Lecture 3A: Generalized associahedra

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Cluster Algebras and Cluster Combinatorics
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Introduction

In this lecture, we'll briefly sketch the story of generalized associahedra, which model cluster patterns of finite type. We'll mostly review results of Fomin and Zelevinsky in (GA) and (CA II)

Indeed, generalized associahedra contribute to a complete classification of cluster algebras of finite type. They also connect cluster algebras of finite type to a collection of finite combinatorial constructions which are studied under the heading of "W-Catalan combinatorics." For more on these connections, see (RSGA).

Root systems are at the heart of generalized associahedra. This is part of the pleasant surprise: the "classical" associahedra were not obviously related to root systems.

Triangulations of a convex polygon

A triangulation of a convex polygon is a decomposition of the polygon into disjoint triangles all of whose vertices are vertices of the polygon.

Equivalently, a triangulation is a maximal collection of noncrossing diagonals of the polygon.

There are

- 2 triangulations of a convex quadrilateral,
- 5 triangulations of a pentagon, and
- 14 triangulations of a hexagon.

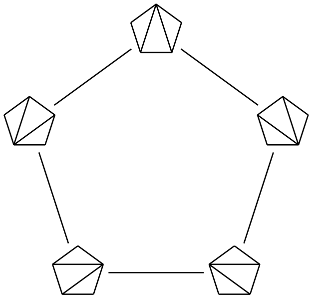
For an $(n+3)$-gon, the number of triangulations is the Catalan number

$$\frac{1}{n+2}\binom{2n+2}{n+1}.$$

Diagonal flips

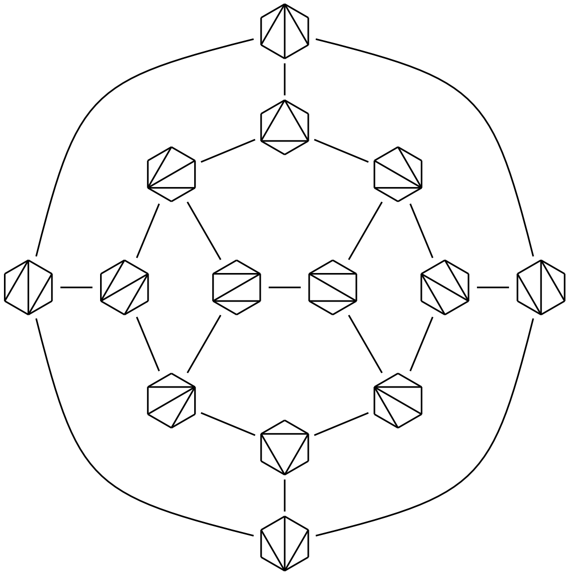
We can organize the triangulations as the vertices of a graph, with edges given by diagonal flips. A flip removes a diagonal to create a quadrilateral, then replaces the removed diagonal with the other diagonal of the quadrilateral.

For example, here is the graph for $n = 2$:



Diagonal flips for $n = 3$

For $n = 3$:



The simplicial associahedron

The graph is connected, although this must be argued. (Much, much more is true.)

There is a natural connection on this graph, in the sense of Lecture 2:

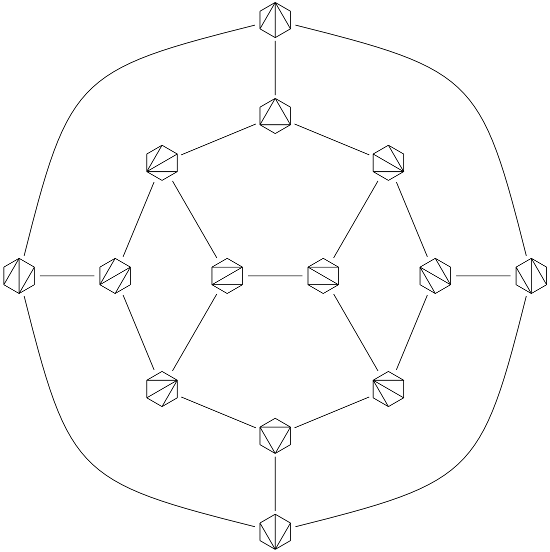
Every edge f connecting two vertices T and T' is equipped with a canonical bijection between the n edges incident to T and the n edges incident to T', fixing f.

The edges incident to T are associated to the diagonals defining T. All of these diagonals, also exist in T', except the diagonal flipped along the edge f. That's the connection.

Since there is a connection, we can define a simplicial complex as before. This complex is the simplicial associahedron.

This can be realized as a complex whose vertices are the diagonals of the polygon. (Can check: Triangulations containing a given diagonal form a connected subgraph.)

The simplicial associahedron for $n = 3$



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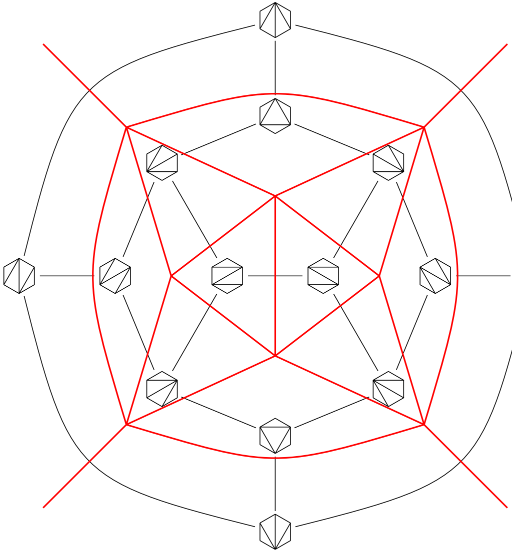

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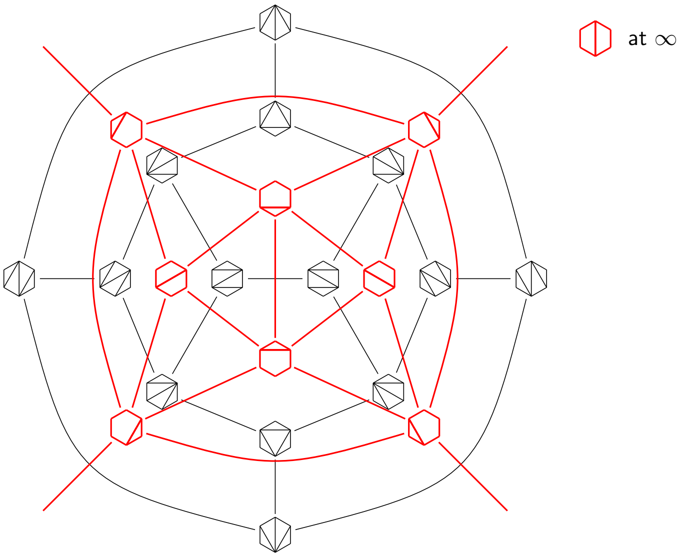

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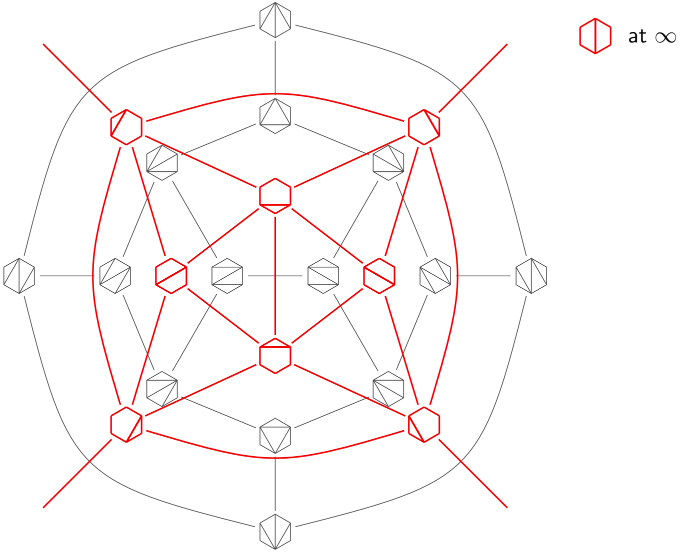

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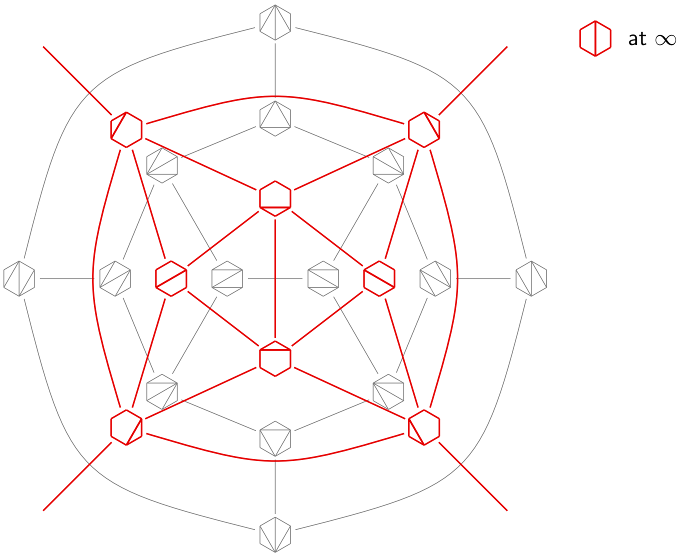

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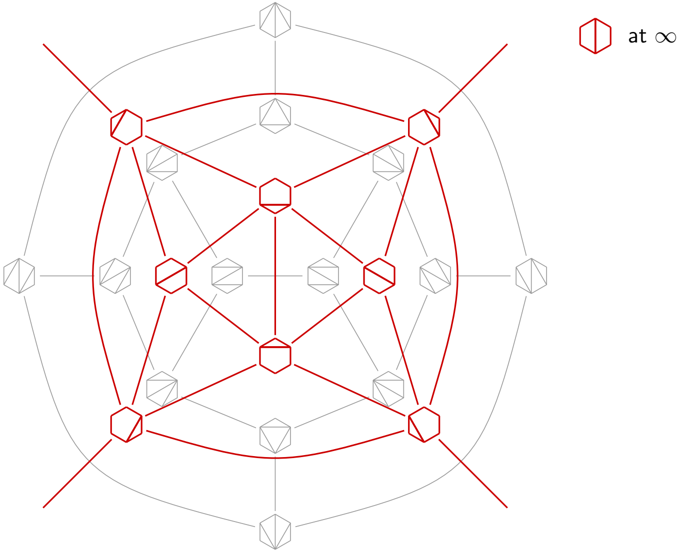

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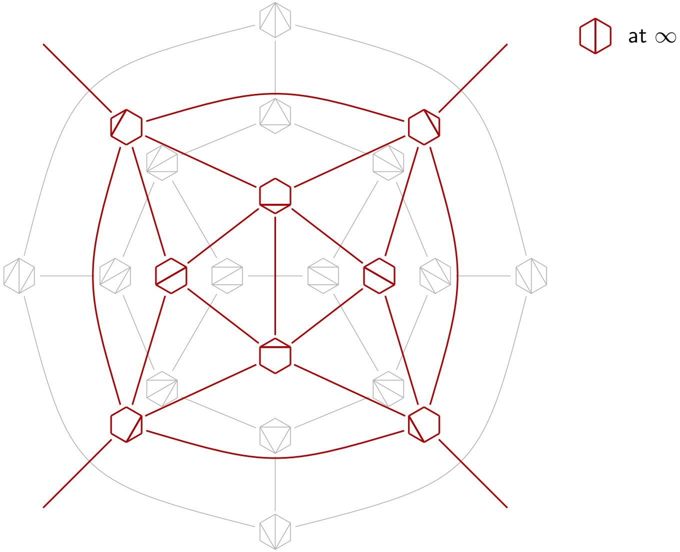

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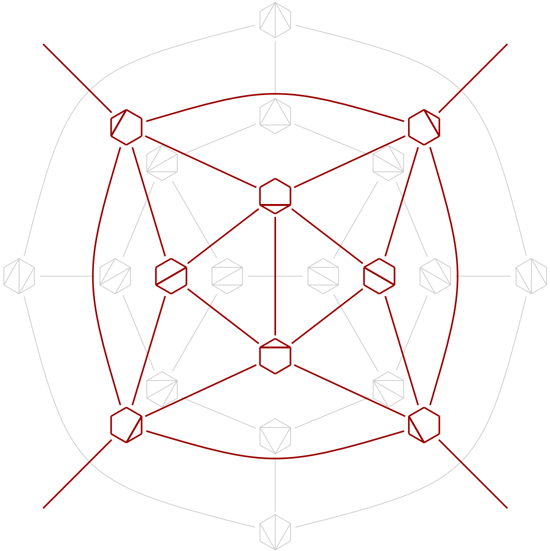

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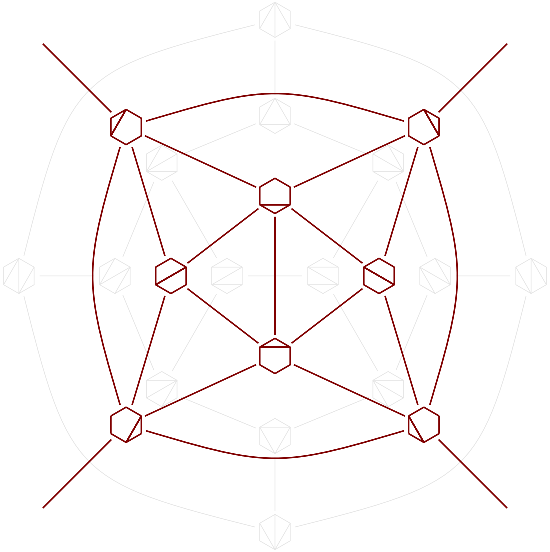

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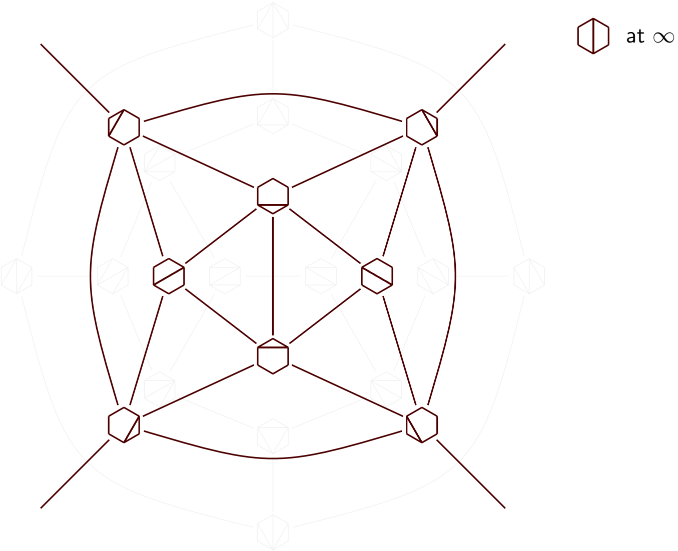

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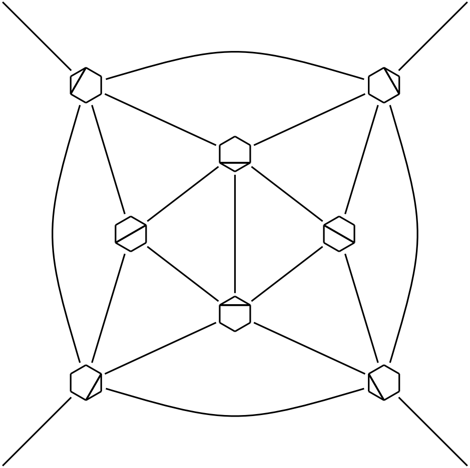

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The simplicial associahedron (continued)

More directly, the simplicial associahedron is the following simplicial complex:

vertices:	diagonals of a convex $(n+3)$-gon
simplices:	partial triangulations of the $(n+3)$-gon (collections of non-crossing diagonals)
maximal simplices:	triangulations of the $(n+3)$-gon (collections of n non-crossing diagonals).

This simplicial complex is homeomorphic to a sphere.

But more is true...

The simplicial associahedron (concluded)

Theorem 3A.1

The simplicial complex described above can be realized as the boundary of an n-dimensional convex polytope.

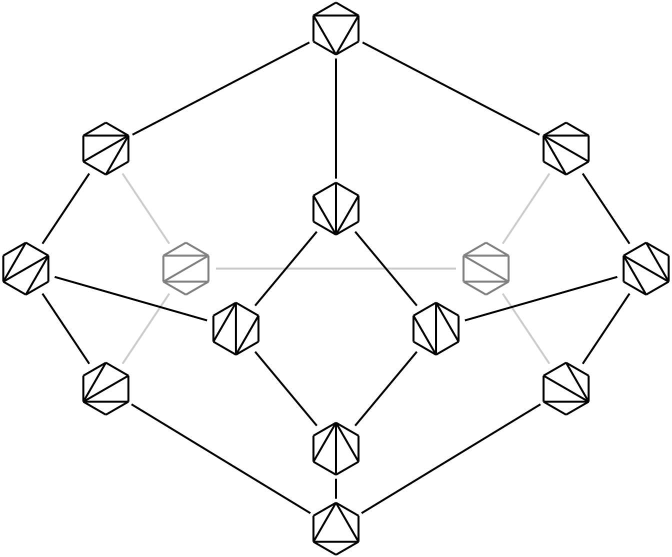
This were proved independently by J. Milnor, M. Haiman, and C. W. Lee. It also follows from the much more general theory of secondary polytopes developed by I. M. Gelfand, M. Kapranov and A. Zelevinsky.

Since it is a polytope, there is a (polar) dual polytope, called the (simple) associahedron or Stasheff polytope.

Simple means that every vertex is incident to exactly n edges.

Starting from the graph on triangulations, we have dualized twice. We conclude that the vertices of the associahedron correspond to triangulations, and that the edges correspond to diagonal flips.

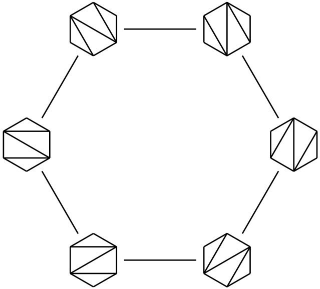
The 3-dimensional associahedron



The cyclohedron

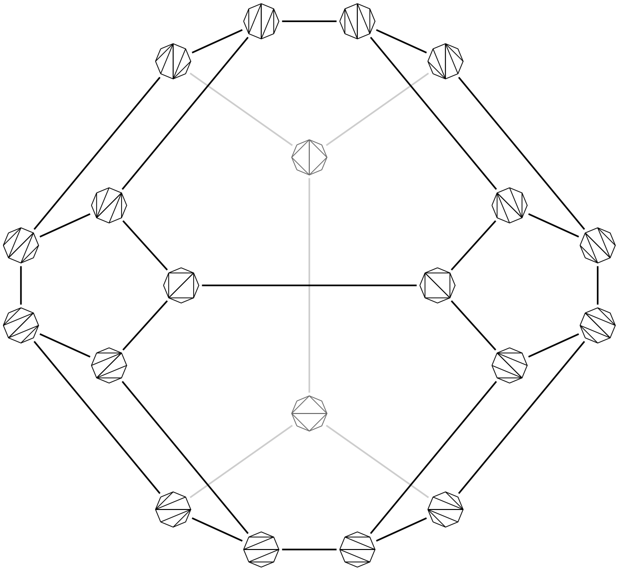
The n-dimensional cyclohedron (or Bott-Taubes polytope) is similar to the associahedron.

The vertices are labeled by centrally-symmetric triangulations of a regular $(2n+2)$-gon. Each edge represents either a diagonal flip of two diameters of the polygon, or a pair of two centrally-symmetric diagonal flips.



The cyclohedron is also a polytope, as was shown independently by M. Markl and R. Simion.

The 3-dimensional cyclohedron



The connection to cluster algebras

So what does all this have to do with cluster algebras?

A lot:

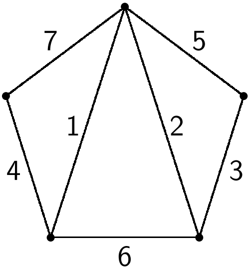
- We'll see that matrix mutation is intimately connected with the combinatorics of triangulations.
- We'll also see that exchange relations relate various lengths associated to triangulations.

The edge-adjacency matrix of a triangulation

Given a triangulation T of the $(n+3)$-gon, number the diagonals arbitrarily $1, \ldots, n$. Number the sides of T by $n+1, \ldots, 2n+3$.

The edge-adjacency matrix $\tilde{B} = (b_{ij})$ if T is the $(2n+3) \times n$ matrix with entries

$$b_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ label two sides the same triangle } \Delta \text{ of } T, \\ & \text{and } (i, j, \cdot) \text{ is a clockwise list of the sides of } \Delta. \\ -1 & \text{if the same holds, } (i, j, \cdot) \text{ a counter-clockwise list.} \\ 0 & \text{otherwise.} \end{cases}$$



$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

The edge-adjacency matrix (concluded)

Exercise 3Aa

Given a fixed numbering $n+1, \ldots, 2n+3$ *of the edges of an* $(n+3)$*-gon* Q*, show that a triangulation of* Q *is determined uniquely by its edge-adjacency matrix.*

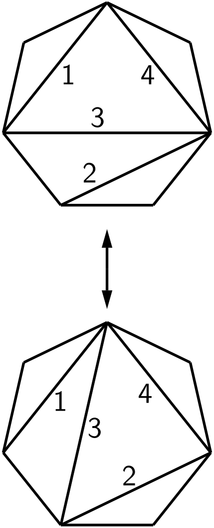
Exercise 3Ab

Show that the top $n \times n$ *submatrix of the edge-adjacency matrix is skew-symmetric. (Thus it is an exchange matrix.)*

Exercise 3Ac

Fix a numbering $n+1, \ldots, 2n+3$ *of the edges of an* $(n+3)$*-gon* Q*. Let* T *and* T' *be triangulations of* Q *related by a diagonal flip. Number the diagonals of* T *and* T' *so that the flipped diagonal is* k *in both, and the non-flipped diagonals have the same labeling in both. Show that the edge-adjacency matrices of* T *and* T' *are related by the matrix mutation* μ_k*.*

Example: diagonal flips and matrix mutation



$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\Big\updownarrow \mu_3$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Exchange matrices in the cyclohedron picture

Similarly, there is a natural way to define edge-adjacency matrices for centrally symmetric triangulations T.

The entries will be 0, ± 1, and ± 2.

Once again centrally-symmetric diagonal flips correspond to matrix mutations.

Exchange relations and Ptolemy's Theorem

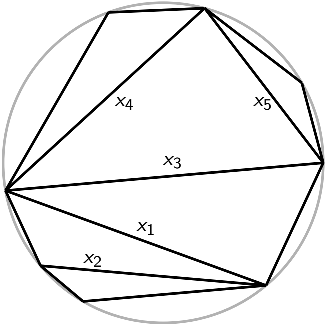
Theorem 3A.2 (Ptolemy, ~ 100)

In an inscribed quadrilateral $ABCD$, the lengths of sides and diagonals satisfy

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

This looks a lot like an exchange relation.

Given an edge-adjacency matrix, inscribe an $(n+3)$-gon in a circle and create a triangulation. Let x_i be the length of the i^{th} diagonal. We can't choose the x_i arbitrarily, but there are no algebraic relations on the x_i, so we can think of them as indeterminates.

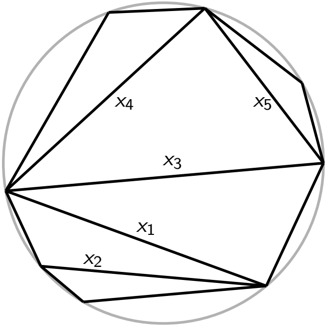


Exchange relations and Ptolemy's Theorem (continued)

We cannot choose the polygon's edge lengths independently, but we'll still think of them as indeterminates for now. We can use Ptolemy's Theorem

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

to write the length of each diagonal as a rational function in $x_1, \ldots, x_n$.



These rational functions are cluster variables in the cluster pattern of geometric type defined by the given edge-adjacency matrix.

Exchange relations and Ptolemy's Theorem (continued)

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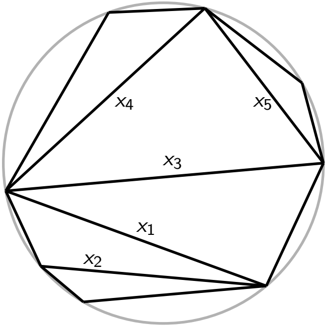

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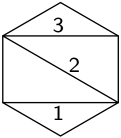
Well, not quite, because the polygon's edge lengths are not all indeterminates. (Some y's depend on x's and other y's.) We get a specialization of the cluster pattern. But cluster variables can be constructed by passing to hyperbolic geometry.

Exercise 3Ad

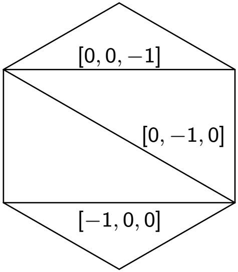
Consider the triangulation shown to the right, with diagonals labeled. Interpret the top square part of its edge-adjacency matrix an the exchange matrix. Take $\mathbf{x} = (x_1, x_2, x_3)$ *and* $\mathbb{P} = \{1\}$*. Find the cluster variables and exchange matrices in the pattern.*



Recommendation: Use a drawing of the diagonal flips graph to organize your calculation of exchange matrices. Use a drawing of the dual simplicial complex to organize your calculation of cluster variables.

Denominator vectors in the associahedron

When you do Exercise 3Ad, you will get denominator vectors:



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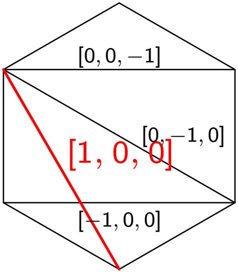

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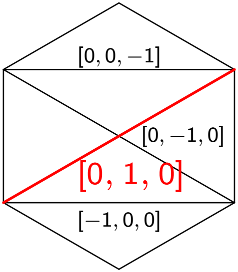

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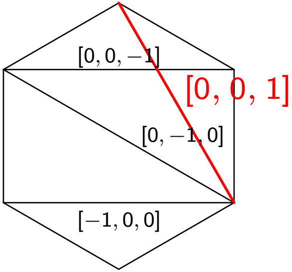

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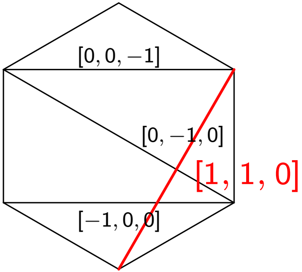

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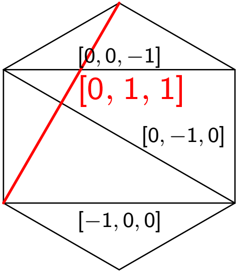

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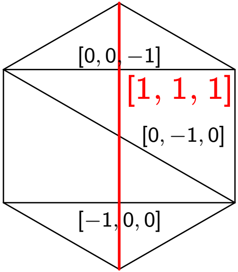

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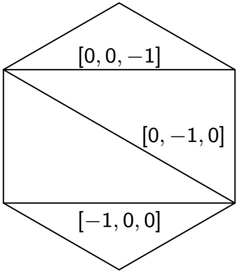

In Exercises 1k.1 and 1m.1, you investigated the root system A_n.
In the case, $n = 3$, the simple roots are

$$\alpha_1 = e_2 - e_1, \quad \alpha_2 = e_3 - e_2, \quad \alpha_3 = e_4 - e_3,$$

and the remaining positive roots are
$\alpha_1 + \alpha_2 = e_3 - e_1, \quad \alpha_2 + \alpha_3 = e_4 - e_2, \quad \alpha_1 + \alpha_2 + \alpha_3 = e_4 - e_1.$

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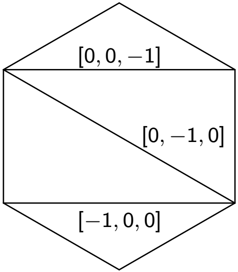

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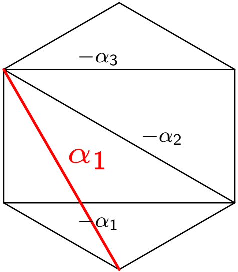

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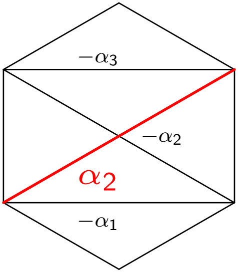

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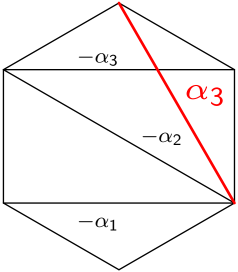

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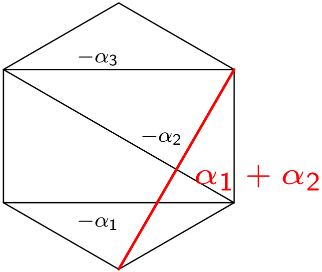

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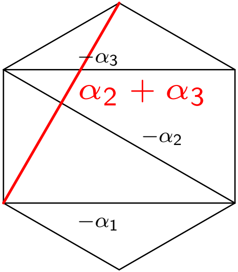

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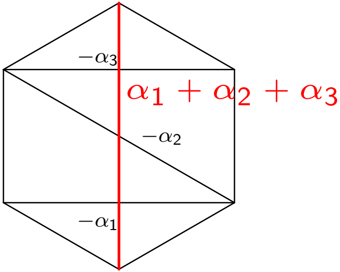


In Exercises 1k.1 and 1m.1, you investigated the root system A_n.
In the case, $n = 3$, the simple roots are

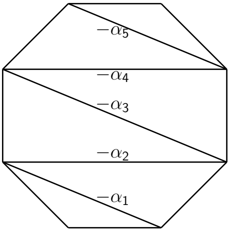
$$\alpha_1 = e_2 - e_1, \quad \alpha_2 = e_3 - e_2, \quad \alpha_3 = e_4 - e_3,$$

and the remaining positive roots are
$\alpha_1 + \alpha_2 = e_3 - e_1, \quad \alpha_2 + \alpha_3 = e_4 - e_2, \quad \alpha_1 + \alpha_2 + \alpha_3 = e_4 - e_1.$

Our denominator vectors (besides the initial 3) are the simple-root coordinates of the positive roots. The initial 3 denominators are simple root coordinates of negative simple roots.

Denominator vectors in the associahedron (continued)

In general, we associate diagonals of an $(n+3)$-gon with almost positive roots in type A_n. (These are roots that are positive or the negatives of simples.) The negative simple roots label the snake as shown.



Each positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ labels the unique diagonal that crosses exactly the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$.

Theorem 3A.3 (Fomin and Zelevinsky, CA II)

Let B be the top-square part of the edge-adjacency matrix for the snake triangulation. Label the diagonals by almost positive roots, as described above. Then there is a graph isomorphism $\Sigma \mapsto T$ from $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ *to the diagonal-flip graph such that the denominator vectors in each Σ are the simple-root coordinates of the roots labeling the diagonals of T.*

Denominator vectors in the associahedron (concluded)

Theorem 3A.6 says:

The associahedron is a combinatorial model for the exchange graph/cluster complex (in particular showing that the "cluster complex" is a valid notion—see Conjecture 2.2) for a particular B.

We want to:

- generalize the associahedron, and
- Show that the cluster algebras given by generalized associahedra are the only cluster algebras of finite type.

Compatibility of almost positive roots

To generalize the associahedron, it is easiest to work with the simplicial associahedron. Recall its description:

vertices:	diagonals of a convex $(n+3)$-gon
simplices:	collections of non-crossing diagonals

But diagonals are in bijection with almost positive roots. We will generalize as follows, for any finite root system Φ:

vertices:	almost positive roots $\Phi_{\geq -1}$
simplices:	collections of "compatible" roots

We just need to define compatibility!

We will write $\beta \parallel \gamma$ to mean that β and γ are compatible.

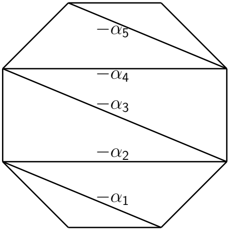
Compatibility of almost positive roots (continued)

To generalize, note three properties of compatibility in the A_n case.

First: Let $\Phi_{\langle i \rangle}$ be the subset of roots in Φ consisting of roots in the positive span of the simple roots, not including α_i.

Then a negative simple root $-\alpha_i$ is compatible with an almost positive root β if and only if $\beta \in \Phi_{\langle i \rangle}$.

Second: If we are using a regular polygon, then the dihedral symmetry of the polygon preserves compatibility.



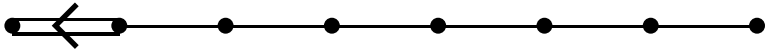
Third: For any diagonal, there is a dihedral symmetry of the polygon that moves that diagonal to the snake.

The first property is already written in general form. How do we generalize "dihedral symmetry" to almost positive roots in a finite root system Φ? We will assume Φ is irreducible.

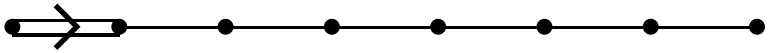
A glance back at the irreducible finite root systems

A_n $(n \geq 1)$

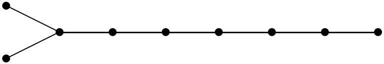
B_n $(n \geq 2)$



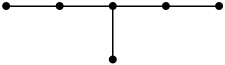
C_n $(n \geq 3)$



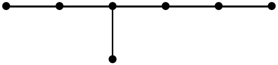
D_n $(n \geq 4)$



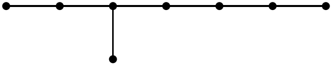
E_6



E_7



E_8



F_4

G_2

Dihedral symmetry of almost positive roots

Every Dynkin diagram of an irreducible root system is a bipartite graph. We can write $[n]$ as a disjoint union of sets I_+ and I_- with the property that

$$\langle \alpha_i^\vee, \alpha_j \rangle = 0 \quad \text{if } i,j \in I_+ \quad \text{or } i,j \in I_-.$$

This implies that the corresponding reflections s_i and s_j commute.

Define involutions $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i \text{ for } i \in I_{-\varepsilon} \\ \prod_{i \in I_\varepsilon} s_i(\alpha) & \text{otherwise.} \end{cases}$$

For example, in type A_2, with $I_+ = \{1\}$ and $I_- = \{2\}$:

$$\underset{\substack{\circlearrowright \\ \tau_-}}{-\alpha_1} \overset{\tau_+}{\longleftrightarrow} \alpha_1 \overset{\tau_-}{\longleftrightarrow} \alpha_1 + \alpha_2 \overset{\tau_+}{\longleftrightarrow} \alpha_2 \overset{\tau_-}{\longleftrightarrow} \underset{\substack{\circlearrowright \\ \tau_+}}{-\alpha_2}$$

Dihedral symmetry of almost positive roots (continued)

Exercise 3Ae

Verify, for Φ of type A_3, that the action of τ_+ on $\Phi_{\geq -1}$ corresponds to a reflection of the hexagon, acting on diagonals. Same for τ_-. Verify that these reflections generate the symmetry group of the hexagon.

Theorem 3A.4 (Fomin and Zelevinsky, CA II)

1. *Every $\langle \tau_-, \tau_+ \rangle$-orbit in $\Phi_{\geq -1}$ has a nonempty intersection with $-\Pi$.*
2. *There is a unique binary relation $\parallel$ on $\Phi_{\geq -1}$ that has the following two properties:*
   - $\alpha \parallel \beta$ *if and only if* $\tau_\varepsilon \alpha \parallel \tau_\varepsilon \beta$, *for* $\varepsilon \in \{+, -\}$
   - $-\alpha_i \parallel \beta$ *if and only if* $\beta \in \Phi_{\langle i \rangle}$

Generalized associahedra

The simplicial generalized associahedron associated to an irreducible root system Φ is the simpicial complex with

vertices:	almost positive roots $\Phi_{\geq -1}$
simplices:	collections of compatible roots

Theorem 3A.5 (Chapoton, Fomin, Zelevinsky (PRGA))

The simplicial generalized associahedron for Φ is dual to the boundary of an n-dimensional simple polytope called the (simple) generalized associahedron for Φ.

For Φ of type A_n: this is the classical associahedron.
For Φ of type B_n: this is the cyclohedron.
For Φ of type D_n: there is a combinatorial construction, similar to triangulations, that realizes the generalized associahedron.
For the other types: ??

Denominator vectors in generalized associahedra

The maximal sets of compatible almost positive roots are called clusters of almost positive roots. The fact that the generalized associahedron is simple and n-dimensional says that each cluster contains exactly n roots. The clusters label the vertices of the simple associahedron.

Given a Cartan matrix A of finite type, define $B_{\text{bip}}(A)$ by

$$b_{ij} = \begin{cases} 0 & \text{if } i = j; \\ a_{ij} & \text{if } i \neq j \text{ and } i \in I_+; \\ -a_{ij} & \text{if } i \neq j \text{ and } i \in I_-, \end{cases}$$

Theorem 3A.6 (Fomin and Zelevinsky, CA II)

Let Φ be finite with Cartan matrix A. There is an isomorphism $\Sigma \mapsto C$ from $\text{Ex}(\mathbf{x}, \mathbf{y}, B_{\text{bip}}(A))$ to the vertex-edge graph of the generalized associahedron for Φ such that the denominator vectors in Σ are the simple-root coordinates of the roots in the cluster C.

Cluster algebras of finite type

Recall: A cluster algebra is of *finite type* if it has finitely many distinct seeds.

Theorem 3A.7 (Fomin and Zelevinsky, CA II)

For a cluster algebra $\mathcal{A}$, the following are equivalent:

(i) *$\mathcal{A}$ is of finite type.*

(ii) *There exists a seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ in the cluster pattern such that $B_t = B_{\text{bip}}(A)$ for a Cartan matrix A of finite type.*

(iii) *There exists a seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ in the cluster pattern such that B_t has Cartan companion of finite type.*

(iv) *$|b^t_{ij} b^t_{ji}| \leq 3$ for every seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ and every $i, j \in [n]$.*

Cluster algebras of finite type (continued)

Some remarks:

- The property of having finite type or not depends only on B.
- Generalized associahedra give us a model for denominator vectors when $B_{\text{bip}}(A)$ is the initial exchange matrix.
- Work by Marsh, Reineke and Zelevinsky (MRZ) leads to a model for denominator vectors when the initial matrix has a Cartan companion of finite type. The denominators are still the almost positive roots, but the compatibility relation is altered.

Cluster algebras of finite type (concluded)

The classification of cluster algebras of finite type proves many structural conjectures in the case of finite type, including:

2.1: The exchange graph $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ depends only on B.

2.2: The simplicial complex defined by the exchange graph can be realized as a "cluster complex." (Also 2.3, which implies 2.2.)

2.5: Positivity of Laurent coefficients for initial B having finite-type Cartan companion.

2.6 (2.7): Different cluster variables (monomials) have different denominator vectors.

A local criterion for finite type

So far, the criteria mentioned for finite type are global, in the sense that you may have to check every seed to check the criterion.

Barot, Geiss and Zelevinsky gave a local criterion.

A quasi-Cartan companion A' of B is any symmetric matrix with 2's on the diagonal and $|a'_{ij}| = |b_{ij}|$ off the diagonal.

The exchange matrix B determines a directed graph $\Gamma(B)$: We orient each edge of the diagram $i \to j$ if $b_{ij} < 0$.

Theorem 3A.8 (BGZ)

A cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is of finite type if and only if

- *B has a quasi-Cartan companion A' such that the symmetrized matrix DA' is positive definite, and*
- *Every chordless cycle of $\Gamma(B)$ is cyclically oriented.*

References

(BGZ) M. Barot, C. Geiss, and A. Zelevinsky, "Cluster algebras of finite type and positive symmetrizable matrices." J. London Math. Soc (2) **73**

(RSGA) S. Fomin and N. Reading, "Root systems and generalized associahedra." IAS/PCMI Lecture Series **13**.

(PRGA) F. Chapoton, S. Fomin, and A. Zelevinsky, "Polytopal realizations of generalized associahedra." Canad. Math. Bull. **54**.

(GA) S. Fomin and A. Zelevinsky, "Y-systems and generalized associahedra." Annals of Math. **158**.

(CA II) S. Fomin and A. Zelevinsky, "Cluster algebras II: Finite type classification." Invent. Math. **154**.

(MRZ) R. Marsh, M. Reineke and A. Zelevinsky, "Generalized associahedra via quiver representations." Transactions AMS **355**.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a half day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

3Ab, 3Ac,

3Ae, 3Ad 3Aa.