Lattice homomorphisms between weak orders and between Cambrian lattices

Nathan Reading
NC State University

Algebraic and Geometric Combinatorics of Reflection Groups
CRM/LaCIM Workshop
UQAM, June 5, 2017

Main points

Recall: Automorphisms of the weak order on a finite Coxeter group are exactly diagram automorphisms.

Surjective lattice homomorphisms between different finite Coxeter groups are diagram homomorphisms: A surjective homomorphism exists from W to W' if and only if the diagram for W' can be obtained from the diagram for W by deleting vertices and/or decreasing edge labels (and thus possibly erasing edges).

Such homomorphisms depend only on their restriction to rank-2 standard parabolic subgroups.

W' is (usually) the quotient of W modulo a homogeneous congruence of degree 2 (or sometimes nonhomog. of degree 3).

From there: surjective hom.s between Cambrian lattices… refinement relations among Cambrian fans… **g**-vector fans… scattering diagrams… hom.s between cluster algebras…

Section 1: Simion's map

The story begins with Simion

At the time of Rodica Simion's untimely death in 2000, she had nearly completed her paper "A type-B associahedron." (Reiner finished preparing it for publication with help from Greene, Matsko, and Devadoss.)

The paper constructs a type-B analog of the usual associahedron, based on centrally symmetric triangulations of a polygon.

(At about the same time, the operads/homotopy theory people were doing the same thing, calling this the cyclohedron. Simion's work was independent and initiated the combinatorial study of this polytope.)

In the paper, Simion defined a map from signed permutations to permutations. That map is the beginning of the story (not just the story of this talk, but the story of this research).

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall \ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $(-6)81(-3)(-4)725$

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall\ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $\bar{6}81\bar{3}\bar{4}725$

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall \ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $(-6)81(-3)(-4)725$

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall \ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $(-6)81(-3)(-4)725$

Extended one-line notation: $\pi_{-n}\cdots\pi_{-2}\pi_{-1}\pi_1\pi_2\cdots\pi_n$.

Example. $(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall \ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1 \pi_2 \cdots \pi_n$.

Example. $(-6)81(-3)(-4)725$

Extended one-line notation: $\pi_{-n} \cdots \pi_{-2} \pi_{-1} \pi_1 \pi_2 \cdots \pi_n$.

Example. $(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$

Consider signed permutations whose (short) one-line notation avoids $1\overline{2}$ and $2\overline{1}$. That is, negative entries (if any) must come first and then positive entries (if any).

Example. $(-6)(-3)(-4)81725$

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \ \forall \ i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $(-6)81(-3)(-4)725$

Extended one-line notation: $\pi_{-n}\cdots\pi_{-2}\pi_{-1}\pi_1\pi_2\cdots\pi_n$.

Example. $(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$

Consider signed permutations whose (short) one-line notation avoids $1\overline{2}$ and $2\overline{1}$. That is, negative entries (if any) must come first and then positive entries (if any).

Example. $(-6)(-3)(-4)81725$

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$?

Some pattern-avoiding signed permutations

Signed permutations: Permutations π of $\pm[n]$ with $\pi_{-i} = -\pi_i \;\forall\; i$.
The Coxeter group B_n is the group of signed permutations.

(Short) one-line notation: $\pi_1\pi_2\cdots\pi_n$.

Example. $(-6)81(-3)(-4)725$

Extended one-line notation: $\pi_{-n}\cdots\pi_{-2}\pi_{-1}\pi_1\pi_2\cdots\pi_n$.

Example. $(-5)(-2)(-7)43(-1)(-8)6 \;\big|\; (-6)81(-3)(-4)725$

Consider signed permutations whose (short) one-line notation avoids $1\overline{2}$ and $2\overline{1}$. That is, negative entries (if any) must come first and then positive entries (if any).

Example. $(-6)(-3)(-4)81725$

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

Some pattern-avoiding signed permutations (continued)

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

A bijection:

Example. $(-6)(-3)(-4)81725$

Some pattern-avoiding signed permutations (continued)

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

A bijection:

Place 0 between the right and left sides of the extended one-line notation.

Example. $(-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436 \mid (-6)(-3)(-4)81725$

Some pattern-avoiding signed permutations (continued)

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

A bijection:

Place 0 between the right and left sides of the extended one-line notation.

Example. $(-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436 \mid (-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436\,0\,(-6)(-3)(-4)81725$

How many permutations in B_n avoid $1\bar{2}$ and $2\bar{1}$? $(n+1)!$

A bijection:

Place 0 between the right and left sides of the extended one-line notation.

Restrict to nonnegative entries

Example. $(-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436 \mid (-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436\,0\,(-6)(-3)(-4)81725$

$436\,0 \qquad 81725$

Some pattern-avoiding signed permutations (continued)

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

A bijection:

Place 0 between the right and left sides of the extended one-line notation.

Restrict to nonnegative entries

Example. $(-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436 \mid (-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436\,0\,(-6)(-3)(-4)81725$

$436\,0 \qquad 81725$

436081725

How many permutations in B_n avoid $1\overline{2}$ and $2\overline{1}$? $(n+1)!$

A bijection:

Place 0 between the right and left sides of the extended one-line notation.

Restrict to nonnegative entries

Add 1 to every entry.

Example. $(-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436 \mid (-6)(-3)(-4)81725$

$(-5)(-2)(-7)(-1)(-8)436\,0\,(-6)(-3)(-4)81725$

$436\,0 \qquad 81725$

436081725

547192836

Variations

What about signed permutations avoiding $\overline{1}2$ and $\overline{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Variations

What about signed permutations avoiding $\overline{1}2$ and $\overline{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

Variations

What about signed permutations avoiding $\bar{1}2$ and $\bar{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8) \mid 81725(-6)(-3)(-4)$

Variations

What about signed permutations avoiding $\overline{1}2$ and $\overline{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8) \mid 81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8)\,0\,81725(-6)(-3)(-4)$

Variations

What about signed permutations avoiding $\overline{1}2$ and $\overline{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8) \mid 81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8)\,0\,81725(-6)(-3)(-4)$

$436 \qquad\qquad\qquad\qquad\qquad 0\,81725$

Variations

What about signed permutations avoiding $\bar{1}2$ and $\bar{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8) \mid 81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8)\,0\,81725(-6)(-3)(-4)$

$436 \qquad\qquad 0\,81725$

436081725

Variations

What about signed permutations avoiding $\bar{1}2$ and $\bar{2}1$? That is, positive entries (if any) first and then negative entries (if any).

Example. $81725(-6)(-3)(-4)$

Again, there are $(n+1)!$ of these. The map is the same.

Example. $81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8) \mid 81725(-6)(-3)(-4)$

$436(-5)(-2)(-7)(-1)(-8)\,0\,81725(-6)(-3)(-4)$

$436 \qquad 0\,81725$

436081725

547192836

A map from signed permutations to permutations

We extend the map to all signed permutation.

Example. $(-6)81(-3)(-4)725 \mapsto 547192836$

A map from signed permutations to permutations

We extend the map to all signed permutation.

Example. $(-6)81(-3)(-4)725 \mapsto 547192836$

Surjective.

A map from signed permutations to permutations

We extend the map to all signed permutation.

Example. $(-6)81(-3)(-4)725 \mapsto 547192836$

Surjective.

Fiber (i.e. preimage) of a permutation is an interval in weak order on B_n. (Bottom and top of interval are the two types of avoiders.)

Example. Fiber of 547192836 is
$[(-6)(-3)(-4)81725, 81725(-6)(-3)(-4)]$.

Reminder: The weak order on B_n is containment of inversion sets of the extended one-line notation.

A map from signed permutations to permutations

We extend the map to all signed permutation.

Example. $(-6)81(-3)(-4)725 \mapsto 547192836$

Surjective.

Fiber (i.e. preimage) of a permutation is an interval in weak order on B_n. (Bottom and top of interval are the two types of avoiders.)

Example. Fiber of 547192836 is
$[(-6)(-3)(-4)81725, 81725(-6)(-3)(-4)]$.

Reminder: The weak order on B_n is containment of inversion sets of the extended one-line notation.

Always when we have a map from a lattice and the fibers are intervals, we should ask ourselves if the fibers are a lattice congruence and if the map is a lattice homomorphism.

Congruences and homomorphisms, combinatorially

Congruence: equivalence relation respecting meet and join.

An equivalence relation $\equiv$ on a finite lattice L is a lattice congruence if and only if the following three conditions hold:

(i) Each equivalence class is an interval in L.

(ii) The map $\pi_\downarrow$ taking each element to the bottom element of its equivalence class is order-preserving.

(iii) The map $\pi^\uparrow$ taking each element to the top element of its equivalence class is order-preserving.

Congruences and homomorphisms, combinatorially

Congruence: equivalence relation respecting meet and join.

An equivalence relation $\equiv$ on a finite lattice L is a lattice congruence if and only if the following three conditions hold:

(i) Each equivalence class is an interval in L.

(ii) The map $\pi_\downarrow$ taking each element to the bottom element of its equivalence class is order-preserving.

(iii) The map $\pi^\uparrow$ taking each element to the top element of its equivalence class is order-preserving.

Homomorphism: Map respecting meet and join.

A surjective map $\eta : L \to L'$ (finite lattices) is a lattice homomorphism if and only if the following two conditions hold:

(i) η is order-preserving.

(ii) For every interval $[x, y]$ in L', the set $\eta^{-1}([x, y])$ is an interval.

A surjective homomorphism from B_n to S_{n+1}

A surjective map $\eta : L \to L'$ (finite lattices) is a lattice homomorphism if and only if the following two conditions hold:

(i) η is order-preserving.

(ii) For every interval $[x, y]$ in L', the set $\eta^{-1}([x, y])$ is an interval.

We can pretty easily convince ourselves that our map from signed permutations to permutations satisfies these conditions.
(Aside: Simion had most of this and could have done all of it.)

A surjective homomorphism from B_n to S_{n+1}

A surjective map $\eta : L \to L'$ (finite lattices) is a lattice homomorphism if and only if the following two conditions hold:

(i) η is order-preserving.

(ii) For every interval $[x, y]$ in L', the set $\eta^{-1}([x, y])$ is an interval.

We can pretty easily convince ourselves that our map from signed permutations to permutations satisfies these conditions.
(Aside: Simion had most of this and could have done all of it.)

Conclusion: We have a surjective lattice homomorphism from the weak order on B_n to the weak order on S_{n+1}. In other words, S_{n+1} is a lattice quotient of B_n.

A surjective homomorphism from B_n to S_{n+1}

A surjective map $\eta : L \to L'$ (finite lattices) is a lattice homomorphism if and only if the following two conditions hold:

(i) η is order-preserving.

(ii) For every interval $[x, y]$ in L', the set $\eta^{-1}([x, y])$ is an interval.

We can pretty easily convince ourselves that our map from signed permutations to permutations satisfies these conditions.
(Aside: Simion had most of this and could have done all of it.)

Conclusion: We have a surjective lattice homomorphism from the weak order on B_n to the weak order on S_{n+1}. In other words, S_{n+1} is a lattice quotient of B_n.

Note: It is much easier to get a surjective lattice homomorphism from the weak order on B_n to the weak order on S_n. That is a parabolic congruence (discussed last week and again soon).

In general?

The simple combinatorics of this map might hide how surprising it is that S_{n+1} is a lattice quotient of B_n.

Are there other examples?

Can we characterize surjective lattice homomorphisms between weak orders?

Equivalently, can we characterize congruences on W such that the quotient is the weak order on some W'?

First we'll need more background.

Section 2: A lattice-theoretic approach

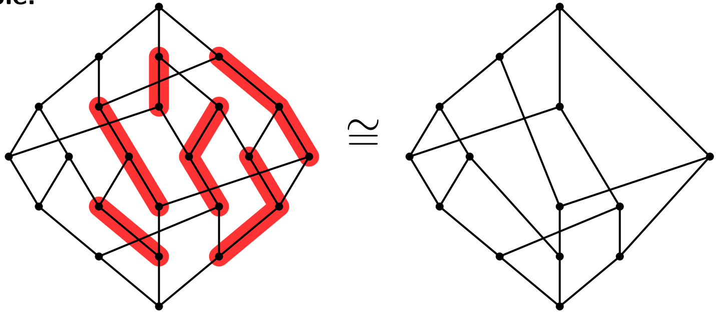
Order-theoretic characterization of a lattice quotient

We saw that congruence classes of a lattice congruence are intervals.

Lattice quotient: The natural lattice structure on the set of congruence classes.

Order-theoretically: The lattice quotient is isomorphic to the subposet induced by the bottom elements of intervals.

Example.



Contracting edges

Write $a \lessdot b$ for a cover relation.

A congruence Θ contracts the edge $a \lessdot b$ if $a \equiv b$ modulo Θ.

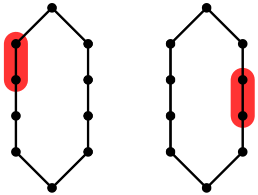
Because congruence classes are intervals, a congruence is completely determined by the edges it contracts.

As one might expect, edges cannot be contracted independently.

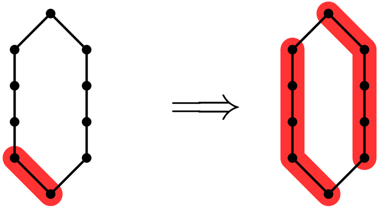
Say $a \lessdot b$ forces $c \lessdot d$ if every congruence contracting $a \lessdot b$ also contracts $c \lessdot d$.

Forcing in a polygon

A “side” edge can be contracted independently. E.g.:



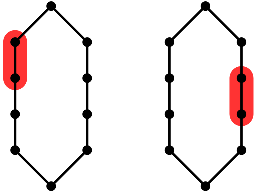
A “bottom” edge forces all side edges and the opposite “top” edge.



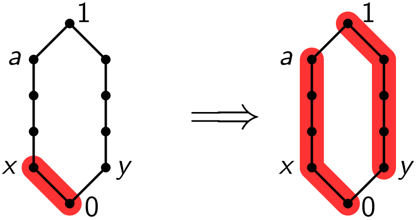
Dually, a “top” edge forces all side edges and the opposite “bottom” edge.

Forcing in a polygon

A "side" edge can be contracted independently. E.g.:



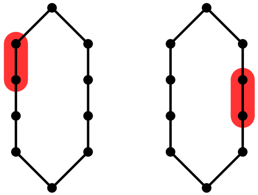
A "bottom" edge forces all side edges and the opposite "top" edge.



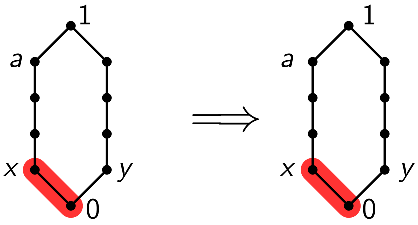
Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in a polygon

A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.

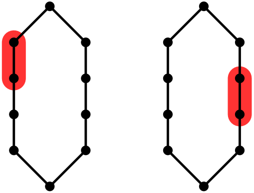


$x \equiv 0$

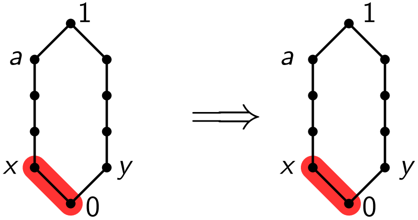
Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in a polygon

A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.

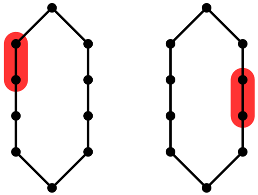


$$\begin{aligned} & x \equiv 0 \\ \Longrightarrow\ & x \vee y \equiv 0 \vee y \end{aligned}$$

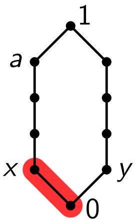
Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in a polygon

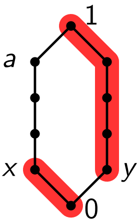
A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.



$\Longrightarrow$

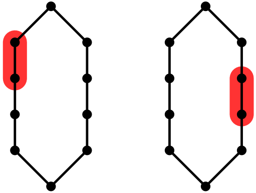


$$\begin{array}{rl} & x \equiv 0 \\ \Longrightarrow & x \vee y \equiv 0 \vee y \\ \text{i.e.} & 1 \equiv y \end{array}$$

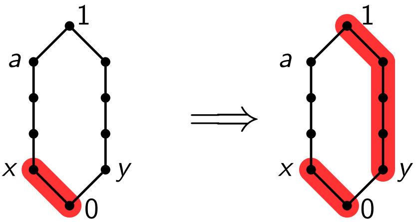
Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in a polygon

A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.

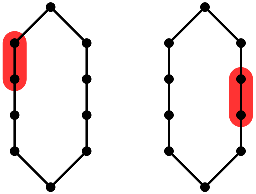


$$
\begin{array}{ll}
 & x \equiv 0 \\
\Longrightarrow & x \vee y \equiv 0 \vee y \\
\text{i.e.} & 1 \equiv y \\
\Longrightarrow & a \wedge 1 \equiv a \wedge y
\end{array}
$$

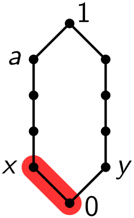
Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in a polygon

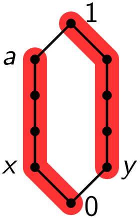
A "side" edge can be contracted independently. E.g.:



A "bottom" edge forces all side edges and the opposite "top" edge.



$\Longrightarrow$



$$
\begin{array}{rl}
 & x \equiv 0 \\
\Longrightarrow & x \vee y \equiv 0 \vee y \\
\text{i.e.} & 1 \equiv y \\
\Longrightarrow & a \wedge 1 \equiv a \wedge y \\
\text{i.e.} & a \equiv 0
\end{array}
$$

Dually, a "top" edge forces all side edges and the opposite "bottom" edge.

Forcing in the weak order

A polygon in the weak order is an interval whose Hasse diagram is a cycle.

Proposition. The forcing relation in the weak order is the transitive closure of the forcing relation in each polygon.

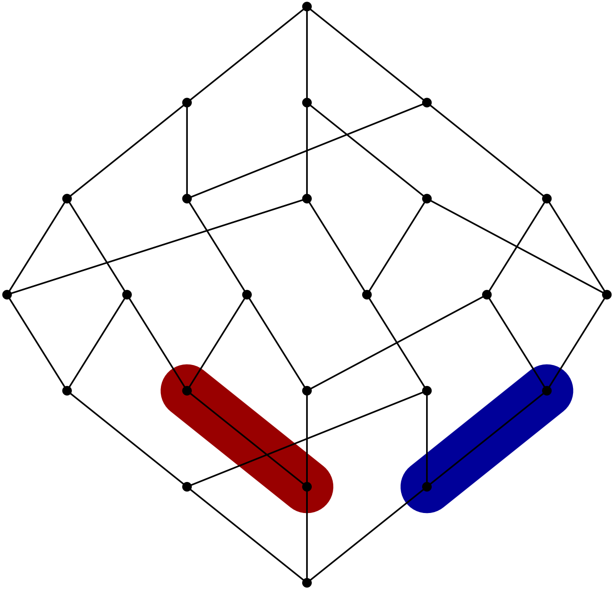
That relation was: bottom edge forces opposite top edge and all side edges (and dually, top edge forces...)

As a result, we can compute examples easily by hand.

Terminology: We'll compute the congruence generated by contracting a set of edges.

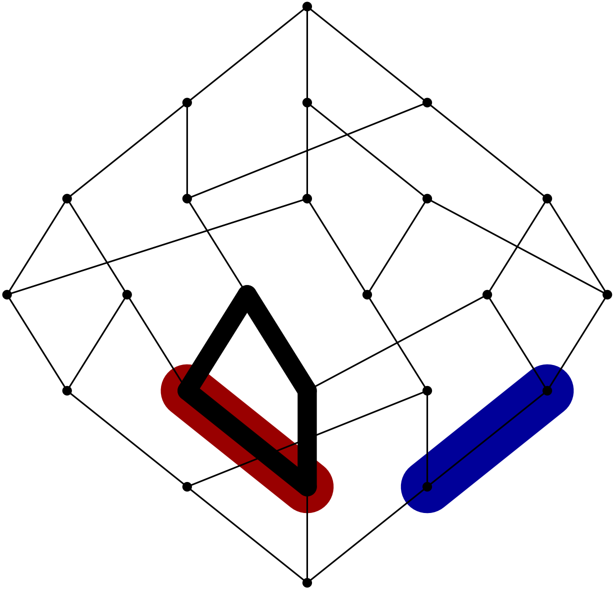
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



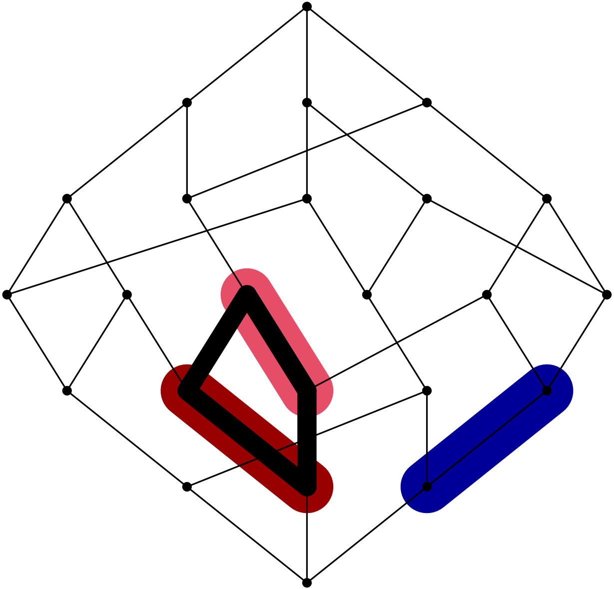
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



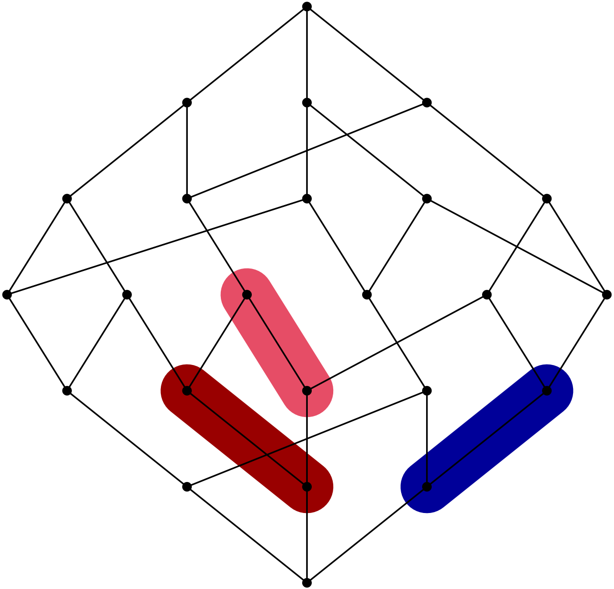
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



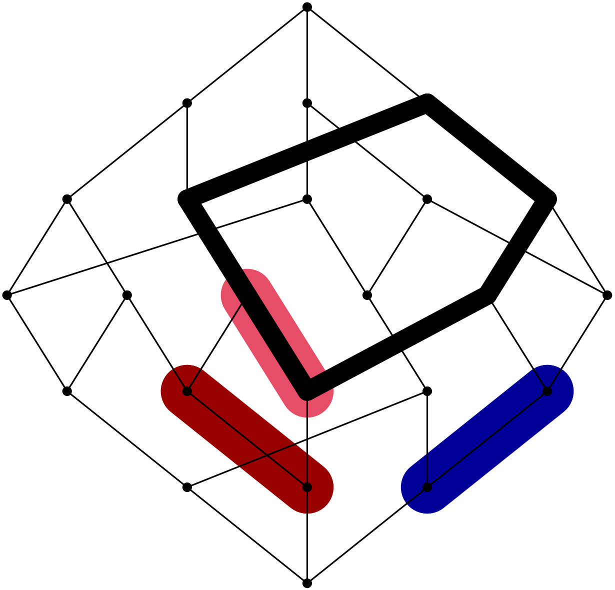
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



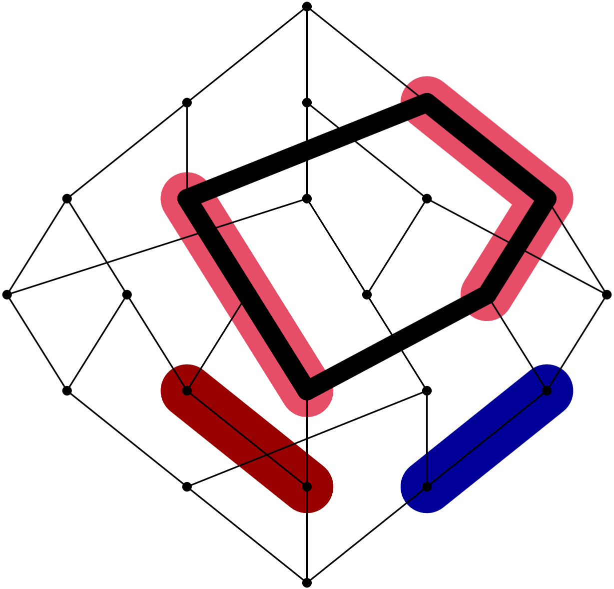
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



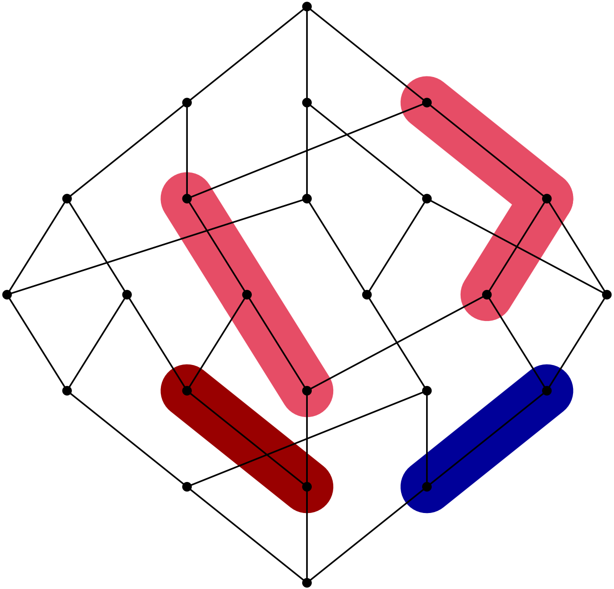
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



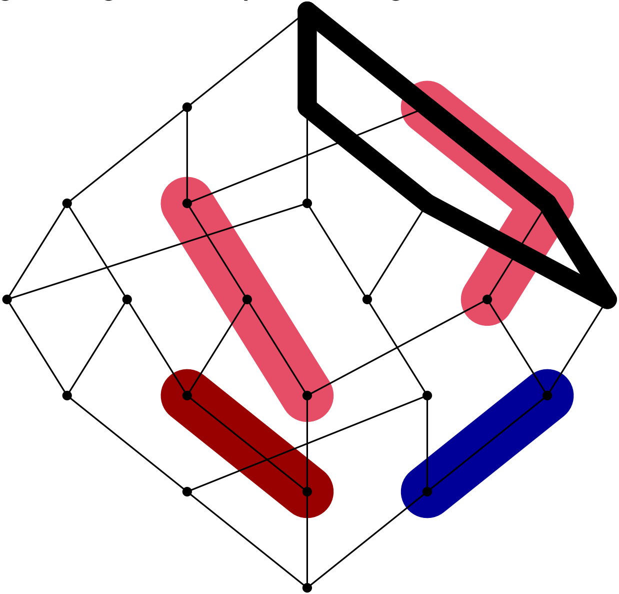
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



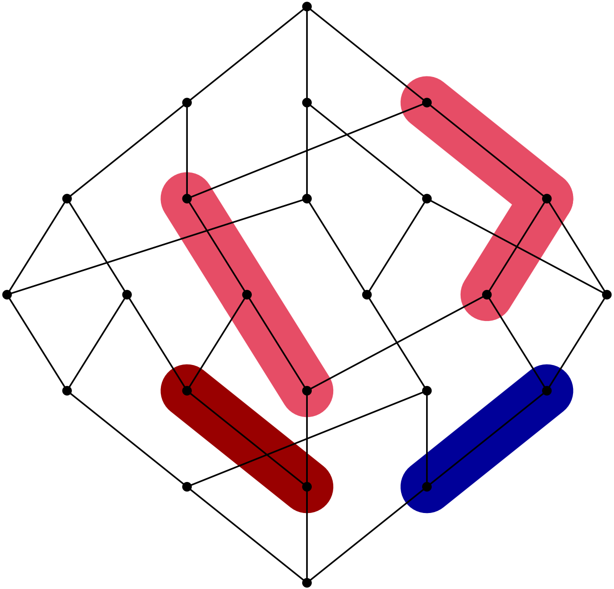
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



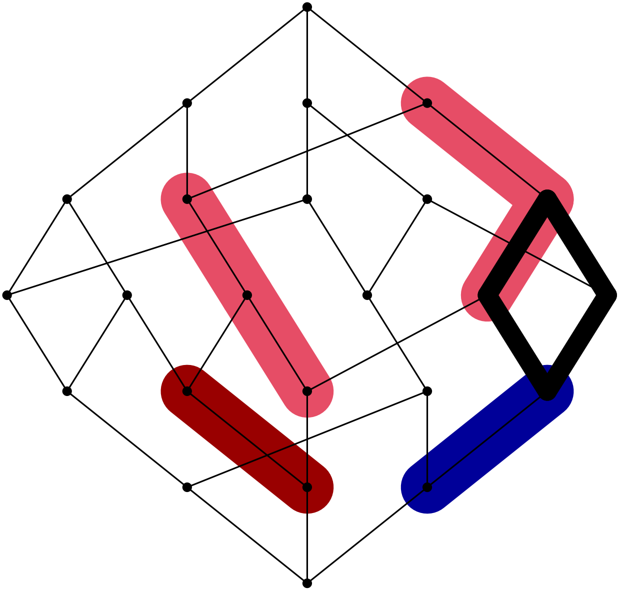
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



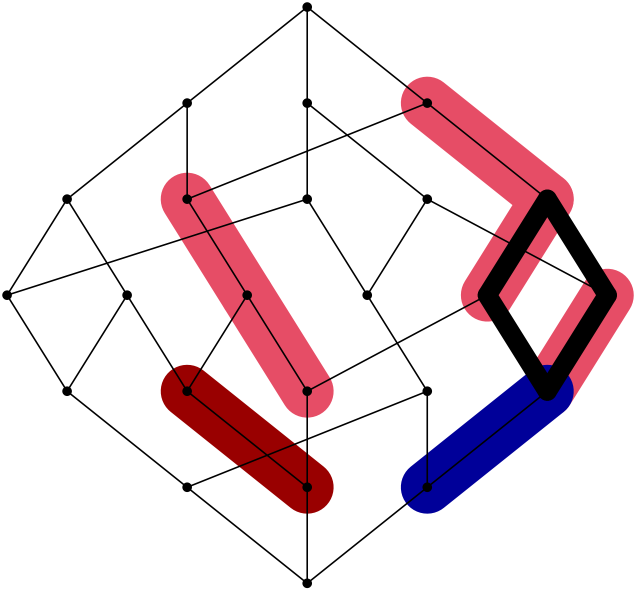
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



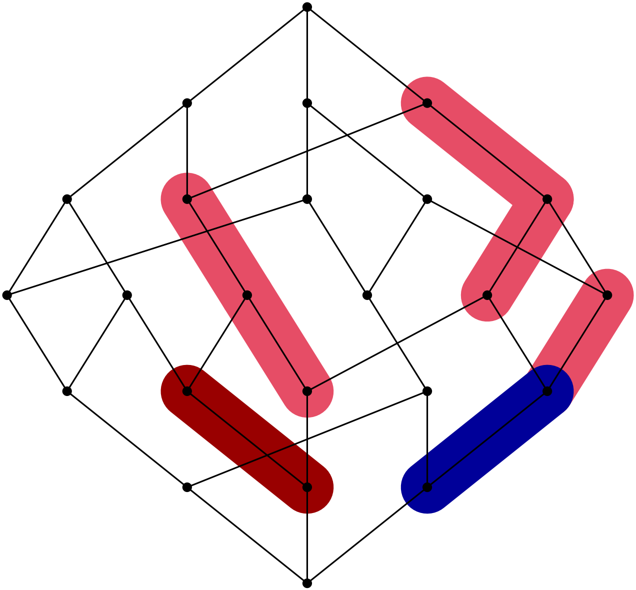
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



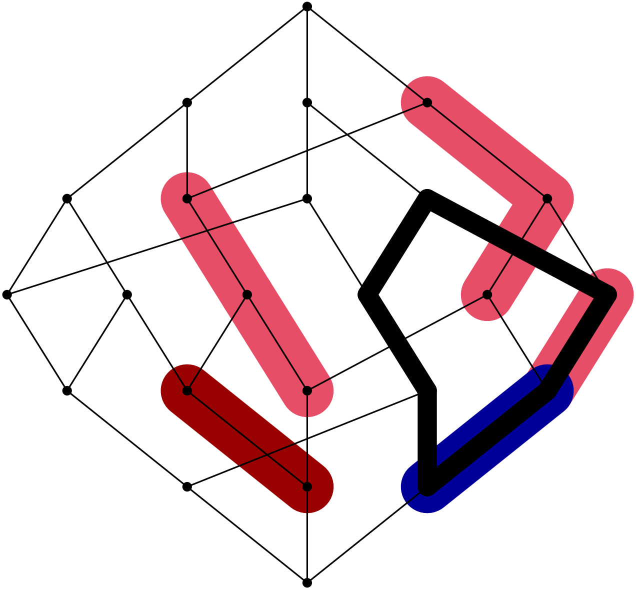
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.

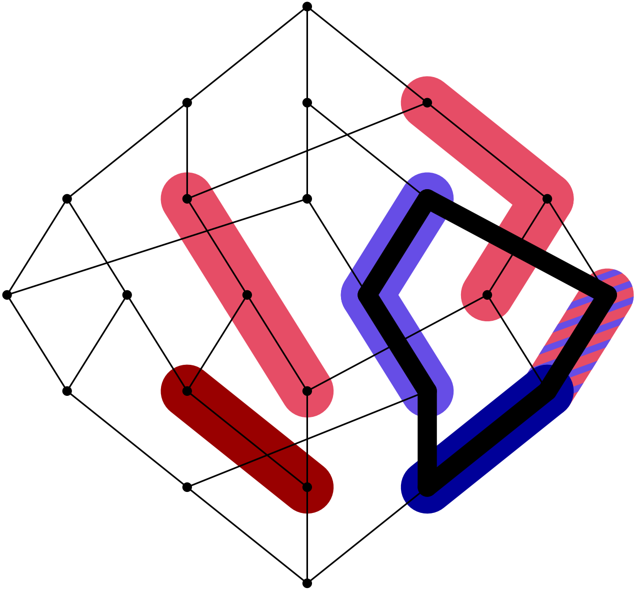


Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.

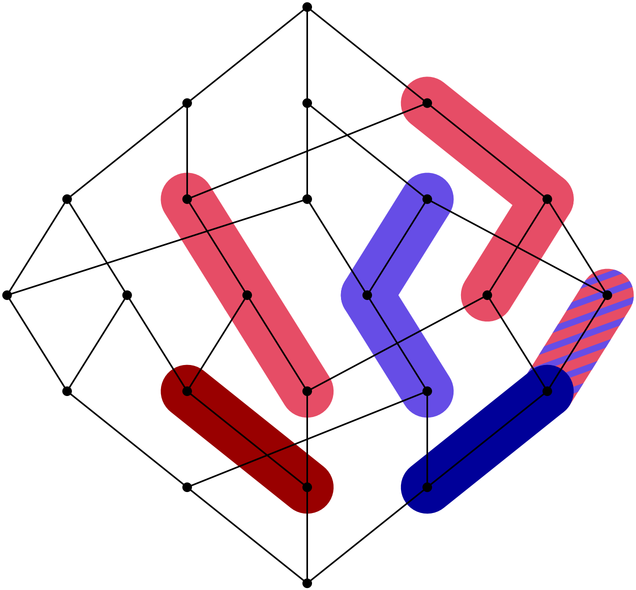


The congruence generated by contracting the **red** and **blue** edges.



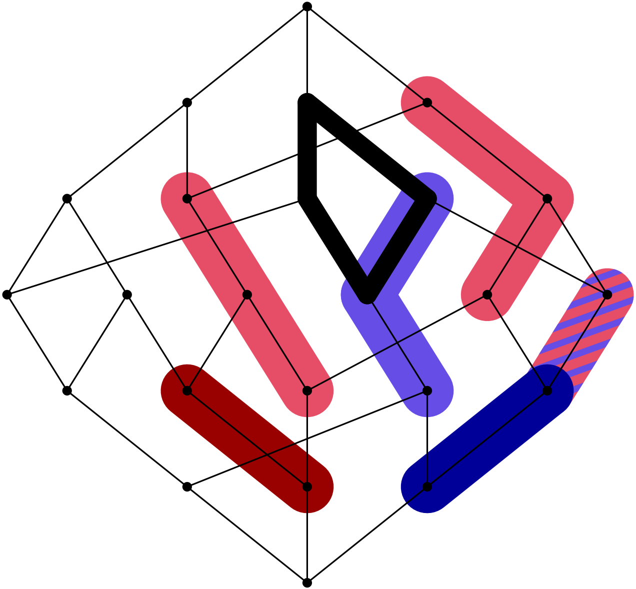
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



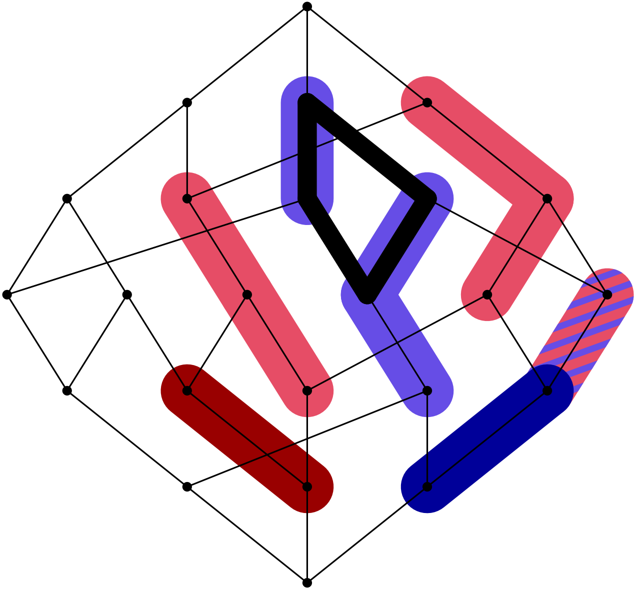
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



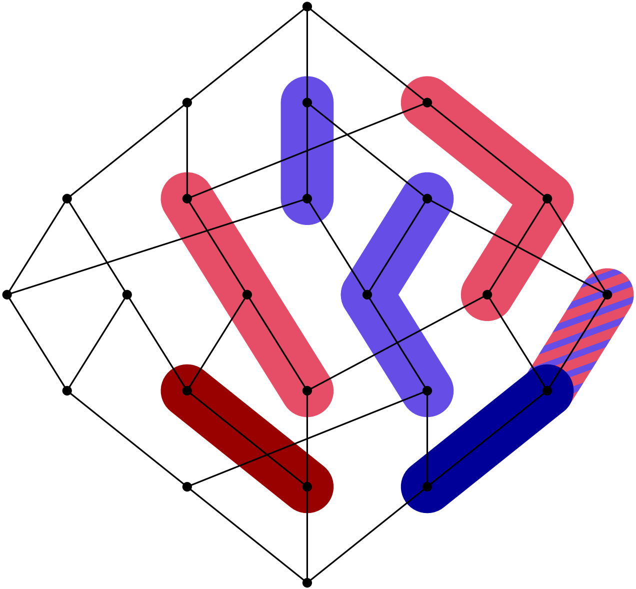
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



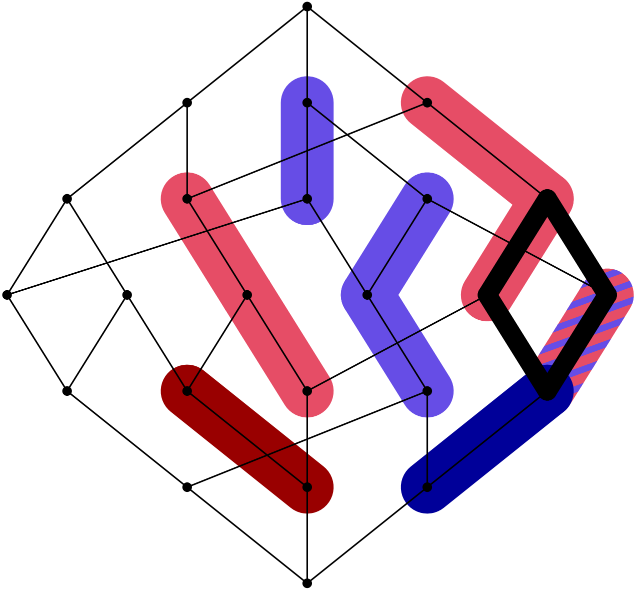
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



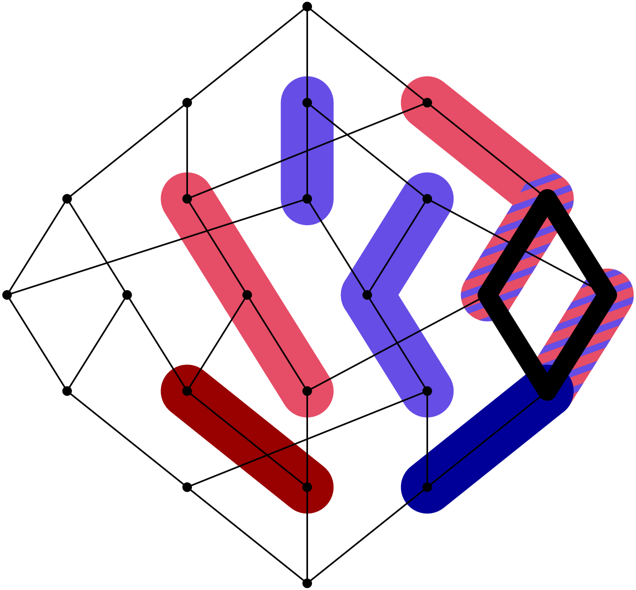
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



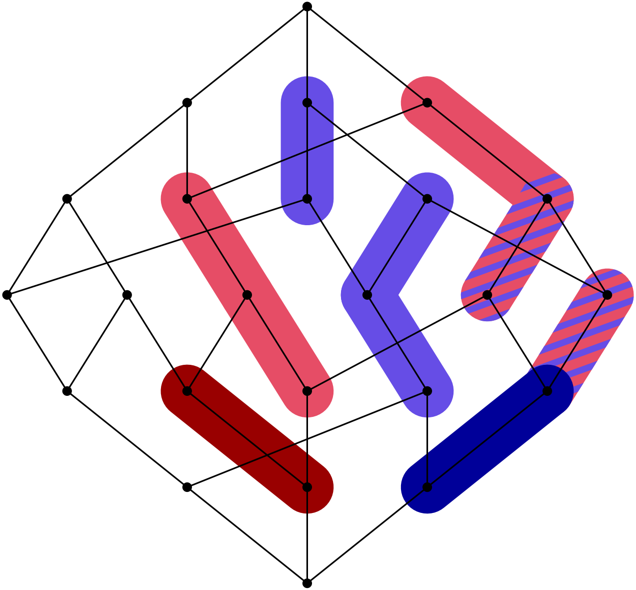
Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



Example of forcing in the weak order

The congruence generated by contracting the **red** and **blue** edges.



Homogeneous congruences

Each edge in the weak order has a degree: The rank of the smallest standard parabolic subgroup containing that edge.

Caveat: This definition has some problems, but it's good enough for this talk. If you want to think more about this, use the definition I gave in the school last week.

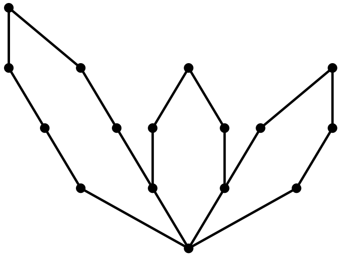
A homogeneous congruence of degree d is a congruence generated by contracting a set of edges of degree d.

For example, a homogenous congruence of degree 1 is generated by contracting edges that look like $1 \lessdot s$.

Important examples of congruences of degree 2 include Cambrian congruences, which make permutohedra into (generalized) associahedra.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

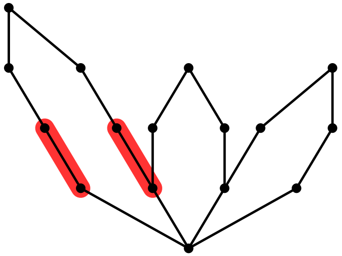


If I want to "turn this into" the weak order on A_4, somehow the octagon needs to turn into a hexagon.

I can do it by contracting edges.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

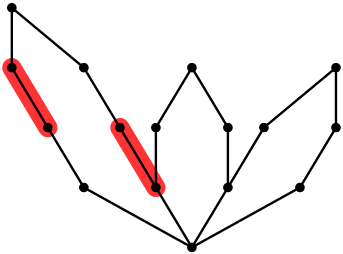


If I want to "turn this into" the weak order on A_4, somehow the octagon needs to turn into a hexagon.

I can do it by contracting edges.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

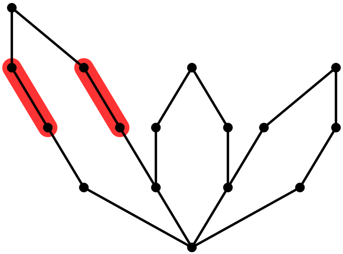


If I want to "turn this into" the weak order on A_4, somehow the octagon needs to turn into a hexagon.

I can do it by contracting edges.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

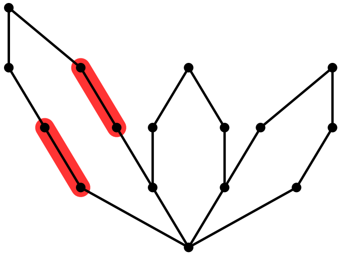


If I want to "turn this into" the weak order on A_4, somehow the octagon needs to turn into a hexagon.

I can do it by contracting edges.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

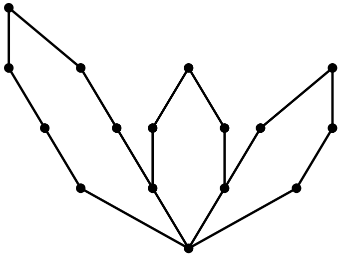


If I want to "turn this into" the weak order on A_4, somehow the octagon needs to turn into a hexagon.

I can do it by contracting edges.

Looking for congruences

The weak order contains, at the bottom, the Coxeter diagram: Vertices are atoms and edges are polygons:

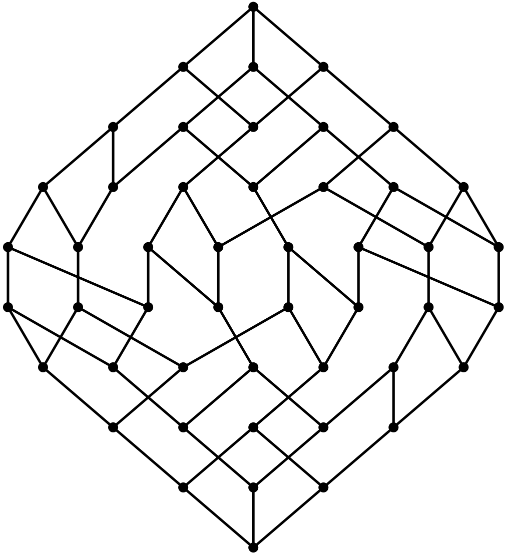


If I want to “turn this into” the weak order on A_4, somehow the octagon needs to turn into a hexagon.

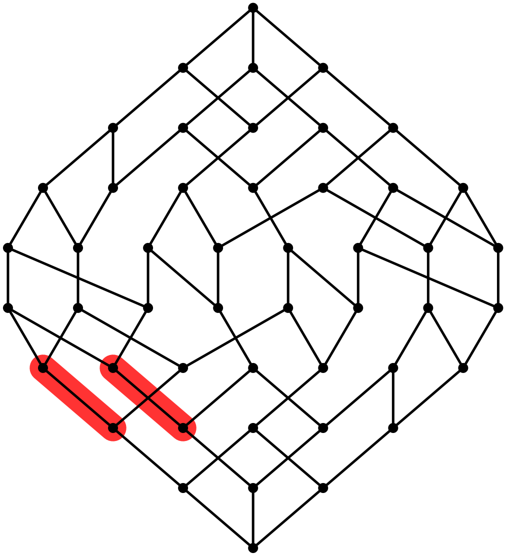
I can do it by contracting edges.

Let’s try it (in lower rank).

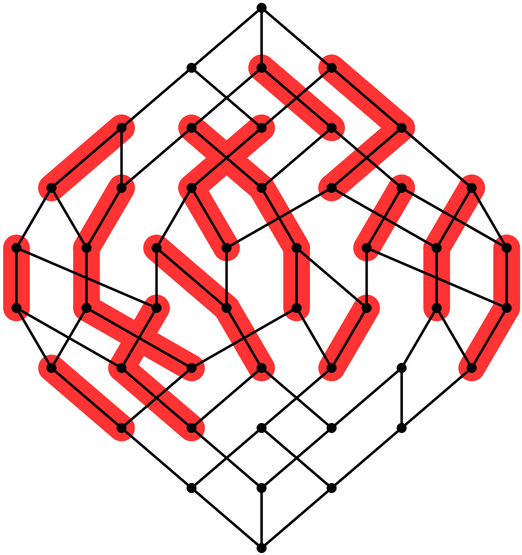
Looking for congruences in rank 3



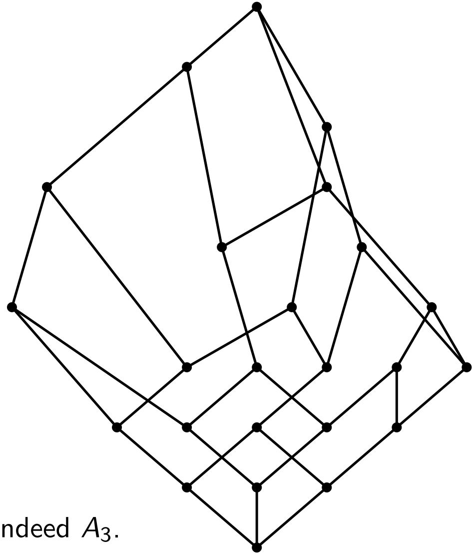
Looking for congruences in rank 3



Looking for congruences in rank 3



Looking for congruences in rank 3



Quotient is indeed A_3.

Looking for congruences in rank 3—Time out

Let's stop and think about this:

We looked at the polygons at the bottom of weak order that encode the Coxeter diagram of B_3.

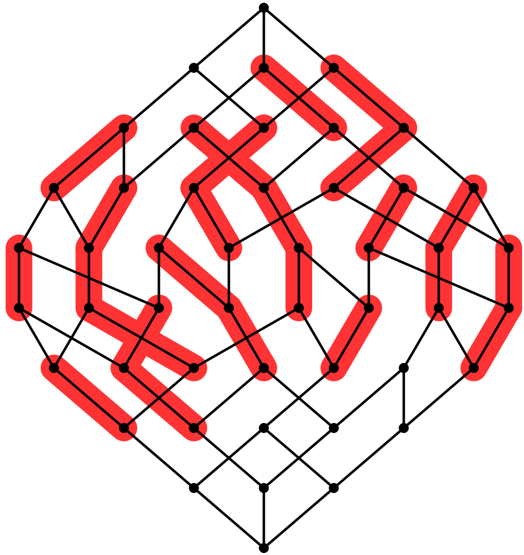
We contracted two side edges to turn the octagon (label 4) into a hexagon (label 3).

Naïvely contracting these edges generates a congruence whose quotient is A_3.

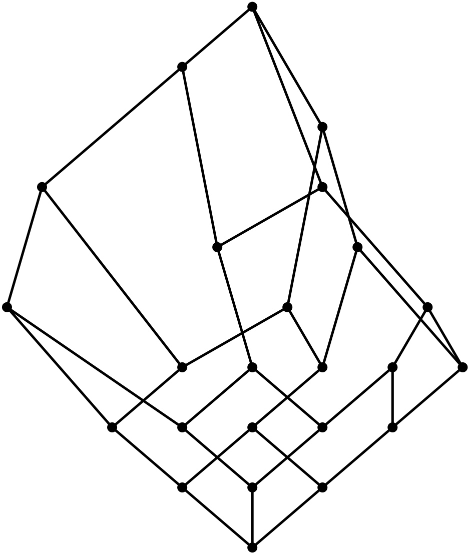
On the one hand, this is surprising. Why should this naïve approach work.

On the other hand, this is consistent with our expectation that constructions in Coxeter groups are determined in rank 2.

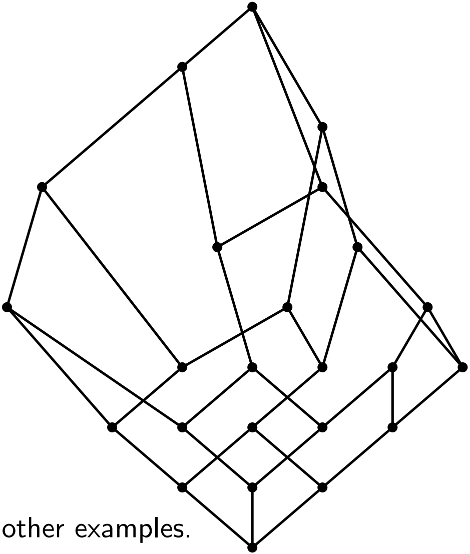
Looking for congruences in rank 3



Looking for congruences in rank 3

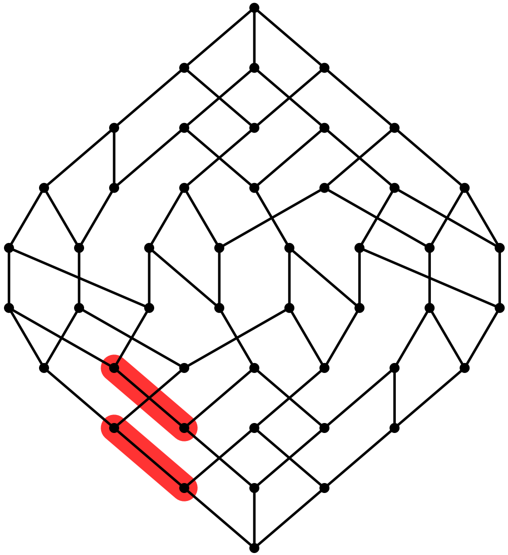


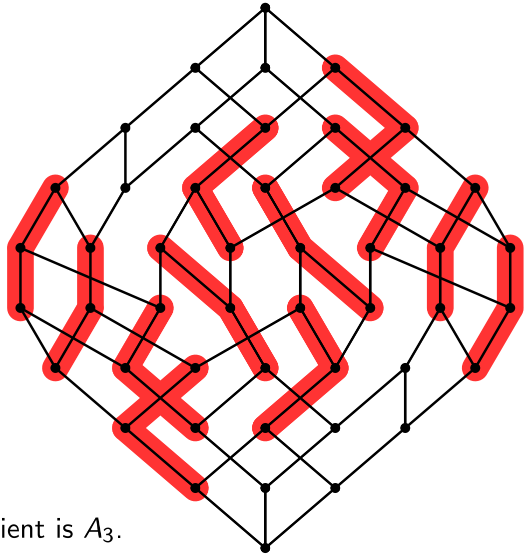
Looking for congruences in rank 3



Let's look at the other examples.

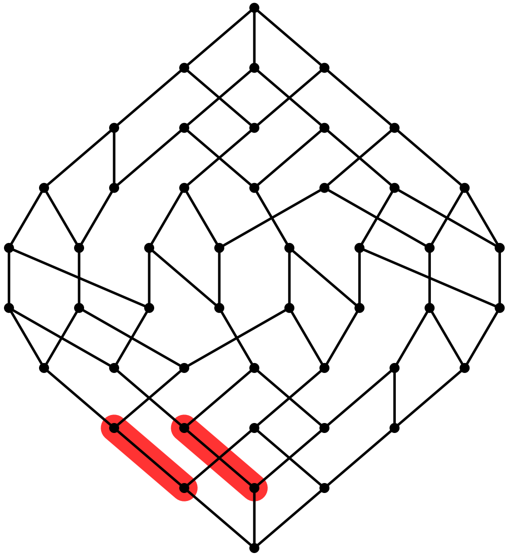
Looking for congruences in rank 3



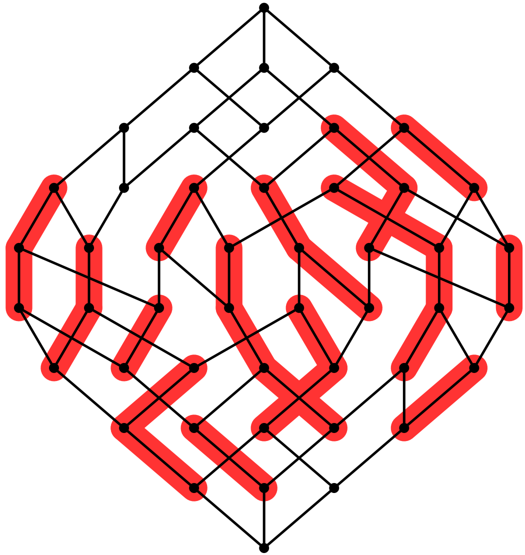


Again, quotient is A_3.

Looking for congruences in rank 3

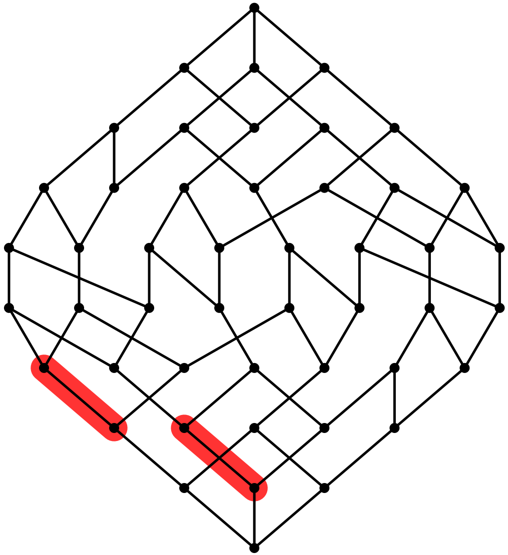


Looking for congruences in rank 3

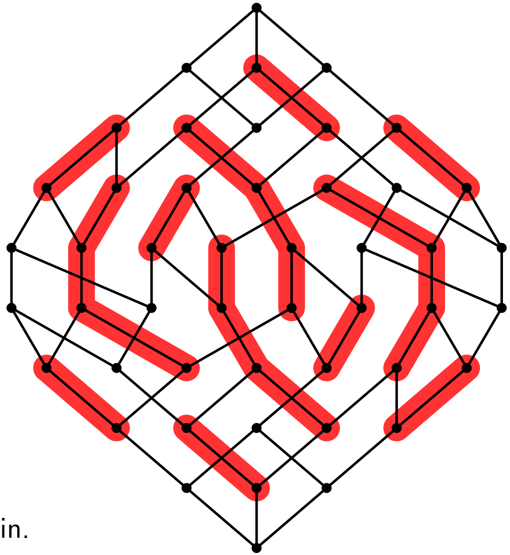


And again.

Looking for congruences in rank 3

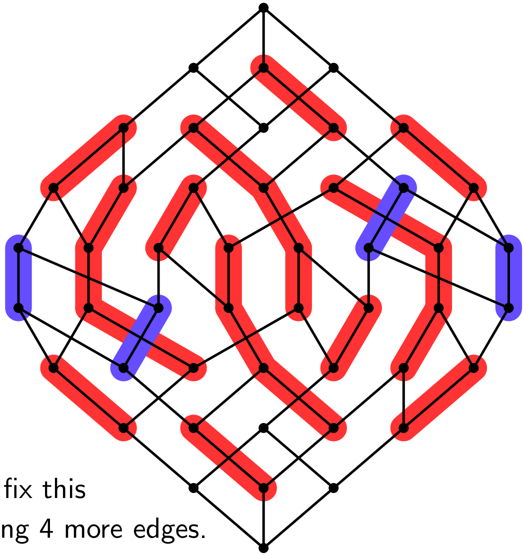


Looking for congruences in rank 3



And not again.

Looking for congruences in rank 3



But we can fix this
by contracting 4 more edges.

Summary of the B_3-to-A_3 examples

All of these are theorems for general B_n-to-A_n.

There are 4 ways to contract side edges of the octagon to get a hexagon.

In 3 of 4 cases, contracting those edges generates a congruence whose quotient is A_3. Thus these are homogeneous congruences of degree 2.

In the remaining case, there are 4 additional edges to contract, but then we still get A_3. This is a non-homogeneous congruence of degree 3. For larger n, the congruence is generated by contracting these edges in the B_3 standard parabolic subgroup of B_n.

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Example. $(-6)81(-3)(-4)725$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Example. $(-6)81(-3)(-4)725$

$(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Example. $(-6)81(-3)(-4)725$

$$\begin{array}{c|c} (-5)(-2)(-7)43(-1)(-8)6 & (-6)81(-3)(-4)725 \\ 43(-1)\quad 6 & 81 \quad 725 \end{array}$$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Example. $(-6)81(-3)(-4)725$

$$
\begin{array}{cccc|cc}
(-5)(-2)(-7)43(-1)(-8)6 & & & & (-6)81(-3)(-4)725 & \\
43(-1) & 6 & & & 81 & 725 \\
\end{array}
$$

$$43(-1)681725$$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Change -1 to 0.

Example. $(-6)81(-3)(-4)725$

$$(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$$

$$43(-1) \quad 6 \quad 81 \quad 725$$

$$43(-1)681725$$

$$430681725$$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Change -1 to 0.

Add 1 to each entry

Example. $(-6)81(-3)(-4)725$

$$(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$$

$$43(-1) \quad 6 \quad 81 \quad 725$$

$$43(-1)681725$$

$$430681725$$

$$541792836$$

Explicit homomorphisms

Simion's map (insert 0, take nonnegative entries, add 1 to each) $\longleftrightarrow$ one of the homogeneous congruences. The homomorphism for the non-homogeneous congruence also has nice combinatorics:

Extract the subsequence of the extended one-line notation consiting of entries ≥ -1.

Change -1 to 0.

Add 1 to each entry

Example. $(-6)81(-3)(-4)725$

$$(-5)(-2)(-7)43(-1)(-8)6 \mid (-6)81(-3)(-4)725$$

$$43(-1) \quad 6 \quad 81 \quad 725$$

$$43(-1)681725$$

$$430681725$$

$$541792836$$

The other two homomorphisms are "hybrids."

Section 3: General theorems

Deleting vertices

Parabolic homomorphism:
Given $J \subseteq S$, let η_J map w to w_J, where w factors as $w_J \cdot {}^J w$.
(If you like: w_J is the largest element of W_J below w.)

Given Coxeter systems (W, S) and (W', S'),

Theorem. Let $\eta : W \to W'$ be a surjective lattice homomorphism and let $J \subseteq S$ be $\{s \in S : \eta(s) \neq 1'\}$. Then η factors as $\eta|_{W_J} \circ \eta_J$, where $\eta|_{W_J}$ (the restriction of η to W_J) restricts to a bijection from J to S'.

That is: η “deletes” vertices from the diagram and then ...

Deleting vertices

Parabolic homomorphism:
Given $J \subseteq S$, let η_J map w to w_J, where w factors as $w_J \cdot {}^J w$.
(If you like: w_J is the largest element of W_J below w.)

Given Coxeter systems (W, S) and (W', S'),

Theorem. Let $\eta : W \to W'$ be a surjective lattice homomorphism and let $J \subseteq S$ be $\{s \in S : \eta(s) \neq 1'\}$. Then η factors as $\eta|_{W_J} \circ \eta_J$, where $\eta|_{W_J}$ (the restriction of η to W_J) restricts to a bijection from J to S'.

That is: η "deletes" vertices from the diagram and then ... does something else.

Proof idea: E.g. for $J = S \setminus \{s\}$, the congruence associated to η_J is the smallest congruence with identity $\equiv s$.

Deleting vertices

Parabolic homomorphism:
Given $J \subseteq S$, let η_J map w to w_J, where w factors as $w_J \cdot {}^J w$.
(If you like: w_J is the largest element of W_J below w.)

Given Coxeter systems (W, S) and (W', S'),

Theorem. Let $\eta : W \to W'$ be a surjective lattice homomorphism and let $J \subseteq S$ be $\{s \in S : \eta(s) \neq 1'\}$. Then η factors as $\eta|_{W_J} \circ \eta_J$, where $\eta|_{W_J}$ (the restriction of η to W_J) restricts to a bijection from J to S'.

That is: η "deletes" vertices from the diagram and then ... does something else.

Proof idea: E.g. for $J = S \setminus \{s\}$, the congruence associated to η_J is the smallest congruence with identity $\equiv s$. Proof idea about to disappear.

Deleting vertices

Parabolic homomorphism:
Given $J \subseteq S$, let η_J map w to w_J, where w factors as $w_J \cdot {}^J w$.
(If you like: w_J is the largest element of W_J below w.)

Given Coxeter systems (W, S) and (W', S'),

Theorem. Let $\eta : W \to W'$ be a surjective lattice homomorphism and let $J \subseteq S$ be $\{s \in S : \eta(s) \neq 1'\}$. Then η factors as $\eta|_{W_J} \circ \eta_J$, where $\eta|_{W_J}$ (the restriction of η to W_J) restricts to a bijection from J to S'.

That is: η "deletes" vertices from the diagram and then . . . does something else.

To see what "something else" is, we reduce to the case where $S = S'$ and η fixes S (for the rest of the talk).

Surjective homomorphisms fixing S

Theorem. Suppose (W, S) and (W', S) are finite Coxeter systems. Then there exists a surjective lattice homomorphism from W to W' (fixing S) if and only if $m'(r, s) \leq m(r, s)$ for each pair $r, s \in S$. If so, then the homomorphism can be chosen so that the associated congruence on W is homogeneous of degree 2.

Proof idea: The "only if" direction (easy): The map fixes S, and so can't make the rank-2 standard parabolics (polygons at the bottom) larger, only smaller (by contracting their side edges).

The "if" direction: Choose side edges to contract and use forcing (geometrized via "shards") to generate congruences. Verify that (possibly after contracting more edges), the quotient is W'. Verify that $\exists$ a choice of side edges to contract s.t. no additional edges need be contracted (so the congruence is homog. of degree 2).

Uniform proof for "erasing edges," then case-by-case, with computations in the remaining exceptional types F_4, H_3, and H_4.

Surjective homomorphisms fixing S (continued)

Theorem. Let W and W' be finite Coxeter groups. A surjective homomorphism from the weak order on W to the weak order on W' is determined entirely by its restrictions to rank-two standard parabolic subgroups.

Proof idea: This is an extension of the previous proof. One checks that, once one chooses which "side edges of polygons" to contract in each rank-2 parabolic subgroup, if the congruence is not already correct, there is at most one way to contract additional edges so that the quotient is W'.

Surjective homomorphisms fixing S (continued)

Theorem. Let (W, S) and (W', S) be finite Coxeter systems with $m'(r,s) \leq m(r,s)$ for each pair $r, s \in S$. For each $r, s \in S$, fix a surjective homomorphism $\eta_{r,s}$ from $W_{\{r,s\}}$ to $W'_{\{r,s\}}$ with $\eta_{r,s}(r) = r$ and $\eta_{r,s}(s) = s$. Then there is exactly one diagram homomorphism $\eta : W \to W'$ such that the restriction of η to $W_{\{r,s\}}$ equals $\eta_{r,s}$ for each pair $r, s \in S$.

Surjective homomorphisms fixing S (continued)

Almost Theorem. Let (W, S) and (W', S) be finite Coxeter systems with $m'(r,s) \le m(r,s)$ for each pair $r, s \in S$. For each $r, s \in S$, fix a surjective homomorphism $\eta_{r,s}$ from $W_{\{r,s\}}$ to $W'_{\{r,s\}}$ with $\eta_{r,s}(r) = r$ and $\eta_{r,s}(s) = s$. Then there is exactly one diagram homomorphism $\eta : W \to W'$ such that the restriction of η to $W_{\{r,s\}}$ equals $\eta_{r,s}$ for each pair $r, s \in S$.

In fact, as stated, the theorem is false, but fails in essentially one case: Going from type H_n to type B_n, one of the nine ways to choose side edges produces a bad congruence. (No matter what additional edges you contract, you can't get the quotient to be B_n.)

Surjective homomorphisms fixing S (continued)

Theorem. Let (W, S) and (W', S) be finite crystallographic Coxeter systems with $m'(r,s) \le m(r,s)$ for each pair $r,s \in S$. For each $r,s \in S$, fix a surjective homomorphism $\eta_{r,s}$ from $W_{\{r,s\}}$ to $W'_{\{r,s\}}$ with $\eta_{r,s}(r) = r$ and $\eta_{r,s}(s) = s$. Then there is exactly one diagram homomorphism $\eta : W \to W'$ such that the restriction of η to $W_{\{r,s\}}$ equals $\eta_{r,s}$ for each pair $r,s \in S$.

(Without crystallography) the theorem fails in essentially one case: Going from type H_n to type B_n, one of the nine ways to choose side edges produces a bad congruence. (No matter what additional edges you contract, you can't get the quotient to be B_n.)

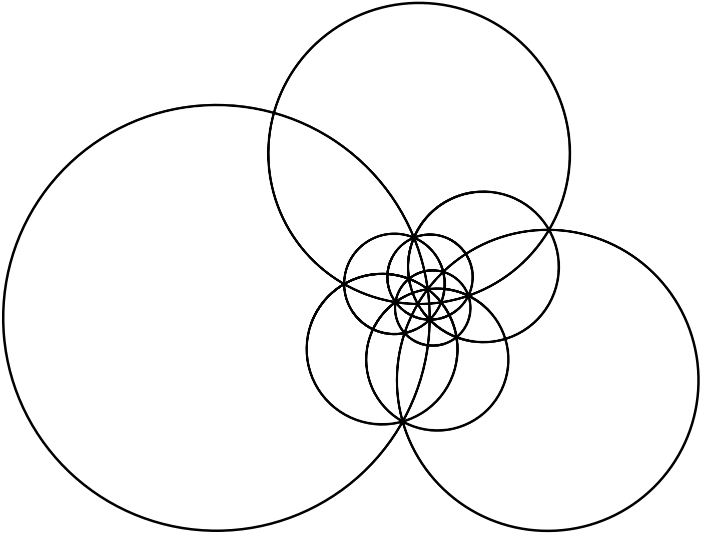
The theorem is true if we restrict W to be crystallographic. Given that we prove the theorem case-by-case and don't use crystallography, this seems artificial, but does give a uniform "existence and uniqueness" statement. $\exists$ a crystallographic proof?

Coarsening the Coxeter fans

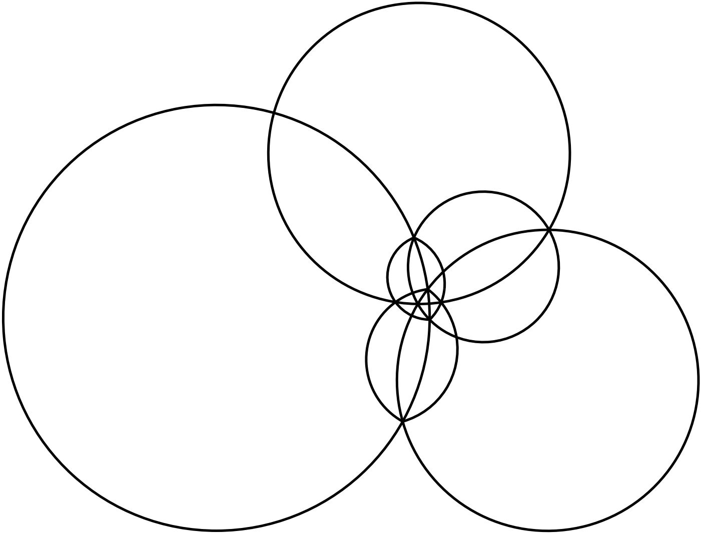
Each lattice congruence defines a complete fan that coarsens the Coxeter fan. (Combine maximal cones of the Coxeter fan according to congruence classes.)

The existence of surjective homomorphisms between different finite Coxeter groups implies Coarsening relations among different Coxeter fans.

Example: Coarsening the Coxeter fan for B_3 to get A_3



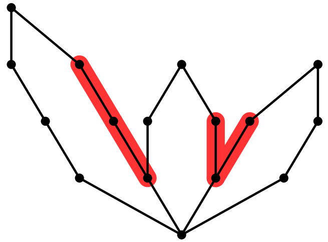
Example: Coarsening the Coxeter fan for B_3 to get A_3



Section 4: Homomorphisms between Cambrian lattices

Cambrian lattices

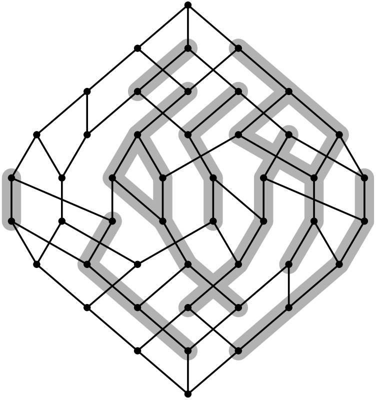
A Cambrian congruence is generated by contracting all edges on one side of each polygon at the bottom of weak order.



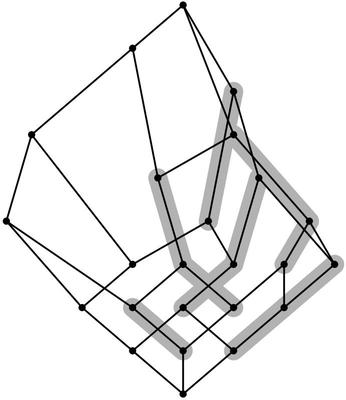
A Cambrian lattice is the quotient of the weak order on W modulo a Cambrian congruence. Its Hasse diagram is the graph of a generalized associahedron, AKA the exchange graph of a cluster algebra of finite type.

Classification of surjective homomorphisms between weak orders $\Longrightarrow$ classification of surjective hom.s between Cambrian lattices.

B_3 and A_3

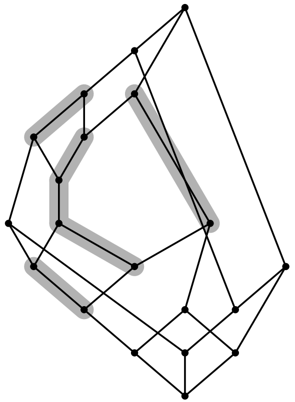


B_3



A_3

B_3 and A_3

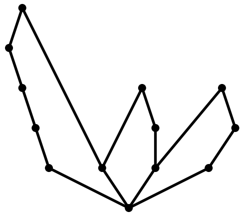


Surjective homomorphisms between Cambrian lattices

The classification of surjective homomorphisms between Cambrian lattices is more uniform.

When we pass from weak order to Cambrian lattice, each "polygon at the bottom" only has side edges on one side.

To decrease $m(r, s)$, we can contract these side edges arbitrarily. These contractions generate the right congruence, with no additional contractions needed (and no restriction to the crystallographic case).



Dominance and coarsening of Cambrian fans

A Cartan matrix A dominates A' if corresponding entries of A' have weakly smaller absolute value.

Example. B_3 and A_3.

Root system for A' is a subset of root system for A.

Dominance and coarsening of Cambrian fans

A Cartan matrix A dominates A' if corresponding entries of A' have weakly smaller absolute value.

Example. B_3 and A_3.

Root system for A' is a subset of root system for A when you identify the simple roots.

Dominance and coarsening of Cambrian fans

A Cartan matrix A dominates A' if corresponding entries of A' have weakly smaller absolute value.

Example. B_3 and A_3.

Root system for A' is a subset of root system for A when you identify the simple roots.

Cambrian fan: Coarsen Coxeter fan according to a Cambrian congruence.

Cambrian fan for (A', c) is obtained from Cambrian fan for (A, c) by removing walls orthogonal to A-roots that are not A'-roots. This implies a relationship among **g**-vector fans, among scattering diagrams, cluster algebras, etc. These relationships seem to be more general than the context ("finite type") where the lattice theory makes sense.

Dominance and coarsening of Cambrian fans

A Cartan matrix A dominates A' if corresponding entries of A' have weakly smaller absolute value.

Example. B_3 and A_3.

Root system for A' is a subset of root system for A when you identify the simple roots.

Cambrian fan: Coarsen Coxeter fan according to a Cambrian congruence.

Cambrian fan for (A', c) is obtained from Cambrian fan for (A, c) by removing walls orthogonal to A-roots that are not A'-roots. This implies a relationship among **g**-vector fans, among scattering diagrams, cluster algebras, etc. These relationships seem to be more general than the context ("finite type") where the lattice theory makes sense. Lattice congruences seem to "know" a lot of combinatorics and algebra.

Edge-erasing congruences in type A_n

Erasing the edge s_k—s_{k+1} in S_n:

$\pi_1\pi_2\cdots\pi_{n+1}$ maps to $(\sigma, \tau) \in S_{k+1} \times S_{n-k+1}$.

σ is the restriction of $\pi_1\pi_2\cdots\pi_{n+1}$ to values $\leq k+1$.

τ is obtained by restricting $\pi_1\pi_2\cdots\pi_{n+1}$ to values $\geq k+1$ and then subtracting k from each value.

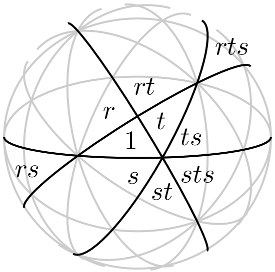
Example. ($n = 7$ and $k = 3$): $58371426 \mapsto (3142, 25413)$.

Bottom elements of the congruence: Permutations in S_{n+1} with no descents ba such that $b > k+1$ and $a < k+1$. These are thus in bijection with $S_{k+1} \times S_{n-k+1}$.

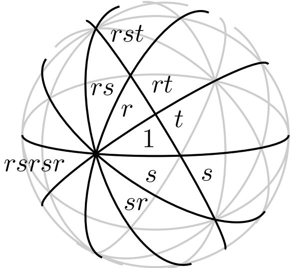
There is a similar description for type B_n.

Edge-erasing congruences in type H_3

$$r \overset{5}{\text{——}} s \text{——} t$$



Erase $r \overset{5}{\text{——}} s$



Erase $s \text{——} t$