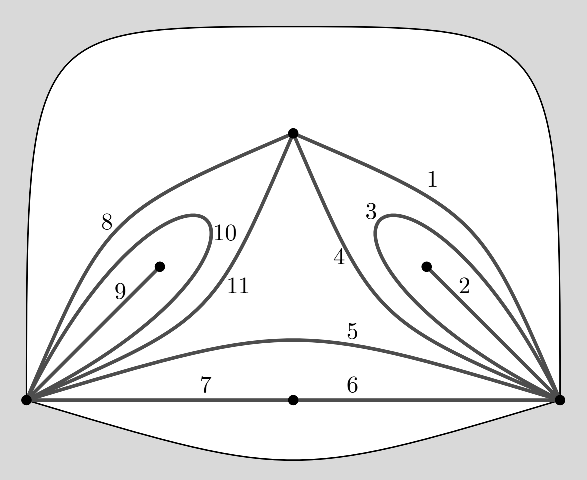
*** Actually, coefficient free cluster variables (i.e. setting all $y_i = 1$) are lambda lengths. For principal coefficients, use laminated lambda lengths.

Example



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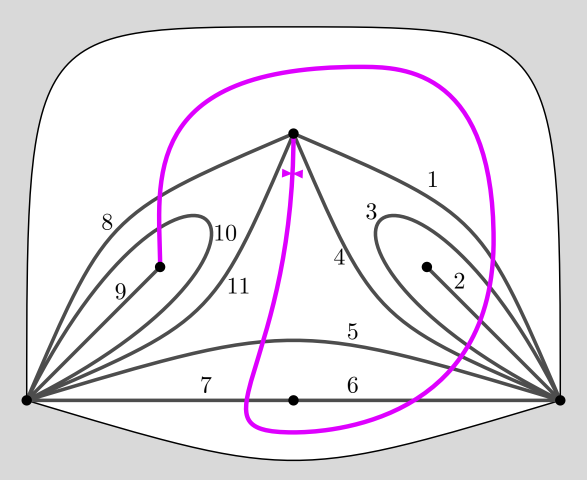

Find the lambda length of this tagged arc.

The previous state of the art

Surfaces model: Fomin, Shapiro, Thurston (FST 2006, FT 2012).

Other main work on main question: Musiker, Schiffler, Williams (MS 2008, MSW 2009, MW 2011, MSW 2011).

MSW give a formula for cluster variables as a weighted sum of perfect matchings on snake graphs, in the case where α has no notches, with Laurent monomials as weights.

When α has notches, there is an extra symmetry condition on matchings or a pair of "compatible" matchings.

Our starting point is an insight about the simpler case, already in the MSW work: There is a distributive lattice structure on the set of perfect matchings of a graph (Propp 2002).

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FTFDL!

Section 2: The theorem

If you see a distributive lattice, you must use FTFDL

By the MSW work, each cluster variable is a sum of Laurent monomials indexed by the elements of a finite distributive lattice, at least in the plain-tagged case.

A finite distributive lattice is the set of downsets in a finite poset.

Q: What is the nicest possible way to get monomials from downsets in a finite poset?

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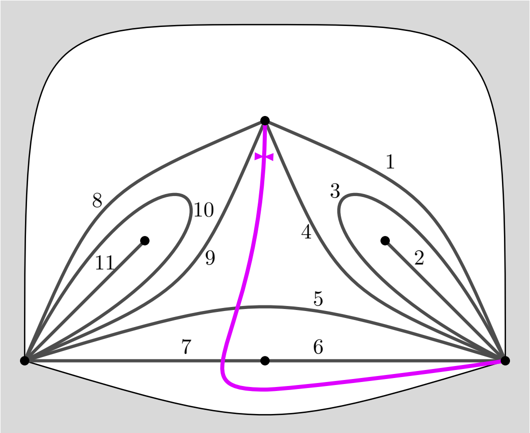
A: Give a monomial weight to each element of the poset. The weight of a downset is the product of the weights of its elements.

Main result (with Pilaud and Schroll): A simple, combinatorial way to construct a weighted poset P_α for any tagged arc α so that the weighted sum of downsets is F-polynomial. (There is already a known formula for the $\mathbf{g}$-vector using shear coordinates). We give a conceptually simple proof (for F and $\mathbf{g}$) using hyperbolic geometry.

Example (how to use the poset P_α, not yet how to make it)



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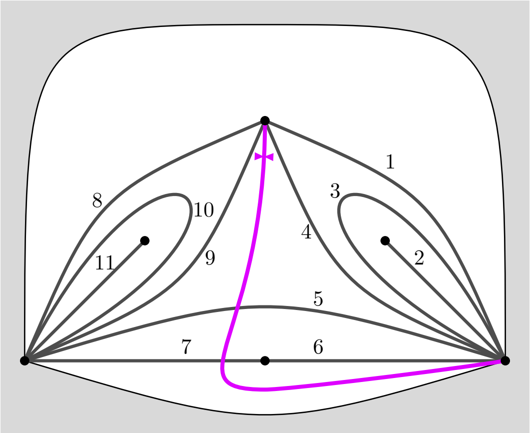

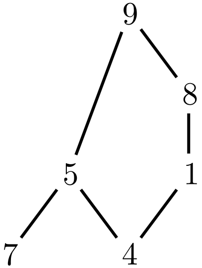

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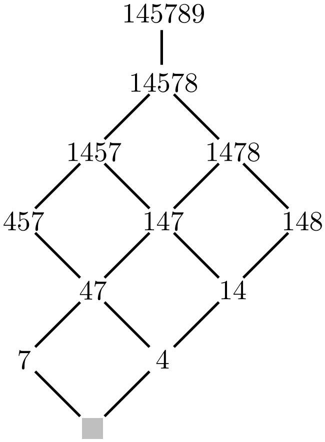

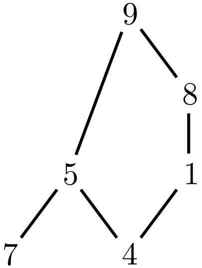

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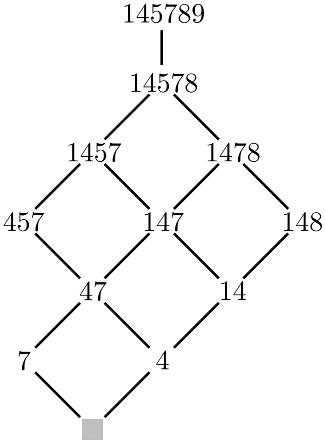

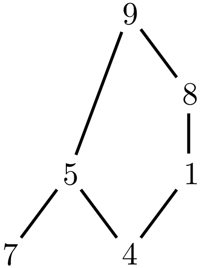

Cluster variable associated to α (i.e. lambda length of α):

$$x_\alpha = \frac{x_5}{x_4}(1 + \hat{y}_4 + \hat{y}_7 + \hat{y}_1\hat{y}_4 + \hat{y}_4\hat{y}_7 + \hat{y}_1\hat{y}_4\hat{y}_7 + \hat{y}_4\hat{y}_5\hat{y}_7 + \hat{y}_1\hat{y}_4\hat{y}_8 + \hat{y}_1\hat{y}_4\hat{y}_5\hat{y}_7 + \hat{y}_1\hat{y}_4\hat{y}_7\hat{y}_8 + \hat{y}_1\hat{y}_4\hat{y}_5\hat{y}_7\hat{y}_8 + \hat{y}_1\hat{y}_4\hat{y}_5\hat{y}_7\hat{y}_8\hat{y}_9)$$

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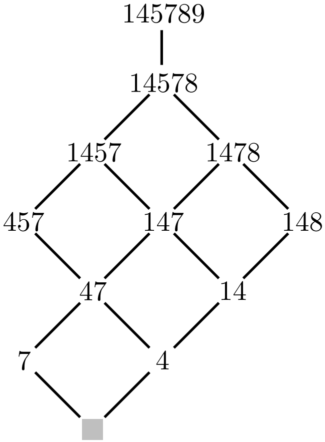
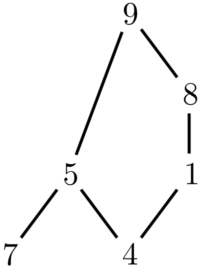


g-vector has nonzero entries $g_4 = -1$ and $g_5 = 1$



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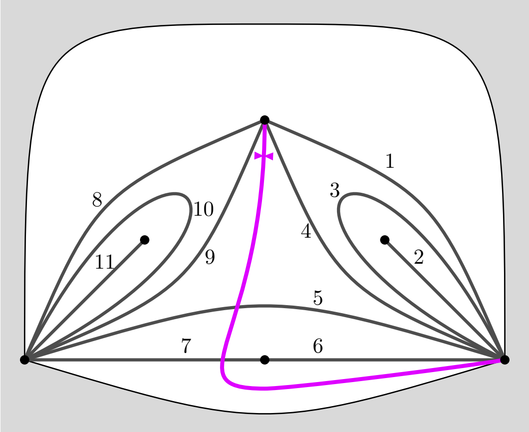

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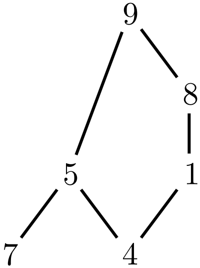

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The weighted poset P_α ("non-degenerate" case)

Non-degenerate case: α is not (a tagged version of) an arc in T.

Follow α through T. Each time α crosses an arc γ of T, we get an element of P_α that is labeled (usually) with $\hat{y}_\gamma$.

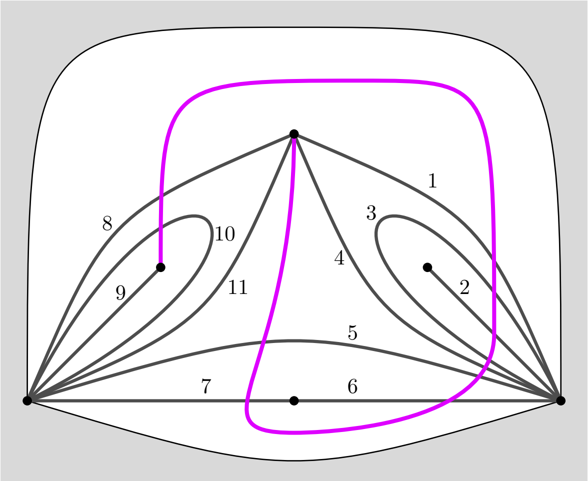
When γ is the interior edge of a self-folded triangle, the label is $\hat{y}_\gamma/\hat{y}_\beta$, where β is the exterior edge.

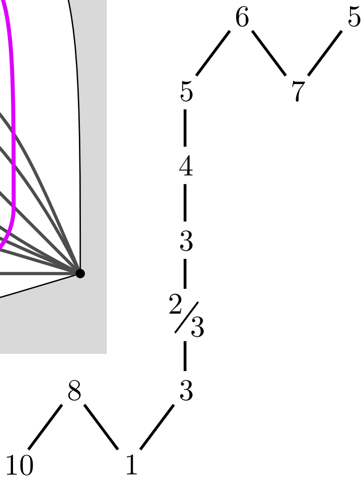
Each new element covers or is covered by the one before.
When we turn right in a triangle, we are going down in the poset.
When we turn left, we are going up.

When α is tagged notched at one or both endpoints, we add chains at those endpoints.

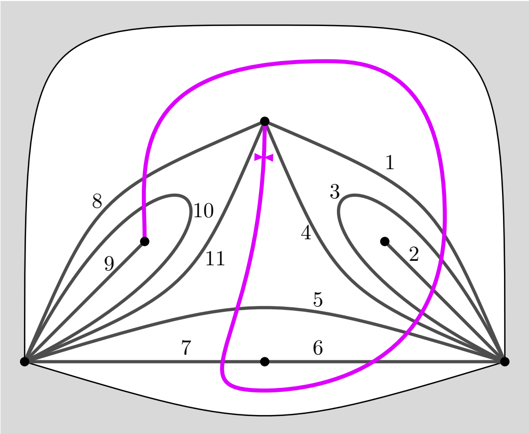
This case also done by Oğuz–Yıldırım. (See also MSW, Wilson).

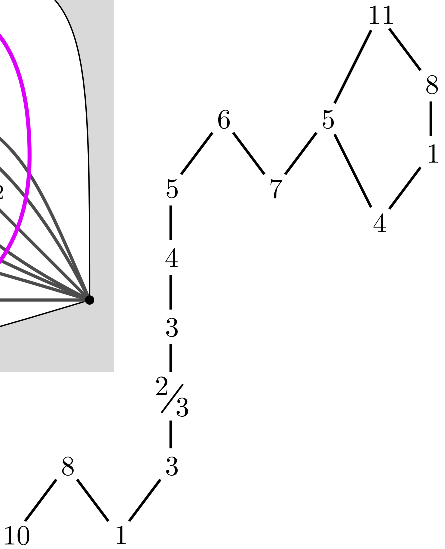
Non-degenerate case (α tagged plain)





Non-degenerate case (α tagged notched at one end)





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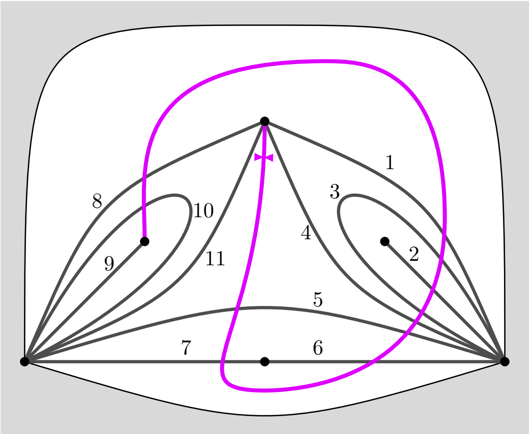
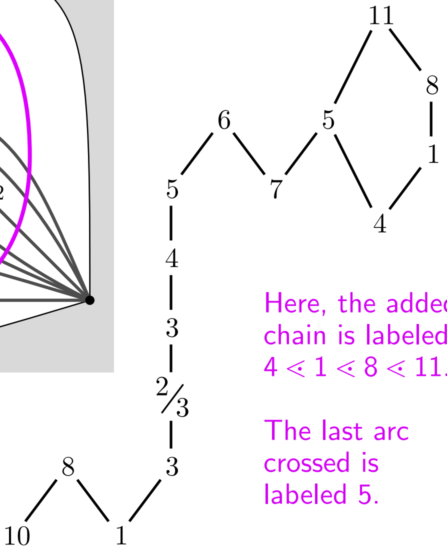


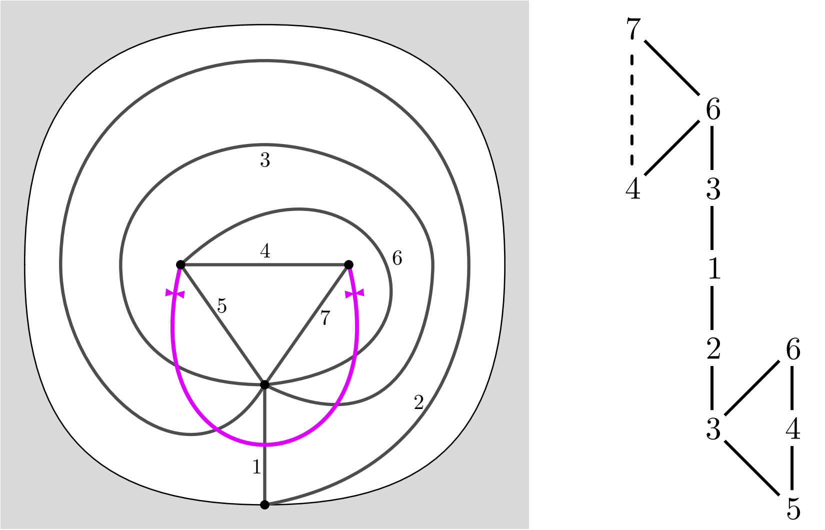
Make the chain that would correspond to an arc tracing around the endpoint. The top of the chain is above the last arc crossed. The bottom of the chain is below the last arc.



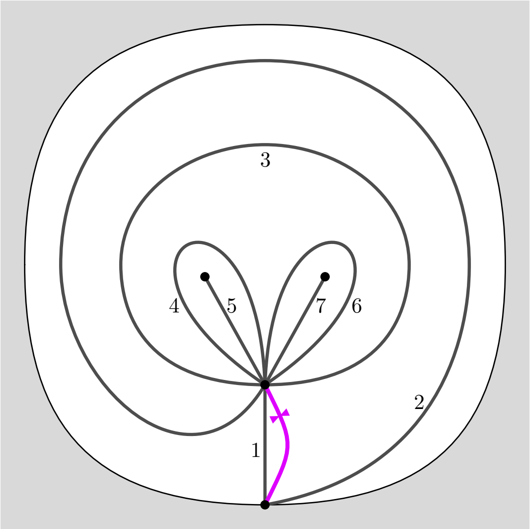
Here, the added chain is labeled $4 \lessdot 1 \lessdot 8 \lessdot 11$.

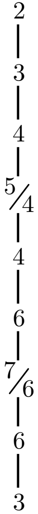
The last arc crossed is labeled 5.

Non-degenerate case (α tagged notched at both ends)



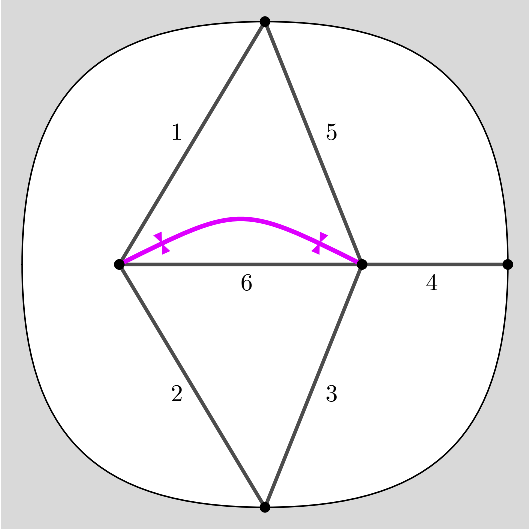
Degenerate case (α notched at one end)

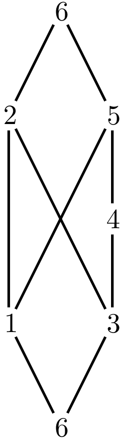




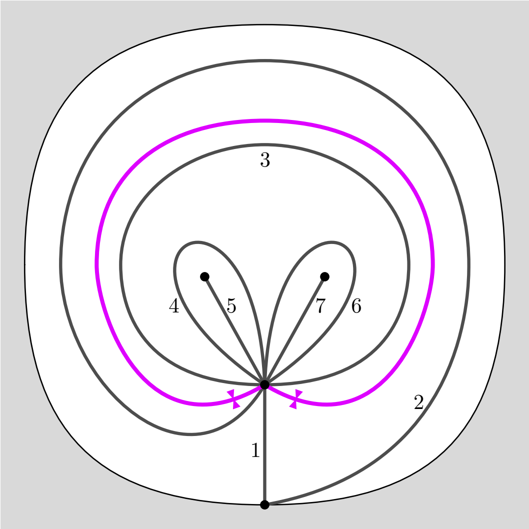
Degenerate means α coincides, up to tagging, with an arc in T.

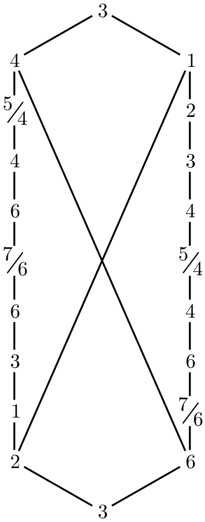
Degenerate case (α notched at both ends)



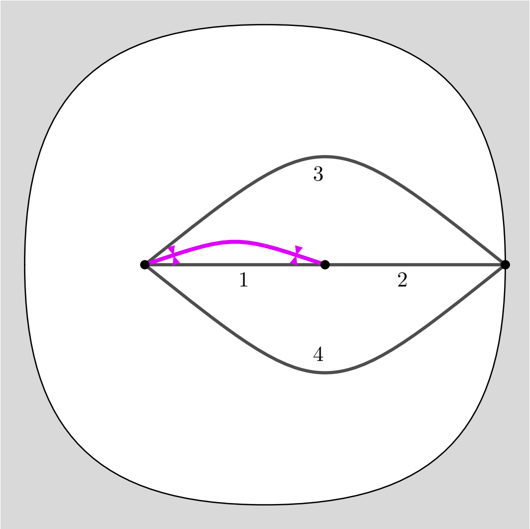


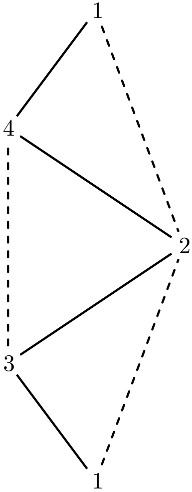
Degenerate case (α notched at both ends, 2nd example)





Degenerate case (α notched at both ends, 3[rd] example)



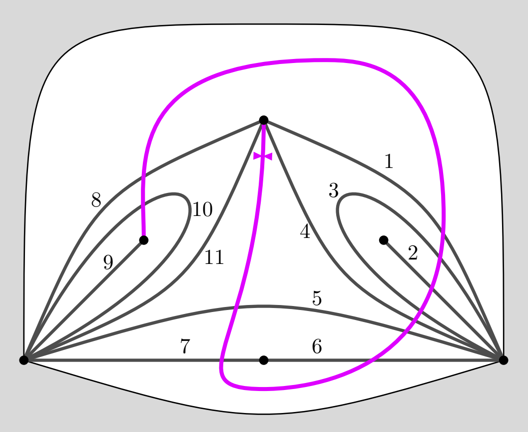


g-Vectors

Proposition. The **g**-vector of x_α is the negative of the shear coordinate vector of $\kappa(\alpha)$.

(Labardini-Fragoso, 2010, Musiker–Schiffler–Williams 2011, R. 2014, Felikson–Tumarkin 2017 + orbifolds, Pilaud–R.–Schroll 2023.)

α



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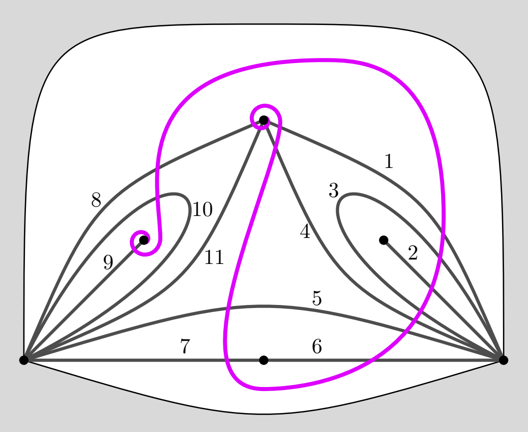
$\kappa(\alpha)$



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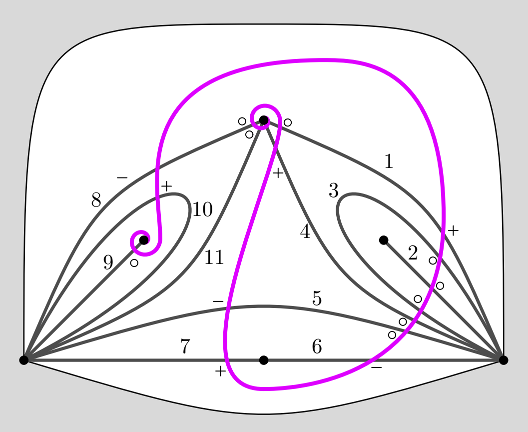

$$x^{\mathbf{g}} = \frac{x_5 x_6 x_8}{x_1 x_4 x_7 x_{10}}$$

Section 3: Proof idea and comments

Proof idea (coefficient-free case)

Want to show:
The cluster variable for a tagged arc α is the weighted sum of downsets in P_α.

If we could show that these weighted sums of downsets satisfy the exchange relations, we would be done.

In practice, it's difficult to have control over exchange relations.
Why? Self-folded triangles, α may intersect the same arc in the triangulation many times, etc.

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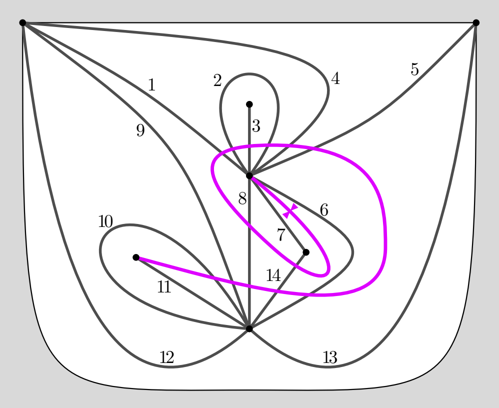
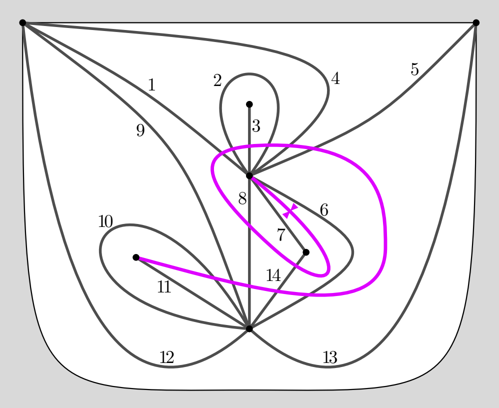
Why? Self-folded triangles, α may intersect the same arc in the triangulation many times, etc.

Instead, use the fact that the cluster variable is a hyperbolic length: Lift α to be a tagged arc α' in a surface without these complications, and lift the hyperbolic metric too. In the new surface, there are uncomplicated exchange relations.

We can induct on the number of elements of P.

Picture of the lifting

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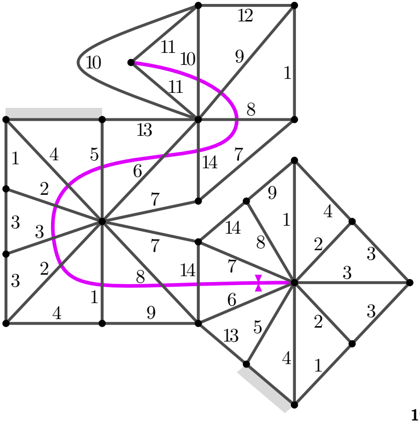
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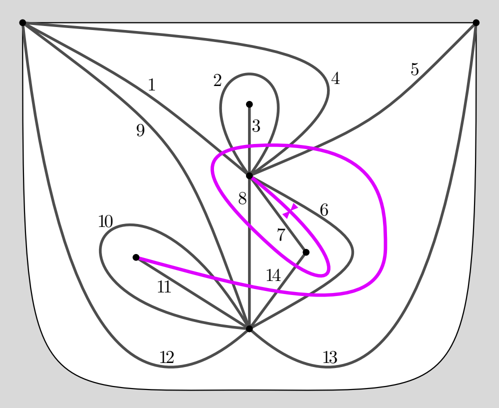
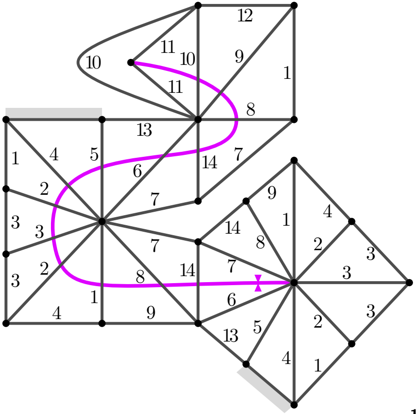
Picture of the lifting





Key point: Lift the hyperbolic metric and the horocycles, not just the combinatorics.

So the arc α and the lift α' have the same lambda length.

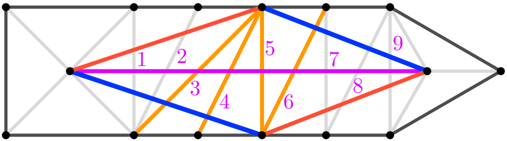
The exchange relation becomes simple combinatorics

Once we lift, $\exists$ many arcs $\gamma \in T'$ such that exchanging α' and γ is

$$F(P_{\alpha'}) = F(P_{\text{blue}}) + \hat{y}_{\text{orange}} \cdot F(P_{\text{red}}).$$

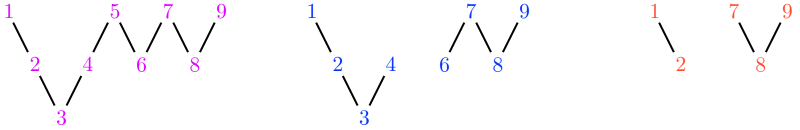
Specifically, if e_γ is the element of $P_{\alpha'}$ labeled $\hat{y}_\gamma$, then

- $F(P_{\text{blue}})$ is the weighted sum of downsets not containing e_γ.
- $\hat{y}_{\text{orange}} \cdot F(P_{\text{red}})$ is weighted sum of downsets containing e_γ.



Arc 5 is γ

$\hat{y}_{\text{orange}} = \hat{y}_3 \hat{y}_4 \hat{y}_5 \hat{y}_6$

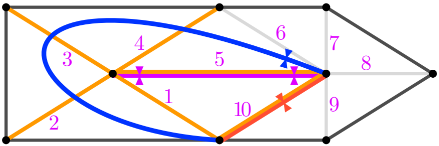


P_α P_{blue} P_{red}

Another exchange relation example

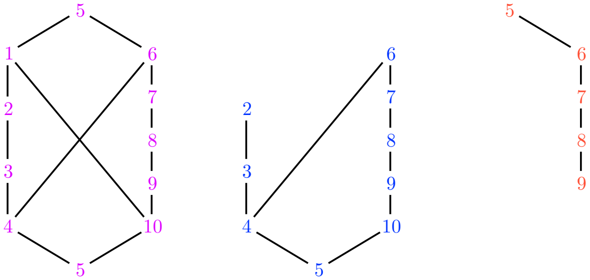
$$F(P_{\alpha'}) = F(P_{\text{blue}}) + \hat{y}_{\text{orange}} \cdot F(P_{\text{red}}).$$

- $F(P_{\text{blue}})$ is the weighted sum of downsets not containing e_γ.
- $\hat{y}_{\text{orange}} \cdot F(P_{\text{red}})$ is weighted sum of downsets containing e_γ.



Arc 1 is γ

$\hat{y}_{\text{orange}} = \hat{y}_1 \hat{y}_2 \hat{y}_3 \hat{y}_4 \hat{y}_5 \hat{y}_{10}$



P_α P_{blue} P_{red}

Comments

Coefficients: Everything I have explained here works for the coefficient-free case (after you set all the y_i to 1). To do principal coefficients, we use laminated lambda lengths (FT 2012). Basically the same proof, once one digests FT's tropical hyperbolic geometry.

Relationship to other work: Insights from the MSW work are very important. Downsets in posets, in puncture-free case, are already in Musiker-Schiffler-Williams (2011) and Çanakçı-Schroll (2021). The non-degenerate case is in Oğuz–Yıldırım (2022). Similar posets are in work of Wilson (2020) and Weng (2023). Exchange (skein) relations are in MSW and Çanakçı–Schiffler.

What's new here:
- Make FTFDL the crucial idea.
- Treat all cases (all tagged arcs, no restrictions on **S** or **M**).
- Simple proof (poset combinatorics + FT's hyperbolic geometry).

Future/ongoing work

Are F-posets the right way to describe cluster variables for other classes of cluster algebras?

- Orbifolds?
- Finite type?

Applications to adjacent areas?

Especially representation theory, where a special case of these poset constructions was the initial motivation for this work.

References

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Musiker–Schiffler–Williams, *Positivity for cluster algebras...*

Musiker–Schiffler–Williams, *Bases for cluster algebras...*

Oğuz–Yıldırım, *Cluster algebras and oriented posets*

Weng, *F-polynomials of Donaldson-Thomas transformations.*

Wilson, *Surface cluster algebra expansion formulae via loop graphs.*