# Counting chains in noncrossing partition lattices

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# Counting chains in noncrossing partition lattices

Classical noncrossing partitions

Coxeter groups and noncrossing partitions

Counting maximal chains

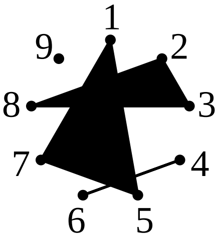
# Classical noncrossing partitions (Kreweras, 1972)

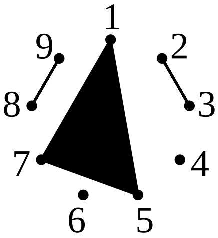
Identify the numbers $1, 2, \ldots n$ with $n$ distinct points in cyclic order on a circle. For each block $B$ of a partition $\pi$, draw the convex polygon whose vertices are the points in $B$. If $|B|$ is 1 or 2, this "polygon" is a point or a line segment.

The partition $\pi$ is noncrossing if and only if in its planar diagram, the blocks are disjoint (i.e. don't cross).

## Example

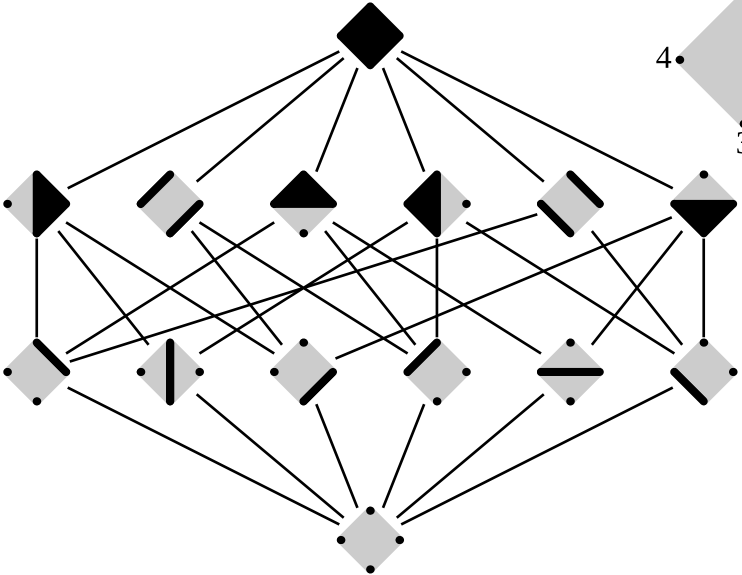
Crossing and noncrossing:

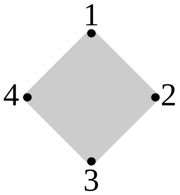


# The classical noncrossing partition lattice

Ordered by refinement of partitions.





# Enumeration

## Noncrossing partitions

Noncrossing partitions of $[n]$ are counted by the famous Catalan numbers

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

## Chains

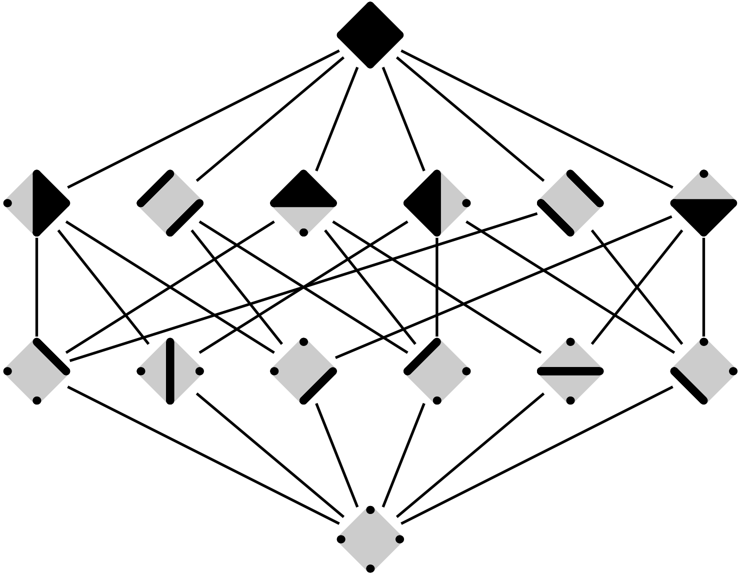
Detailed enumeration formulas exist counting chains (totally ordered subsets) in the noncrossing partition lattice according to the set of ranks visited. (Edelman, 1980).

## Maximal chains

There are $n^{n-2}$ maximal chains. There is a nice bijection with parking functions (Stanley, 1997).

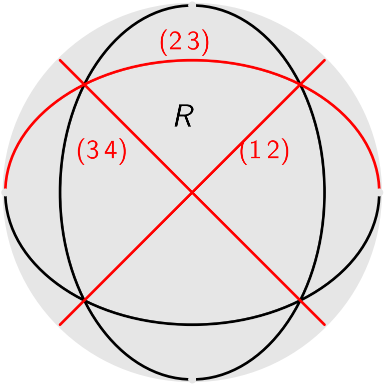
# Example

$4^{4-2} = 16$ maximal chains for $n = 4$.



Finite groups $W$ generated by (Euclidean) orthogonal reflections.
Examples: symmetry groups of regular polytopes, Weyl groups.
Coxeter arrangement $\mathcal{A} = \{$All reflecting hyperplanes for $W\}$.
Simple reflections: Fix a connected component $R$ of the complement of $\bigcup \mathcal{A}$. Let $S$ be the set of reflections in the facet-hyperplanes of $R$.

$W = S_4$:



# Coxeter groups

Coxeter group: a group with a presentation of the form

$$\langle S \mid s^2 = 1,\ (st)^{m(s,t)} = 1 \rangle.$$

Finite Coxeter groups $\leftrightarrow$ finite reflection groups.

Coxeter diagram: encodes the presentation.

- Vertex set: $S$
- Edges: $s$—$t$ when $m(s,t) > 2$, labeled by $m(s,t)$ when $m(s,t) > 3$.

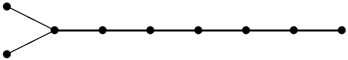
Irreducible Coxeter group: a Coxeter group having a connected diagram.

# Classification of irreducible finite Coxeter groups

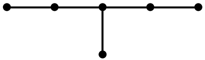
$A_n$ $(n \geq 1)$

$B_n$ $(n \geq 2)$

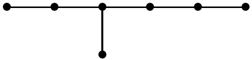

$D_n$ $(n \geq 4)$



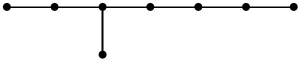
$E_6$



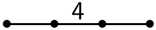
$E_7$



$E_8$

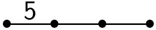


$F_4$

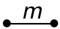



$H_3$


$H_4$

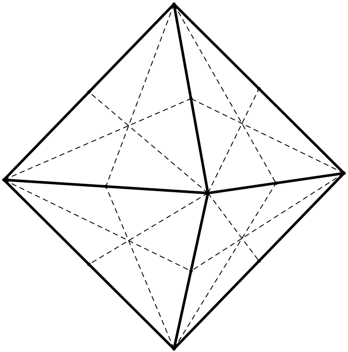



$I_2(m)$ $(m \geq 5)$



# Types A, B and D

- $A_{n-1} = S_n$: Reflecting hyperplanes are $x_i = x_j$ for $i \neq j$.
- $B_n$ (the symmetry group of the $n$-cube or $n$-octohedron): Reflecting hyperplanes are $x_i = 0$ and $x_i = \pm x_j$ for $i \neq j$.



- $D_n$: Reflecting hyperplanes are $x_i = \pm x_j$ for $i \neq j$.

# Intersection lattices

Intersection lattice of the Coxeter arrangement: intersections of sets of hyperplanes, ordered by reverse inclusion.
Key point: The intersection lattice for $S_n$ is the lattice of partitions.

$S_9$ example:

$$\boxed{\begin{array}{c} \bigcap\{x_1 = x_2, x_2 = x_8, x_3 = x_7, x_3 = x_9, x_5 = x_6\} \\ \updownarrow \\ \{1,2,8\},\{3,7,9\},\{4\},\{5,6\}. \end{array}}$$

$B_n$ intersection lattice: partitions of $\pm[n]$ fixed by $i \mapsto -i$, at most one block containing a pair $(i, -i)$.

$B_9$ example:

$$\boxed{\begin{array}{c} \bigcap\{x_1 = 0, x_2 = 0, x_2 = -x_7, \\ x_4 = -x_9, x_5 = x_7, x_6 = x_8\} \\ \updownarrow \\ \{\pm 1, \pm 2, \pm 5, \pm 7\},\{3\},\{-3\}, \\ \{4,-9\},\{-4,9\},\{6,8\},\{-6,-6\}. \end{array}}$$

# Coxeter element

Coxeter element: $c := s_1 s_2 \cdots s_n$ for a permutation $s_1, s_2, \ldots, s_n$ of $S$.
Coxeter number: $h :=$ the order of $c$ in $W$.
Exponents: positive integers $e_j$ such that $\exp(2\pi i e_j / h)$ is an eigenvalue of $c$.

Example ($A_n = S_{n+1}$):

- Coxeter elements are $(n+1)$ cycles.
- $h = n + 1$.
- Exponents: $1, 2, \ldots n$.

# NC partition lattices for any finite reflection group.

(Athanasiadis, Biane, Bessis, Brady, Reiner, Watt, 1997–2003)

Let $T$ be the set of reflections in $W$. For $w \in W$, write $w = t_1 t_2 \cdots t_k$ for $t_i \in T$, minimizing $k$. Set

$$l(w) := k.$$

(This is not the usual "length function.")

Set $u \preceq uv$ if and only if $l(uv) = l(u) + l(v)$.

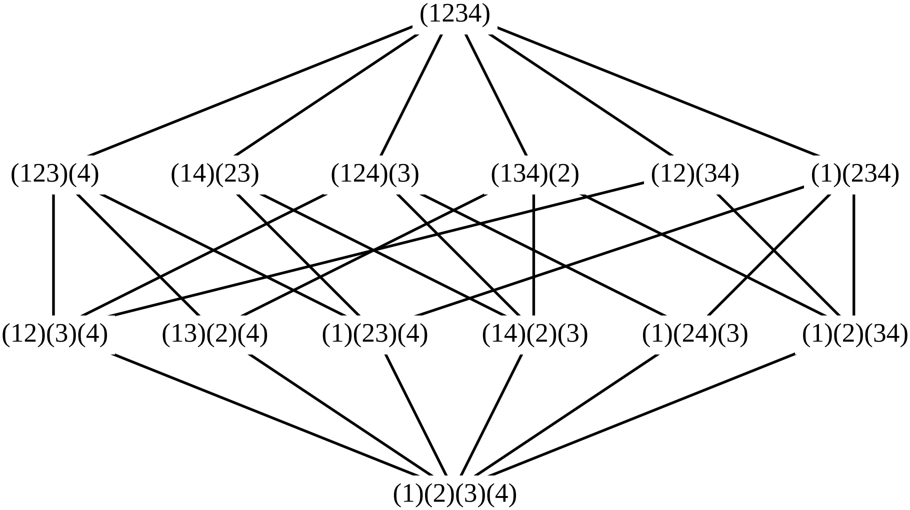
The noncrossing partition lattice for $W$ is the interval $[1, c]_{\preceq}$. (Different choices of $c$ give isomorphic lattices.)

Given $x = t_1 t_2 \cdots t_k \in [1, c]_{\preceq}$, define a "type-W partition:"

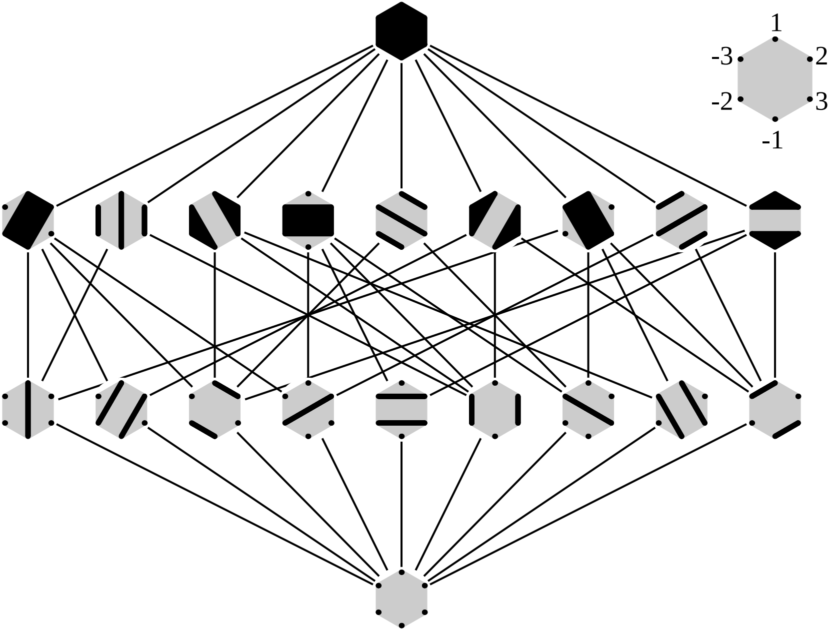
$$U_x := H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_k}.$$

# Example: $A_3 = S_4$ with $c := (1234)$

Cycles $\leftrightarrow$ blocks.



# Example: $B_3$ with $c = (1\,2\,3\,{-1}\,{-2}\,{-3})$



# Garside structure

The lattice $[1, c]$, with a certain edge labeling is a Garside structure. From this, one obtains:

- A presentation of $W$ as a group generated by the whole set $T$ of reflections.
- A monoid structure for the associated Artin group.
- A finite Eilenberg-Maclane ($K(\pi, 1)$) space for the Artin group.

The Artin group is the fundamental group of the complement of the complexification of the Coxeter arrangement, with presentation

$$\langle S \mid (st)^{m(s,t)} \rangle.$$

# The $W$-Catalan number

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}.$$

For $W = A_n = S_{n+1}$, we have $e_i = i$ and $h = n + 1$, so $\mathrm{Cat}(A_{n-1}) = C_n$, the usual Catalan number.

For $W = B_n$, we have $e_i = 2i - 1$ and $h = 2n$, so $\mathrm{Cat}(B_n)$ is $\binom{2n}{n}$.

| $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $H_3$ | $H_4$ | $I_2(m)$ |
|---|---|---|---|---|---|---|---|
| $\frac{3n-2}{n}\binom{2n-2}{n-1}$ | 833 | 4160 | 25080 | 105 | 32 | 280 | $m+2$ |

# $W$-Catalan numbers count:

- $W$-noncrossing partitions (Various researchers, 1997-2003).
- Conjugacy classes of elements of finite order in Lie groups (Djoković, 1980).
- $W$-orbits in $Q/(h+1)Q$, where $Q$ is the root lattice (Haiman, 1994).
- Antichains in the root poset (Postnikov, 1996).
- Positive regions of the Shi arrangement (Shi, 1997).
- The vertices of the $W$-associahedron (Fomin, Zelevinsky 2003).
- Coxeter-sortable elements of $W$ (R., 2005).

# Counting maximal chains

Let $\mathrm{M}(W)$ be the number of maximal chains in the noncrossing partition lattice of $W$. This number is interesting because

- It is the number of top-dimensional cells in the Eilenberg-Maclane space mentioned above.
- It is the number of "reduced words" for $c$ in the alphabet of reflections.
- It generalizes the number of parking functions.

Theorem (R., 2004.)

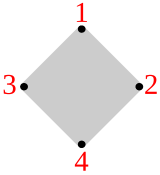
*Let $W$ be a finite irreducible Coxeter group with simple generators $S$. Then*

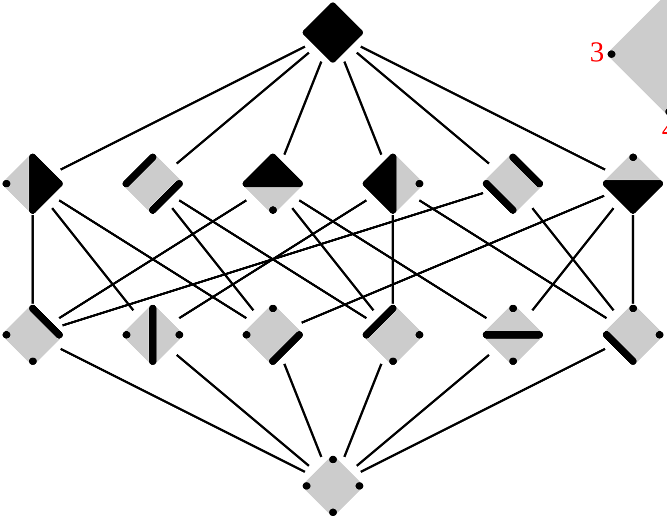
$$\mathrm{M}(W) = \frac{h}{2} \sum_{s \in S} \mathrm{M}(W_{\langle s \rangle}).$$

$W_{\langle s \rangle}$ is the "standard parabolic" subgroup generated by $S - \{s\}$. For reducible groups, use standard techniques for products.

# Example: $W = A_3$ with $c = (1\,2\,4\,3)$

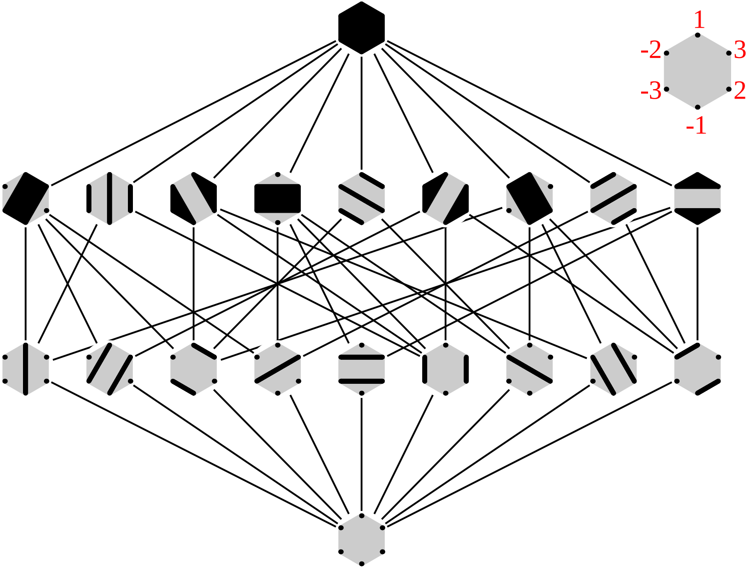
Count maximal chains according to whether the element covering the identity is in the orbit of (1 2), (2 3) or (3 4).



# Example: $B_3$ with $c = (1\,3\,2\,{-1}\,{-3}\,{-2})$

$S = \{(1\ \ -1),\ (1\ 2)(-2\ \ -1),\ (2\ 3)(-3\ \ -2)\}.$



# The Coxeter plane (Steinberg, 1950)

For $W$ **irreducible**, let $S = S_+ \cup S_-$ be a bipartition of the Coxeter diagram. Define a bipartite Coxeter element: $c := c_- c_+$, where

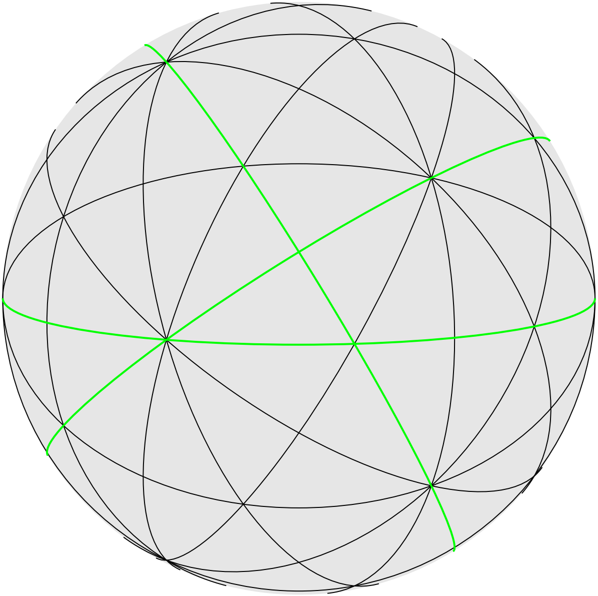
$$c_+ = \left( \prod_{s \in S_+} s \right) \quad \text{and} \quad c_- = \left( \prod_{t \in S_-} t \right).$$

The Coxeter plane is a 2-dimensional plane $P$ such that:

- $P$ is fixed (as a set) by $\langle c_+, c_- \rangle$.
- $\langle c_+, c_- \rangle$ acts on $P$ as a dihedral reflection group.
- $c$ is a rotation through $1/h$ of a turn.
- $P$ is spanned by lines $L_+$ and $L_-$.
- $H_t$ contains $L_\epsilon$ if and only if $t = s \in S_\epsilon$, for $\epsilon \in \{+, -\}$.

(Note: this construction breaks down in the reducible case.)

# Example: $W = H_3$ $(h = 10)$



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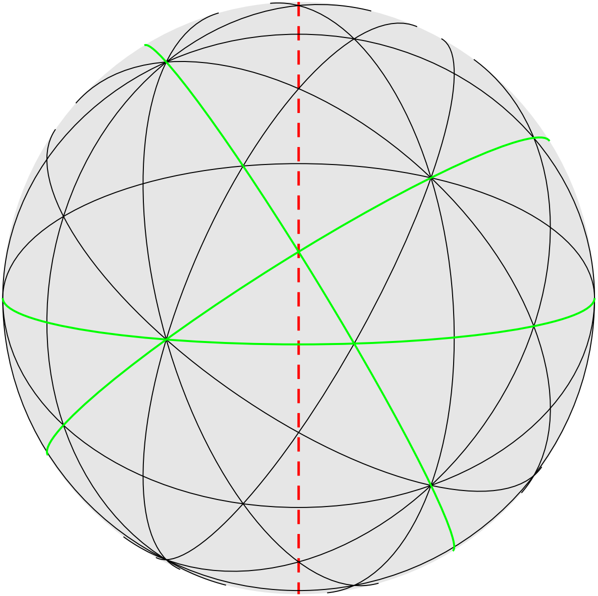

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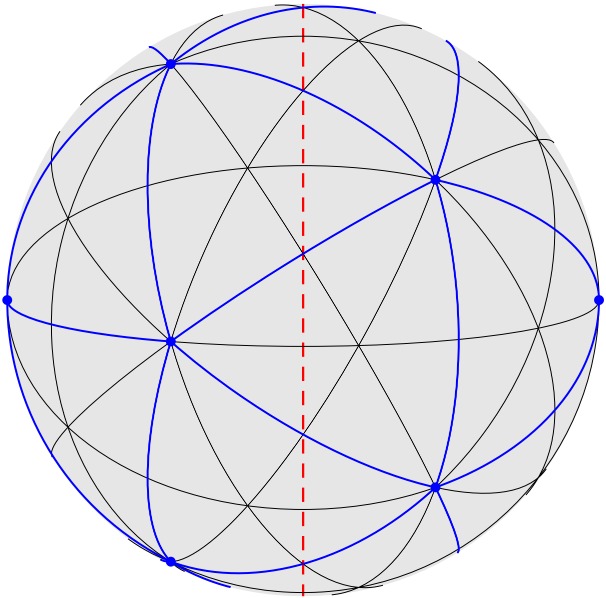

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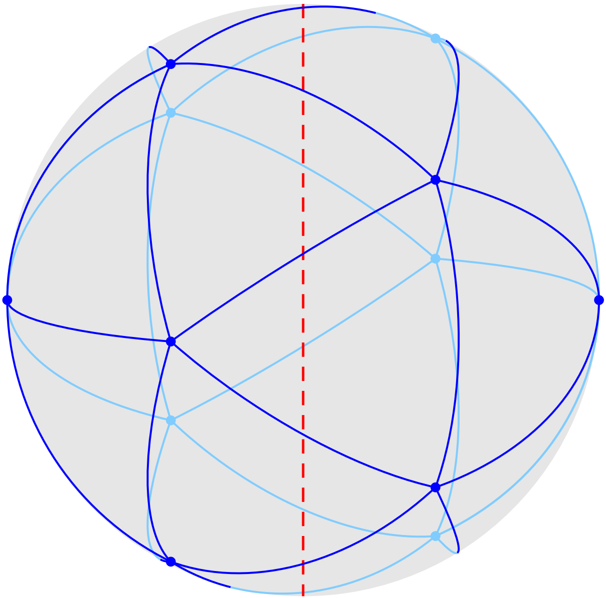

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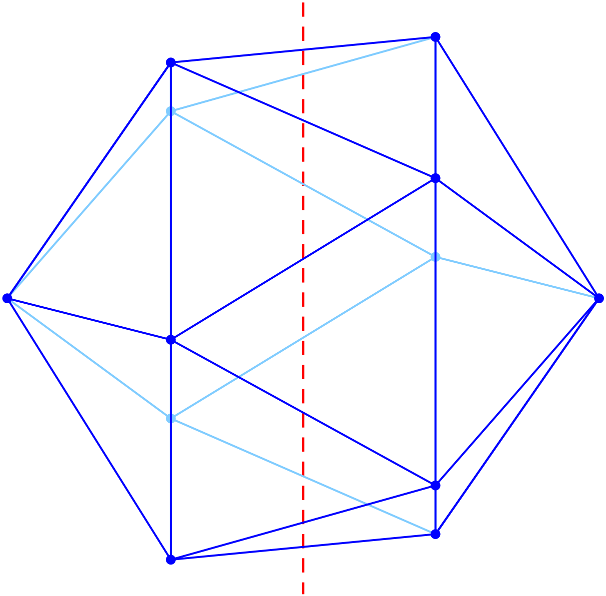

**Proposition (Steinberg, 1950)**

*For any $t \in T$, the orbit of $t$ under conjugation by $c$ either:*

(i) *has $h/2$ elements and intersects $S$ in a single element, or*

(ii) *has $h$ elements and intersects $S$ in a two-element set.*

**Proof.**

The previously-mentioned properties of the Coxeter plane. $\square$

A better-known consequence: $W$ has $\dfrac{nh}{2}$ reflections.

($n = |S|$ and $h =$ Coxeter number.)

# Proof idea for $\mathrm{M}(W) = \dfrac{h}{2} \sum_{s \in S} \mathrm{M}(W_{\langle s \rangle})$

Count maximal chains by "rotating" (conjugating by the Coxeter element) until the element covering the identity is in $S$.

Each $c$-conjugacy orbit under rotation has $h/2$ total reflections per simple reflection.

The rest of the chain is identified with a maximal chain in $W_{\langle s \rangle}$.

Analogous to a method used by Fomin and Zelevinsky to prove facts about clusters and generalized associahedra.

(Technical detail: In fact we act alternately by $c_-$ and $c_+$ and treat $s$ differently depending on whether it is in $S_+$ or $S_-$.)

# The numbers $\mathrm{M}(W)$

| $A_n$ | $B_n$ | $D_n$ | $I_2(m)$ |
|---|---|---|---|
| $(n+1)^{n-1}$ | $n^n$ | $2(n-1)^n$ | $m$ |

| $E_6$ | $E_7$ | $E_8$ | $F_4$ | $H_3$ | $H_4$ |
|---|---|---|---|---|---|
| 41472 | 1062882 | 37968750 | 432 | 50 | 1350 |

For $A_n$, $B_n$ and $D_n$, the recursion is solved by Abel's identity.

The numbers $\mathrm{M}(W)$ are given by a simple formula, due to Chapoton:

$$\mathrm{M}(W) = \frac{n!\, h^n}{|W|}.$$

I don't know how to solve my recursion in general to give this formula. But the recursion is the best way to prove this formula type-by-type, and (as far as I know) the only known way to prove it without asking the computer to do brute-force counting.

# Broader results

The method of "rotation by $c$" can be applied more broadly. In particular, we can give recursions for:

- The number of edges in the $W$-noncrossing partition lattice, leading to a uniform formula:

$$\mathsf{E}(W) = \frac{nh}{h+2}\,\mathsf{Cat}(W) = \frac{nh}{|W|}\prod_{i=2}^{n}(h+e_i+1).$$

- The number of reduced words in the alphabet of reflections.

The proofs also generalize to the setting of $m$-divisible $W$-noncrossing partitions.