Dominance phenomena: Mutation, scattering and cluster algebras

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Vingt ans d'algèbres amassées
CIRM, Luminy
21 mars 2018

Section 1. Dominance phenomena

Dominance relations between exchange matrices

$B = [b_{ij}]$ dominates $B' = [b'_{ij}]$ if, for all i, j,

- b_{ij} and b'_{ij} weakly agree in sign (i.e. $b_{ij}b'_{ij} \geq 0$) and
- $|b_{ij}| \geq |b'_{ij}|$.

Example. $B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Question: What are the consequences of dominance for structures that take an exchange matrix as input?

I'll address that question by presenting some "dominance phenomena."

Four phenomena

Suppose B and B' are exchange matrices and B dominates B'.
In many cases:

Phenomenon I

The identity map from $\mathbb{R}^B$ to $\mathbb{R}^{B'}$ is mutation-linear.

Phenomenon II

$\mathcal{F}_B$ refines $\mathcal{F}_{B'}$. *(mutation fans)*

Phenomenon III

$\mathrm{ScatFan}(B)$ *refines* $\mathrm{ScatFan}(B')$. *(cluster scattering fans)*

Phenomenon IV

There is an injective, $\mathbf{g}$-vector-preserving ring homomorphism from $\mathcal{A}_\bullet(B')$ to $\mathcal{A}_\bullet(B)$. *(principal coefficients cluster algebras)*

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In many cases (not the same cases for all four phenomena):

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Why phenomena?

- There are counterexamples.
- I don't know necessary and sufficient conditions for the phenomena.
- Yet there are theorems that give compelling and surprising examples.

Goal: Establish that something real and nontrivial is happening, with an eye towards two potential benefits:

- Researchers from the various areas will apply their tools to find more examples, necessary and/or sufficient conditions for the phenomena, and/or additional dominance phenomena.
- The phenomena will lead to insights in the various areas where matrix mutation, scattering diagrams, and cluster algebras are fundamental.

Phenomenon I

In many cases, the identity map from $\mathbb{R}^B$ to $\mathbb{R}^{B'}$ is mutation-linear.

One way to understand this:

$$\begin{array}{r} \text{exchange matrix} \\[2em] \text{coefficient rows}\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \end{array} \begin{bmatrix} & B & \\ & & \\ - & - & - \\ - & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

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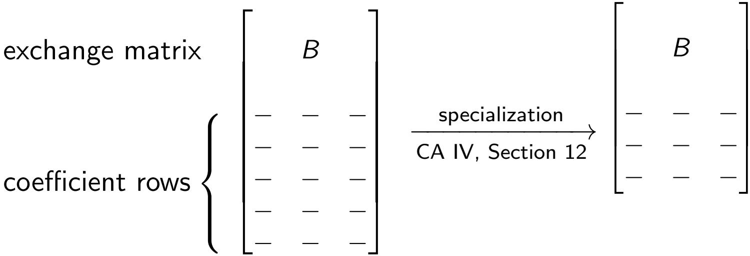

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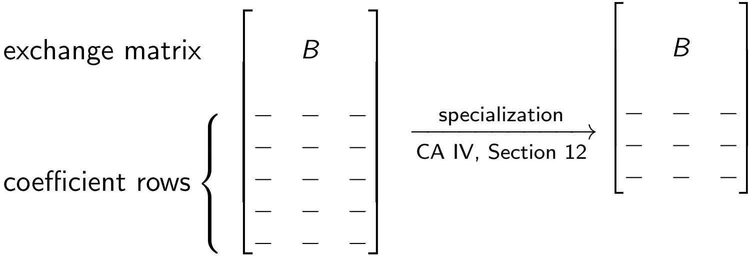

A mutation-linear map $\mathbb{R}^B$ to $\mathbb{R}^{B'}$ induces a functor

$$(\text{geometric cluster algebras for } B,\ \text{specialization})$$
$$\downarrow$$
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A mutation-linear map $\mathbb{R}^B$ to $\mathbb{R}^{B'}$ induces a functor

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Phenomena II and III (refinement of fans)

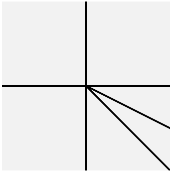
In many cases,

- the mutation fan $\mathcal{F}_B$ refines the mutation fan $\mathcal{F}_{B'}$.
- the cluster scattering fan $\mathrm{ScatFan}(B)$ refines the cluster scattering fan $\mathrm{ScatFan}(B')$.

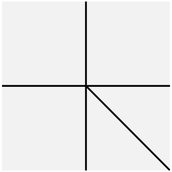
Aside: **Theorem** (R., 2017). A consistent scattering diagram with minimal support cuts space into a fan.

In finite type, both $\mathcal{F}_B$ and $\mathrm{ScatFan}(B)$ coincide with the **g**-vector fan^T, the normal fan to a generalized associahedron.

Example: cyclohedron and associahedron.



$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$



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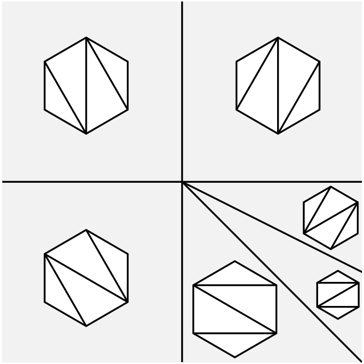
2-cyclohedron & 2-associahedron



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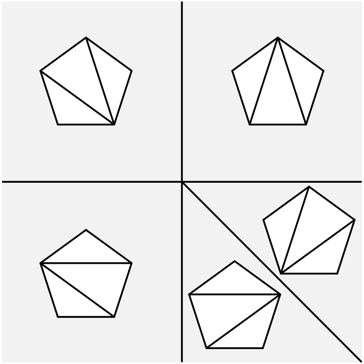

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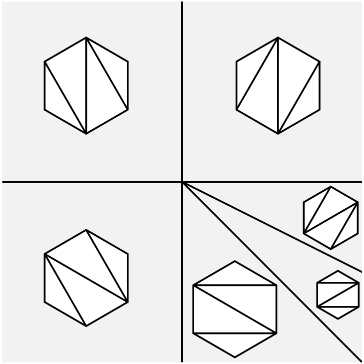

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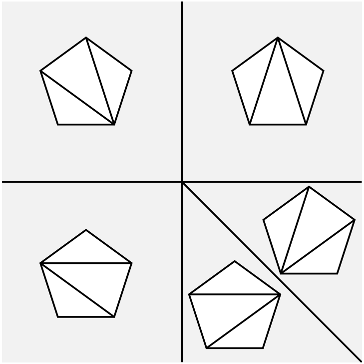
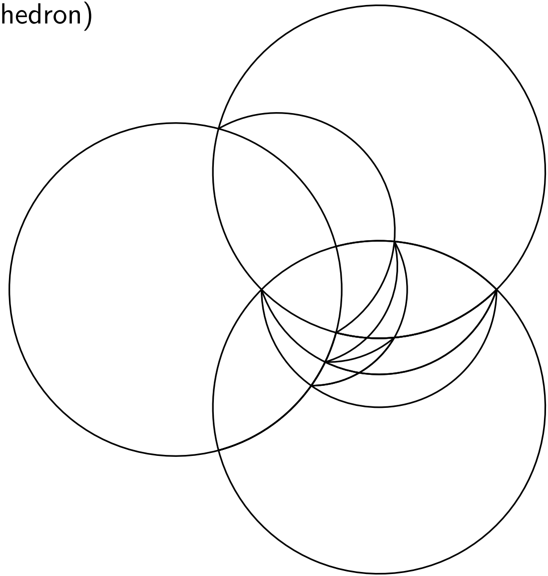


$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Aside: Can we understand this on the level of triangulations?

3-cyclohedron & 3-associahedron: $B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$\mathcal{F}_B$ (cyclohedron)



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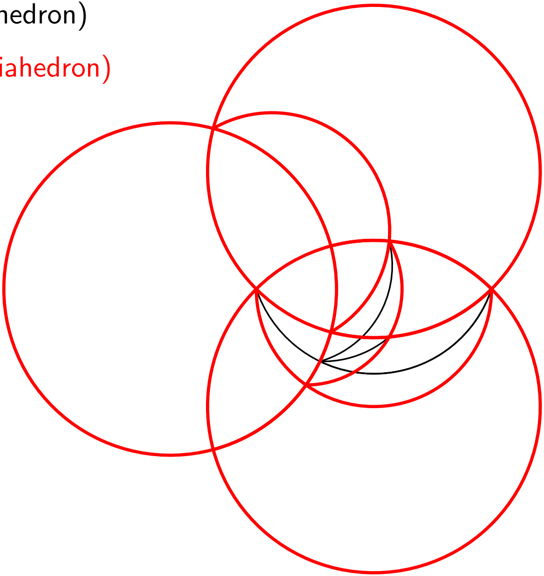

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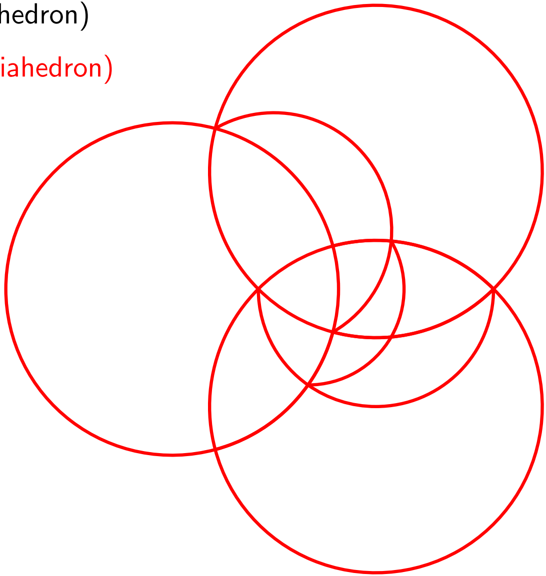

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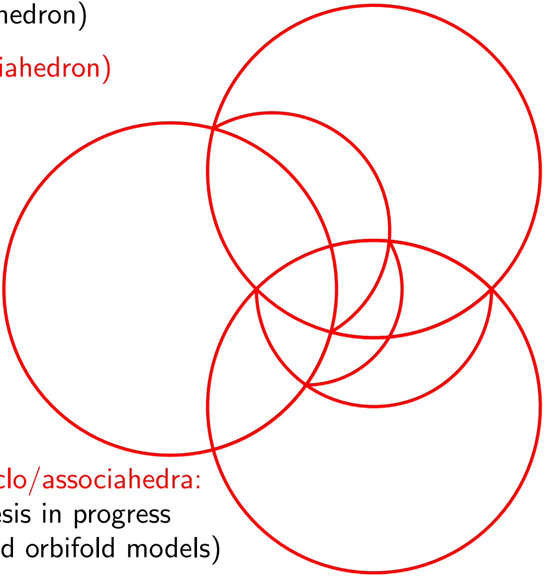
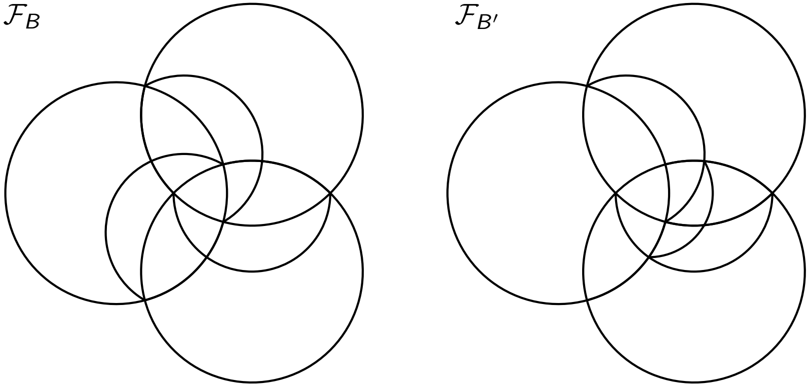


General cyclo/associahedra:
S. Viel, thesis in progress
(surface and orbifold models)

Non-Example: $B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$



These are normal fans to two different 3-associahedra.

Phenomenon IV

In many cases, there is an injective, $\mathbf{g}$-vector-preserving ring homomorphism from $\mathcal{A}_\bullet(B')$ to $\mathcal{A}_\bullet(B)$ (principal coefficients cluster algebras).

Remarks:

- Phenomenon is known* to occur for B acyclic of finite type.
- There is a nice description of the homomorphism (where it sends initial cluster variables and coefficients).
- In some cases, including acyclic finite type, the map sends cluster variables to cluster variables (or "ray theta functions" to ray theta functions).
- Sending cluster variables to cluster variables is suggested by Phenomena II and III (fan refinement).
- Coefficients—and specifically principal ones—are crucial.

Section 2. Refinement

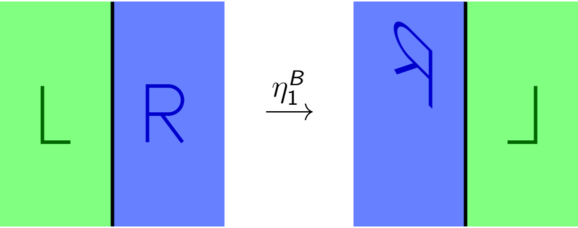
Mutation maps $\eta^B_{\mathbf{k}}$

Let $\widetilde{B}$ be $\left[\begin{smallmatrix} B \\ \mathbf{a} \end{smallmatrix}\right]$ (i.e. B with an extra row $\mathbf{a} \in \mathbb{R}^n$).
For $\mathbf{k} = k_q, k_{q-1}, \ldots, k_1$, define $\eta^B_{\mathbf{k}}(\mathbf{a})$ to be the last row of $\mu_{\mathbf{k}}(\widetilde{B})$.

Example: $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ a_1 & a_2 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -a_1 & ? \end{bmatrix}$$

$$? = \begin{cases} a_2 & \text{if } a_1 \leq 0 \\ a_2 + a_1 & \text{if } a_1 \geq 0 \end{cases}$$



The mutation fan

Define an equivalence relation $\equiv^B$ on $\mathbb{R}^n$ by setting

$$\mathbf{a}_1 \equiv^B \mathbf{a}_2 \quad \Longleftrightarrow \quad \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_1)) = \mathbf{sgn}(\eta_{\mathbf{k}}^B(\mathbf{a}_2)) \quad \forall \mathbf{k}.$$

$\mathbf{sgn}(\mathbf{a})$ is the vector of signs $(-1, 0, +1)$ of the entries of $\mathbf{a}$.

B-classes: equivalence classes of $\equiv^B$.
B-cones: closures of B-classes.

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Mutation fan for B:
The collection $\mathcal{F}_B$ of all B-cones and all faces of B-cones.

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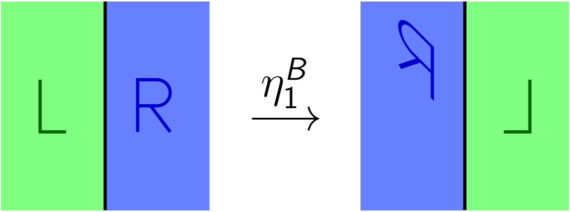
Theorem (R., 2017). ScatFan(B) refines $\mathcal{F}_B$.

Conjecture. For rank ≥ 3, they coincide iff B mutation-finite.

Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



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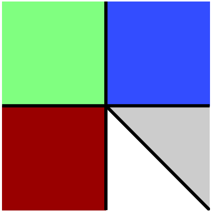
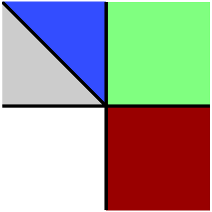


$\xrightarrow{\eta_1^B}$



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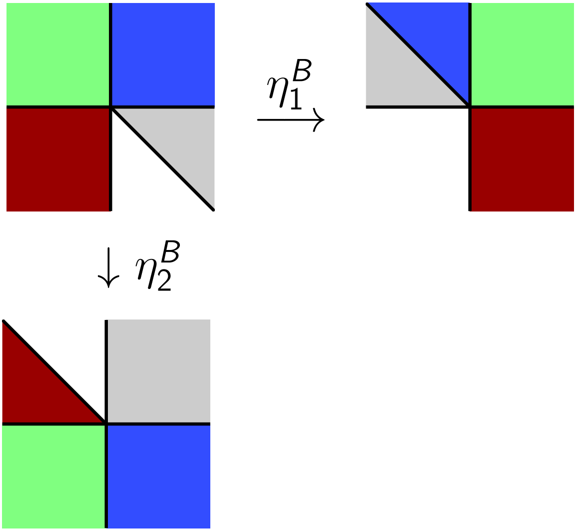

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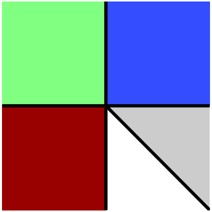

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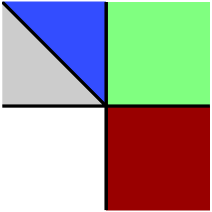
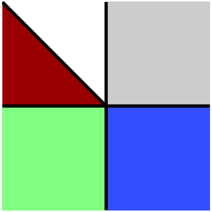
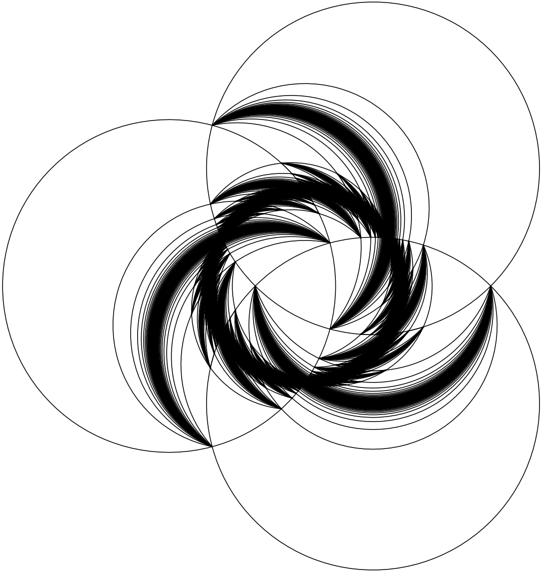


$\downarrow \eta_2^B$



Each of the 5 maximal cones shown in the top-left picture is a B-cone.

Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ (Markov quiver)



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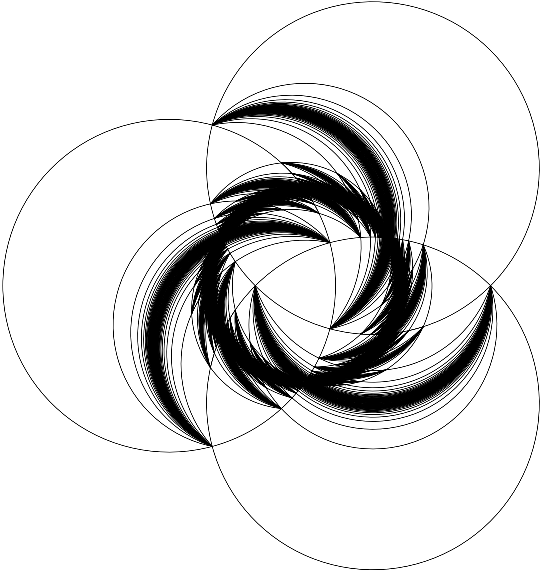

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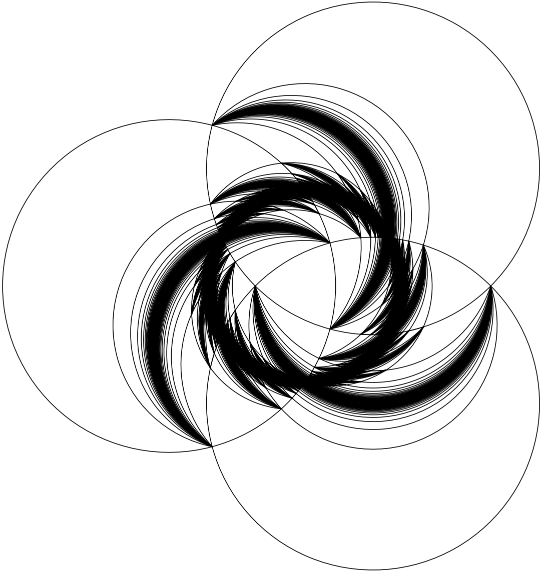

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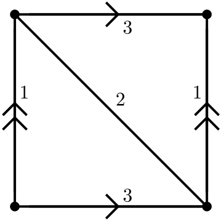
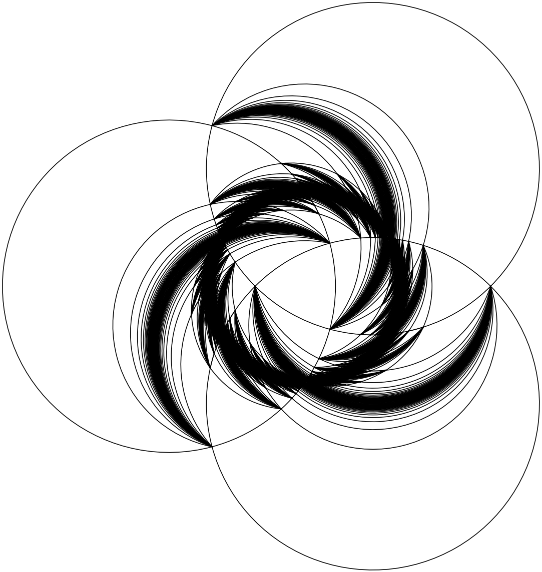
We'll discuss Phenomenon II in two models: Cambrian fans and surfaces (orbifolds).



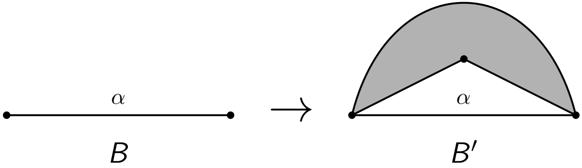
$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Maximal cones in the mutation fan are given by triangulations and more general configurations that include closed curves.

(Shear coordinates of quasi-laminations)



Resecting a triangulated surface on an edge



Theorem. (R., 2013) Assuming the Null Tangle Property, B dominates B' and $\mathcal{F}_B$ refines* $\mathcal{F}_{B'}$.

Null Tangle Property: Known for some surfaces, probably true for many more (or maybe all?).

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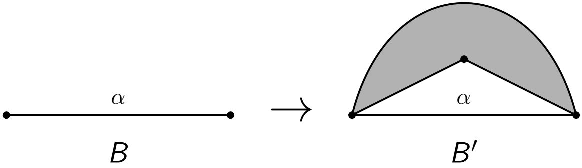

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* "rational parts" of these fans.

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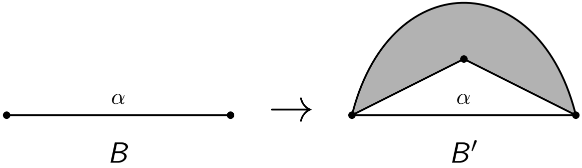

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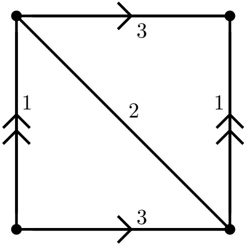
Orbifold model: Extends surfaces model to cover more general non-skew-symmetric cases.

Shira Viel, 2017: Constructs mutation fan for an orbifold. She defines orbifold resection, and proves Phenomenon II. (E.g. cyclohedron fan refines associahedron fan.)

Example

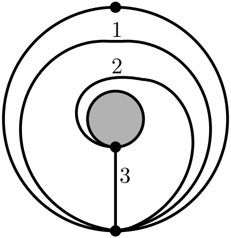
Resect arc 1 then arc 3.

Torus



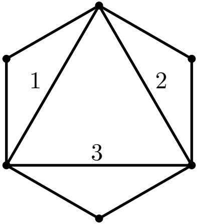
$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Annulus



$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

Hexagon



$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

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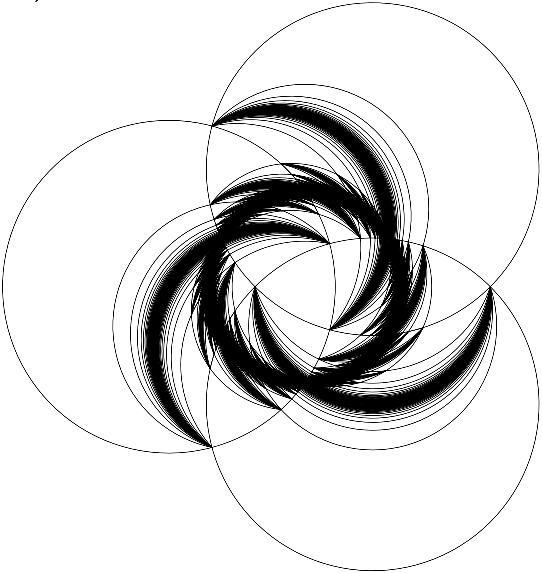
$\mathcal{F}_B$ (torus)



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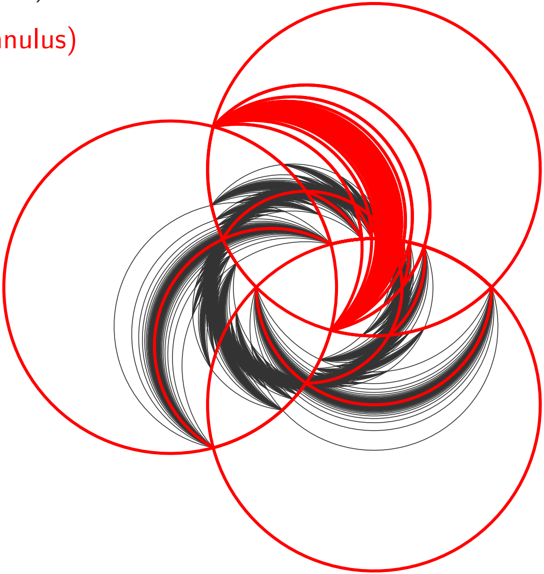

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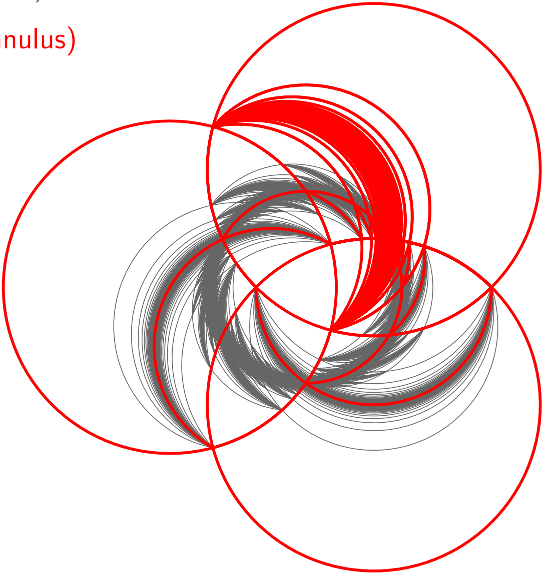
$\mathcal{F}_{B'}$ (annulus)



Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

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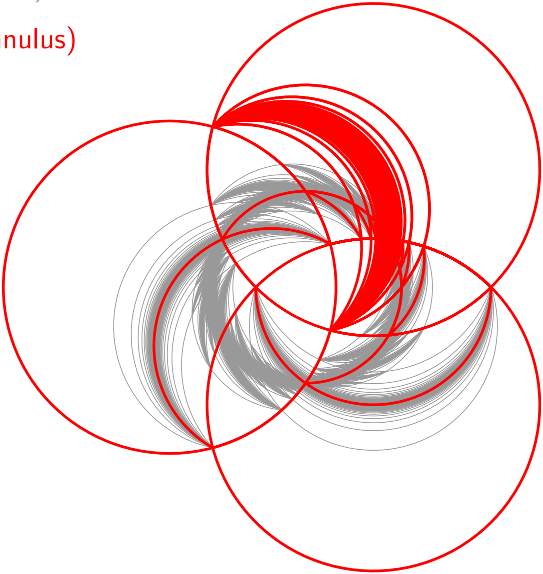

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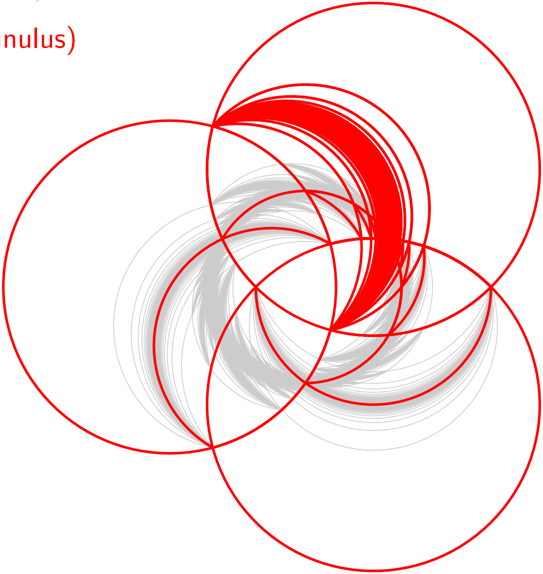

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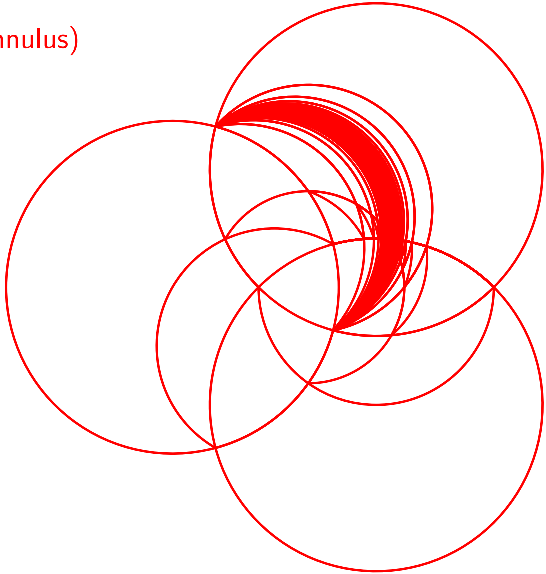

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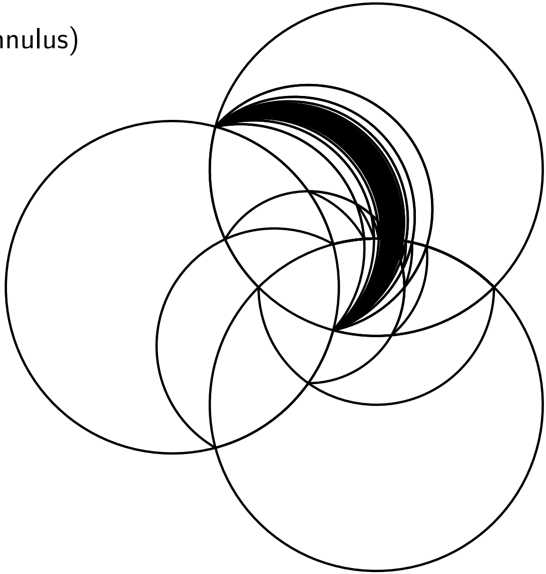

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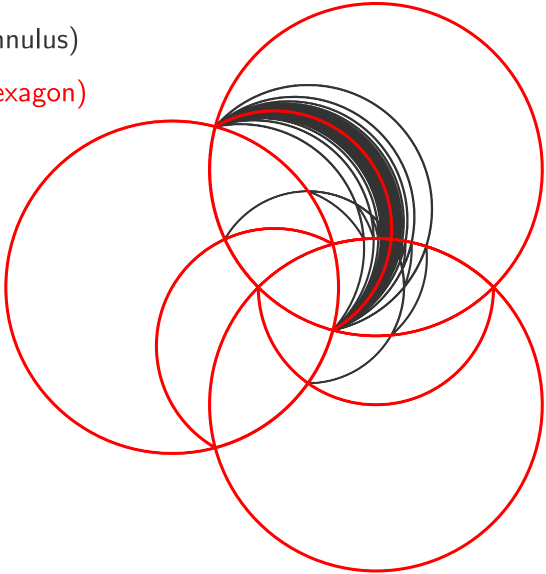

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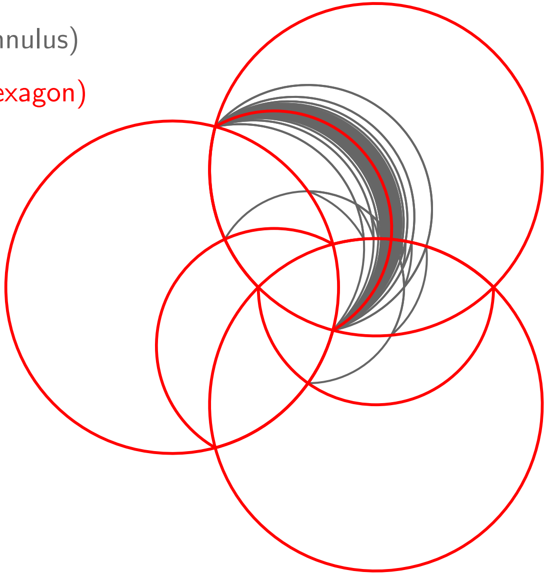

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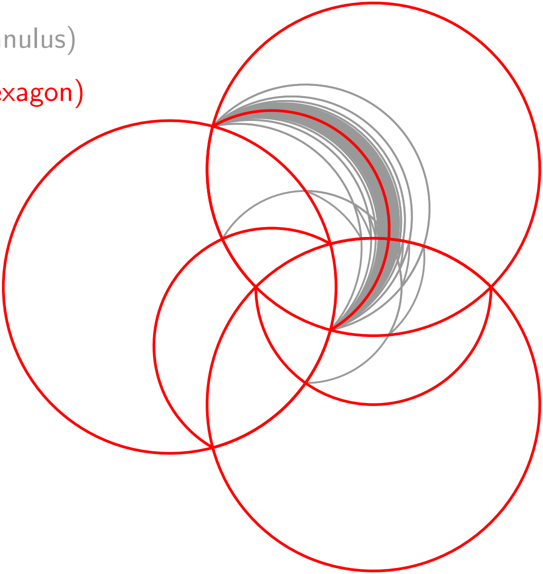

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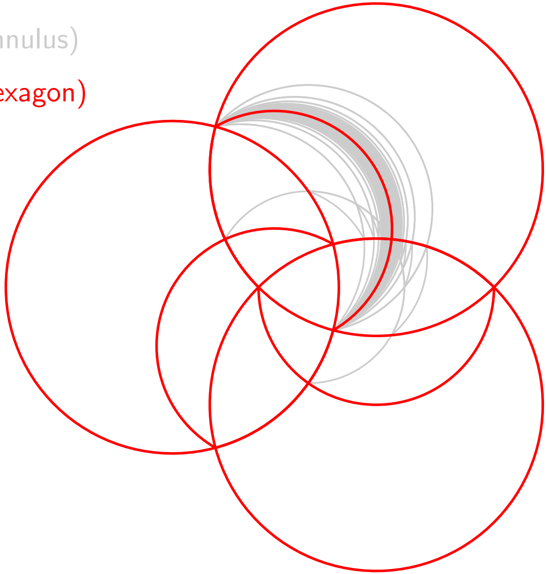

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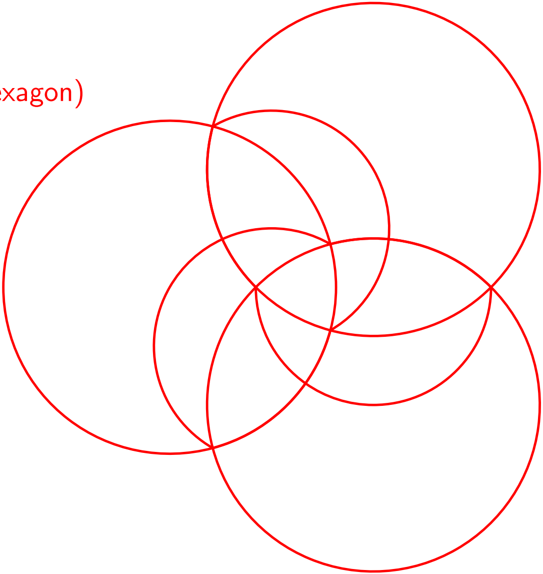

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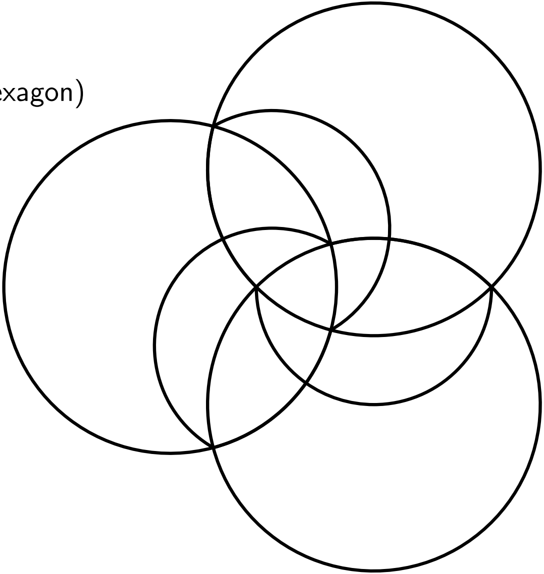

Finite acyclic type: Cambrian fans

Each B defines a Cartan matrix A.

$$\text{E.g.} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \to \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Coxeter fan: Defined by the reflecting hyperplanes of the Coxeter group W associated to A. Maximal cones $\leftrightarrow$ elements of W.

Cambrian fan: A certain coarsening of the Coxeter fan.
Two ways to look at this:

- Coarsen according to a certain lattice congruence on W.
- Coarsen according to the combinatorics of "sortable elements."

For S_n, this is the normal fan to the usual associahedron.
(In general, generalized associahedron.)

Cambrian fans and mutation fans

For B acyclic of finite type, $\mathcal{F}_B$ is a Cambrian fan. (Key technical point: identify fundamental weights with standard basis vectors.)

Theorem (R., 2013). For B acyclic of finite type, $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$ if and only if B dominates B'.

Dominance relations among exchange matrices imply dominance relations among Cartan matrices. So the theorem is a statement that refinement relations exist among Cambrian fans when we decrease edge-labels (or erase edges) on Coxeter diagrams.

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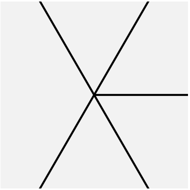

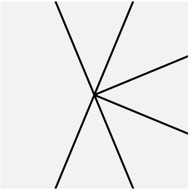

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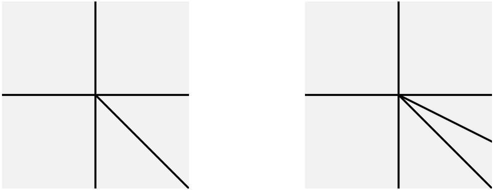


Lattice homomorphisms between Cambrian lattices

The Cambrian lattice Camb_B is:

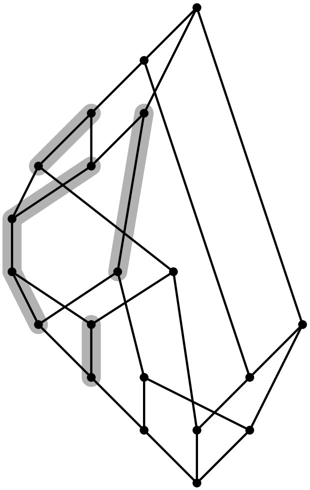
- A partial order on maximal cones in the Cambrian fan $\mathcal{F}_B$. The fan and the order interact very closely.
- A lattice quotient—and a sublattice—of the weak order on the finite Coxeter group associated to B.

To prove the refinement of fans:

- Show that there is a surjective lattice homomorphism from Camb_B to $\text{Camb}_{B'}$.
- Appeal to general results on lattice homomorphisms and fans.

Theorem (R., 2012). Such a surjective lattice homomorphism exists for all acyclic, finite-type B, B' with B dominating B'.

Example: A_3 Tamari is a lattice quotient of B_3 Tamari



Lattice homomorphisms between weak orders

To find a surjective lattice homomorphism $\text{Camb}_B \to \text{Camb}_{B'}$:

Find a surjective lattice homomorphism between the corresponding weak orders.

Theorem (R., 2012). If (W, S) and (W', S) are finite Coxeter systems such that W dominates W', then the weak order on W' is a lattice quotient of the weak order on W.

Dominance here means that the diagram of W' is obtained from the diagram of W by reducing edge-labels and/or erasing edges.

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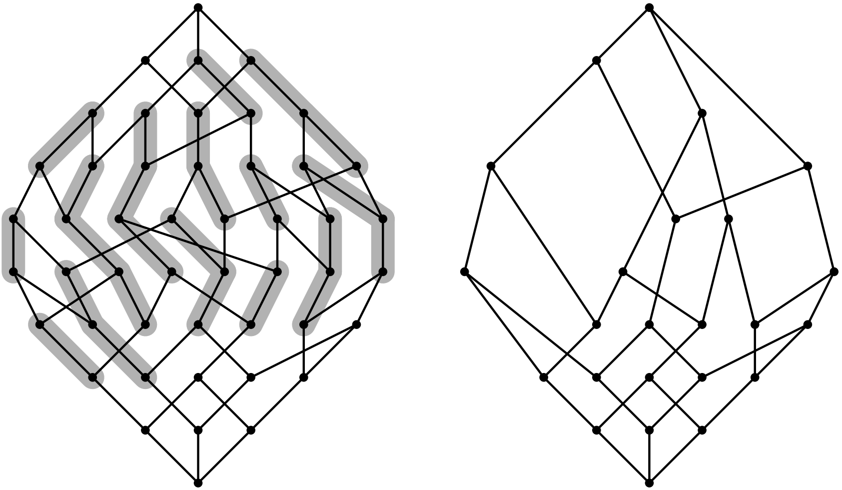
This theorem is the origin of the study of the dominance relation on exchange matrices.

A research theme: Lattice theory of the weak order on finite Coxeter groups "knows" a lot of combinatorics and representation theory.

Example: A_3 as a lattice quotient of B_3



Section 3. Ring homomorphisms

Ring homomorphisms of cluster algebras (finite type)

Rays of the mutation fan $\mathcal{F}_B$ are in bijection with cluster variables.

If $\mathcal{F}_B$ refines $\mathcal{F}_{B'}$, there is an inclusion

$$\{\text{rays of } \mathcal{F}_{B'}\} \hookrightarrow \{\text{rays of } \mathcal{F}_B\}$$

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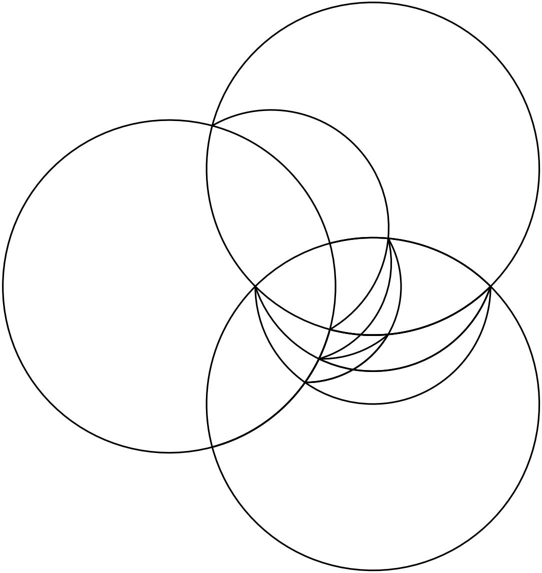
Let's look at a picture...

Ring homomorphisms of cluster algebras (finite type)



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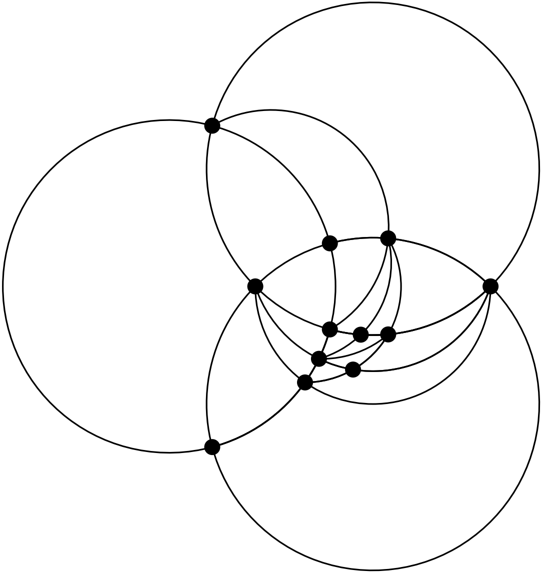

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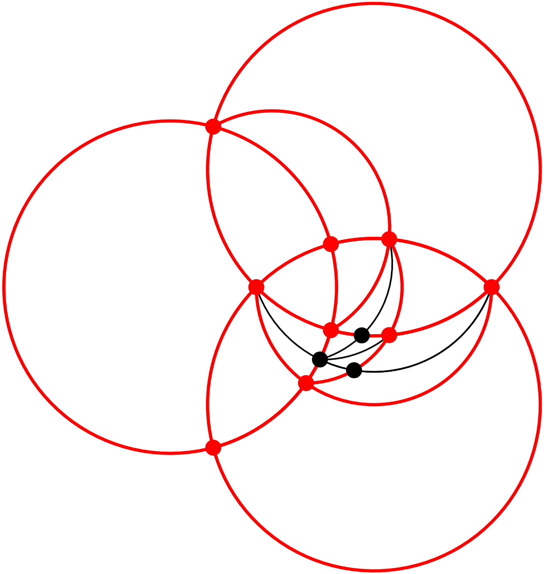

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Therefore there is a natural injective map on cluster variables.

Theorem* (Reading 2017, Viel, thesis in progress). This injection extends to a $\mathbf{g}$-vector-preserving injective homomorphism from $\mathcal{A}_\bullet(B')$ to $\mathcal{A}_\bullet(B)$. The map sends initial cluster variables to initial cluster variables and on the tropical (coefficient) variables, it is

$$y'_k \mapsto y_k z_k$$

where z_k is the cluster monomial whose $\mathbf{g}$-vector is the k^{th} column of B minus the k^{th} column of B'.

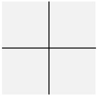
Remarks on ring homomorphisms (finite type)

- Structure-preserving maps (ring structure and **g**-vectors).
- Close algebraic relationships between cluster algebras with different exchange matrices of the same rank were not previously known.

- The homomorphism sends y'_k to where it needs to go to preserve **g**-vectors.

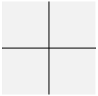
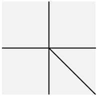
- Proof idea: the map defined on the initial cluster variables is obviously a homomorphism to something, and is injective (check the Jacobian matrix). Check that it sends cluster variables to cluster variables.
- Equivalently, the map sends $\hat{y}'_k$ to $\hat{y}_k$ times the F-polynomial of z_k and we check that it sends F-polynomials of cluster variables to F-polynomials of cluster variables.

Rank-2 examples



$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
x_1	1
x_2^{-1}	$1 + \hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$
x_2	1

Rank-2 examples

		
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1 x_2^{-1}$		$1 + \hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
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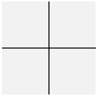 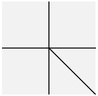

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x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

$$\begin{aligned} \hat{y}_1 &\mapsto \hat{y}_1 \\ \hat{y}_2 &\mapsto \hat{y}_2(1 + \hat{y}_1) \end{aligned}$$

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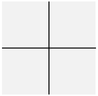 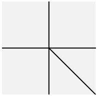

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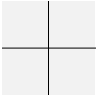 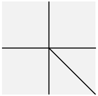 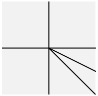

$\downarrow \mathbf{g}$ $\quad B \rightarrow$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$
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x_2^{-1}	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$
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	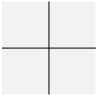	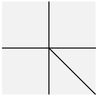	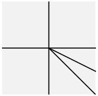
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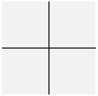 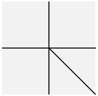 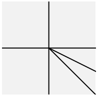

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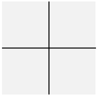
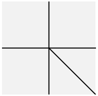
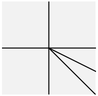
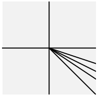
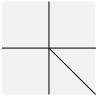
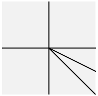
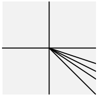
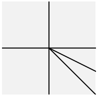
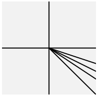
x_1	1	1	1
$x_1^2 x_2^{-1}$			$1+\hat{y}_2$
$x_1 x_2^{-1}$		$1+\hat{y}_2$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$
x_2^{-1}	$1+\hat{y}_2$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$
x_1^{-1}	$1+\hat{y}_1$	$1+\hat{y}_1$	$1+\hat{y}_1$
x_2	1	1	1

				
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1	1	1
$x_1^3 x_2^{-1}$				$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$			$1 + \hat{y}_2$	$1 + \hat{y}_2 +$
$x_1^3 x_2^{-2}$				$1 + 2\hat{y}_2 + \hat{y}_2^2 +$
$x_1 x_2^{-1}$		$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 +$
x_2^{-1}	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$	$1 + \hat{y}_2 +$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1	1	1

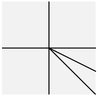
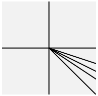
Rank-2 examples

			
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1	1
$x_1^3 x_2^{-1}$			$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$		$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$			$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1 \hat{y}_2^2$ …
$x_1 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}$ …
x_2^{-1}	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1 \hat{y}_2 + 3$ …
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1	1

Rank-2 examples

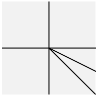
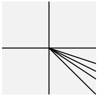
		
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1 \hat{y}_2^2 + 3\hat{y}_1^2 \hat{y}_2^2 + \hat{y}_1^3 \hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1^2 \hat{y}_2 + \hat{y}_1^3 \hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Rank-2 examples

		
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1 \hat{y}_2^2 + 3\hat{y}_1^2 \hat{y}_2^2 + \hat{y}_1^3 \hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1^2 \hat{y}_2 + \hat{y}_1^3 \hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

$$\begin{aligned} \hat{y}_1 &\mapsto \hat{y}_1 \\ \hat{y}_2 &\mapsto \hat{y}_2(1 + \hat{y}_1) \end{aligned}$$

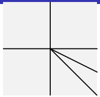
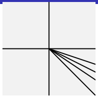
Rank-2 examples

		
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1\hat{y}_2^2 + 3\hat{y}_1^2\hat{y}_2^2 + \hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2 + \hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:
There are **g**-vector preserving homomorphisms whenever

- B is of finite or affine type, or
- B' is of finite type

Rank-2 examples

$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1\hat{y}_2^2 + 3\hat{y}_1^2\hat{y}_2^2 + \hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2 + \hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:
There are **g**-vector preserving homomorphisms whenever

- B is of finite or affine type, or
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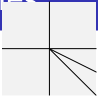
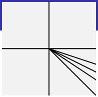
Rank-2 examples

		
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3x_2^{-1}$		$1+\hat{y}_2$
$x_1^2x_2^{-1}$	$1+\hat{y}_2$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$
$x_1^3x_2^{-2}$		$1+2\hat{y}_2+\hat{y}_2^2+3\hat{y}_1\hat{y}_2+3\hat{y}_1\hat{y}_2^2+3\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2$
$x_1x_2^{-1}$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$	$1+\hat{y}_2+3\hat{y}_1\hat{y}_2+3\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1+\hat{y}_1$	$1+\hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:

There are **g**-vector preserving homomorphisms whenever

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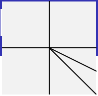
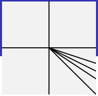
Rank-2 exampl

$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1\hat{y}_2^2 + 3\hat{y}_1^2\hat{y}_2^2 + \hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2 + \hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:

There are **g**-vector preserving homomorphisms whenever

- B is of finite or affine type, or
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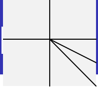
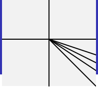
In these cases, cluster variables are sent to cluster variables (or

Rank-2 exampl



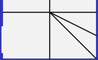


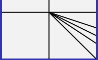
$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1\hat{y}_2^2 + 3\hat{y}_1^2\hat{y}_2^2 + \hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2 + \hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:
There are **g**-vector preserving homomorphisms whenever

- B is of finite or affine type, or
- B' is of finite type.

In these cases, cluster variables are sent to cluster variables (or “any theta functions”) unless $B = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$ and $B' = \begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}$ with





$\downarrow \mathbf{g}$ $\quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3x_2^{-1}$		$1+\hat{y}_2$
$x_1^2x_2^{-1}$	$1+\hat{y}_2$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$
$x_1^3x_2^{-2}$		$1+2\hat{y}_2+\hat{y}_2^2+3\hat{y}_1\hat{y}_2+3\hat{y}_1\hat{y}_2^2+3\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2$
$x_1x_2^{-1}$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$	$1+\hat{y}_2+3\hat{y}_1\hat{y}_2+3\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1+\hat{y}_1$	$1+\hat{y}_1$
x_2	1	1

Summary of what I know in rank-2**:**

There are **g**-vector preserving homomorphisms whenever

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In these cases, cluster variables are sent to cluster variables (or "ray theta functions") unless $B = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$ and $B' = \begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}$ with $cd = -3$ and $1 \notin \{|a|, |b|\}$.

Rank-2 examples

$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1\hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1\hat{y}_2^2 + 3\hat{y}_1^2\hat{y}_2^2 + \hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1\hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1\hat{y}_2 + 3\hat{y}_1^2\hat{y}_2 + \hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:

There are **g**-vector preserving homomorphisms whenever

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$\downarrow \mathbf{g}$ $\quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1+\hat{y}_2$
$x_1^2 x_2^{-1}$	$1+\hat{y}_2$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$
$x_1^3 x_2^{-2}$		$1+2\hat{y}_2+\hat{y}_2^2+3\hat{y}_1\hat{y}_2+3\hat{y}_1\hat{y}_2^2+3\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2$
$x_1 x_2^{-1}$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$	$1+\hat{y}_2+3\hat{y}_1\hat{y}_2+3\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2$
x_1^{-1}	$1+\hat{y}_1$	$1+\hat{y}_1$
x_2	1	1

Summary of what I know in rank-2:
There are **g**-vector preserving homomorphisms whenever

- B is of finite or affine type, or
- B' is of finite type.

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Homomorphisms may exist in additional cases

Rank-2 examples

$\downarrow \mathbf{g} \quad B \rightarrow$	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$
x_1	1	1
$x_1^3 x_2^{-1}$		$1 + \hat{y}_2$
$x_1^2 x_2^{-1}$	$1 + \hat{y}_2$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$
$x_1^3 x_2^{-2}$		$1 + 2\hat{y}_2 + \hat{y}_2^2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1 \hat{y}_2^2 + 3\hat{y}_1^2 \hat{y}_2^2 + \hat{y}_1^3 \hat{y}_2^2$
$x_1 x_2^{-1}$	$1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2$	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$
x_2^{-1}	$1 + \hat{y}_2 + 2\hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2$	$1 + \hat{y}_2 + 3\hat{y}_1 \hat{y}_2 + 3\hat{y}_1^2 \hat{y}_2 + \hat{y}_1^3 \hat{y}_2$
x_1^{-1}	$1 + \hat{y}_1$	$1 + \hat{y}_1$
x_2	1	1

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$x_1x_2^{-1}$	$1+\hat{y}_2+\hat{y}_1\hat{y}_2$	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$
x_2^{-1}	$1+\hat{y}_2+2\hat{y}_1\hat{y}_2+\hat{y}_1^2\hat{y}_2$	$1+\hat{y}_2+3\hat{y}_1\hat{y}_2+3\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2$
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Homomorphisms may exist in additional cases.

The proof in the surfaces case (finite type)

Cluster variables	$\longleftrightarrow$	(tagged) arcs
Coefficient variables	$\longleftrightarrow$	"elementary laminations"

Strategy: Consider

- A homomorphism ν sending initial cluster variables to initial cluster variables and sending coefficients to coefficients times cluster monomials (as before).
- A map χ sending each cluster variable to the cluster variable with the same $\mathbf{g}$-vector and treating coefficients like ν.

ν and χ agree on initial cluster variables and coefficients.

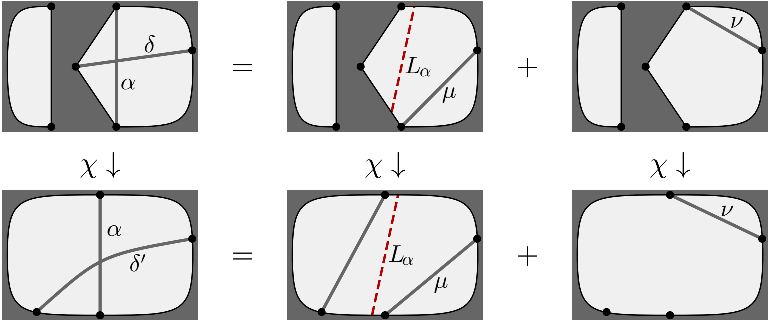
Thus, if we show that χ sends each exchange relation to a valid relation, we can conclude that χ is the restriction of ν (which in particular maps to the cluster algebra).

The proof in the surfaces case (continued)

χ sends each cluster variable to the cluster variable with the same $\mathbf{g}$-vector, sends coefficients to coefficients times cluster monomials.

Want: χ sends each exchange relation to a valid relation.

Example:

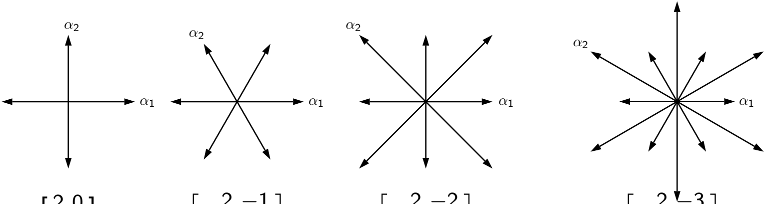


Aside: Dominance on Cartan matrices

A Cartan matrix $A = [a_{ij}]$ dominates a Cartan matrix $A' = [a'_{ij}]$

$$|a_{ij}| \geq |a'_{ij}| \text{ for all } i, j.$$

Theorem (R., 2018) If A dominates A' then $\Phi(A') \subseteq \Phi(A)$.



$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

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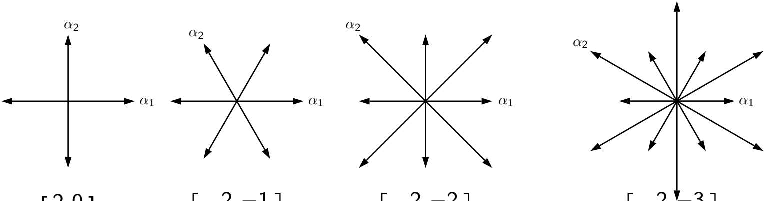

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... but only if you do it right.

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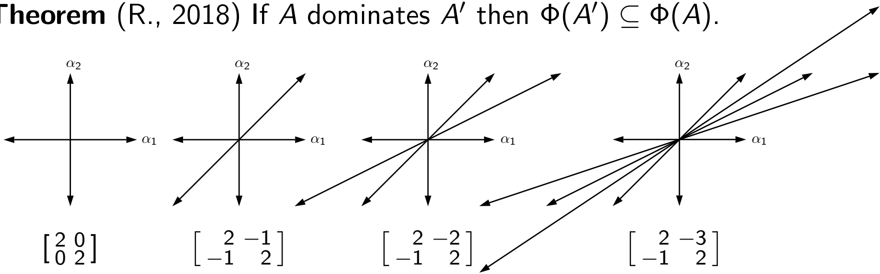

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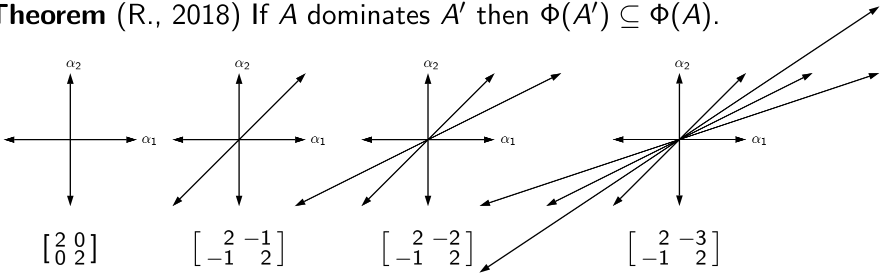

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- Same simple roots in both root systems
- Include imaginary roots

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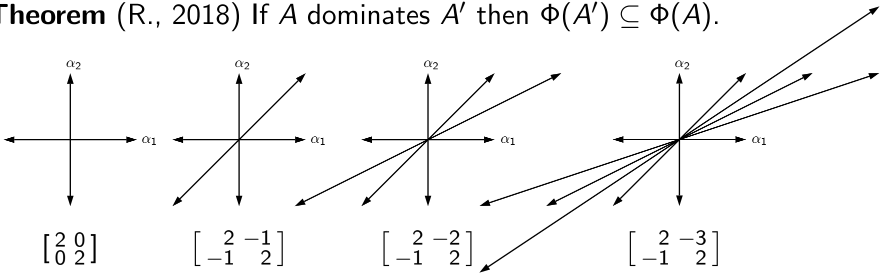

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Proof: Kac-Moody Lie algebras (Serre relations)

Dominance phenomena scorecard (B dominates B')

Phenomenon	**Cases where it is known**
I & II (μ-linearity and mutation fan refinement)	• acyclic finite type (& affine soon with Stella?) • resection of surfaces ($\mathbb{Q}$ versions) • erasing arrows to disconnect the quiver • fully characterized in rank 2 (occurs and fails)
III (scattering fan refinement)	• acyclic finite type (& affine soon with Stella?) • finite type surfaces (& more soon with Muller?) • occurs always* in rank 2
IV ($\mathbf{g}$-vector-preserving ring homomorphisms)	• acyclic finite type • rank 2, B finite or affine type • rank 2, B' finite type • some non-acyclic surfaces of finite type

arXiv:1802.10107

Thanks for listening.