A fan for every cluster algebra

Nathan Reading
NC State University

Combinatorics and beyond: the many facets
of Sergey Fomin's mathematics
University of Michigan
November 8-11, 2018

Cluster algebras of finite type

g-Vector fans

Other fan constructions

Scattering diagrams/fans

Combinatorial models

Opening remarks

- Thanks to Sergey and to the organizers
- Some of the recent/ongoing work I'll mention is joint with David Speyer. Some is joint with Salvatore Stella. Some is joint with Greg Muller and Shira Viel.
- Assumptions about the audience
- A complete fan for every cluster algebra.

Opening remarks

- Thanks to Sergey and to the organizers
- Some of the recent/ongoing work I'll mention is joint with David Speyer. Some is joint with Salvatore Stella. Some is joint with Greg Muller and Shira Viel.
- Assumptions about the audience
- A complete fan for every cluster algebra.

Opening remarks

- Thanks to Sergey and to the organizers
- Some of the recent/ongoing work I'll mention is joint with David Speyer. Some is joint with Salvatore Stella. Some is joint with Greg Muller and Shira Viel.
- Assumptions about the audience
- A complete fan for every ~~cluster algebra~~.

Opening remarks

- Thanks to Sergey and to the organizers
- Some of the recent/ongoing work I'll mention is joint with David Speyer. Some is joint with Salvatore Stella. Some is joint with Greg Muller and Shira Viel.
- Assumptions about the audience
- A complete fan for every ~~cluster algebra~~ exchange matrix.

Section 1: Cluster algebras of finite type

(Principal coefficients) Cluster algebras

Start with an initial seed consisting of initial cluster of cluster variables $x_1, \ldots x_n$ and a skew-symmetric integer matrix B.

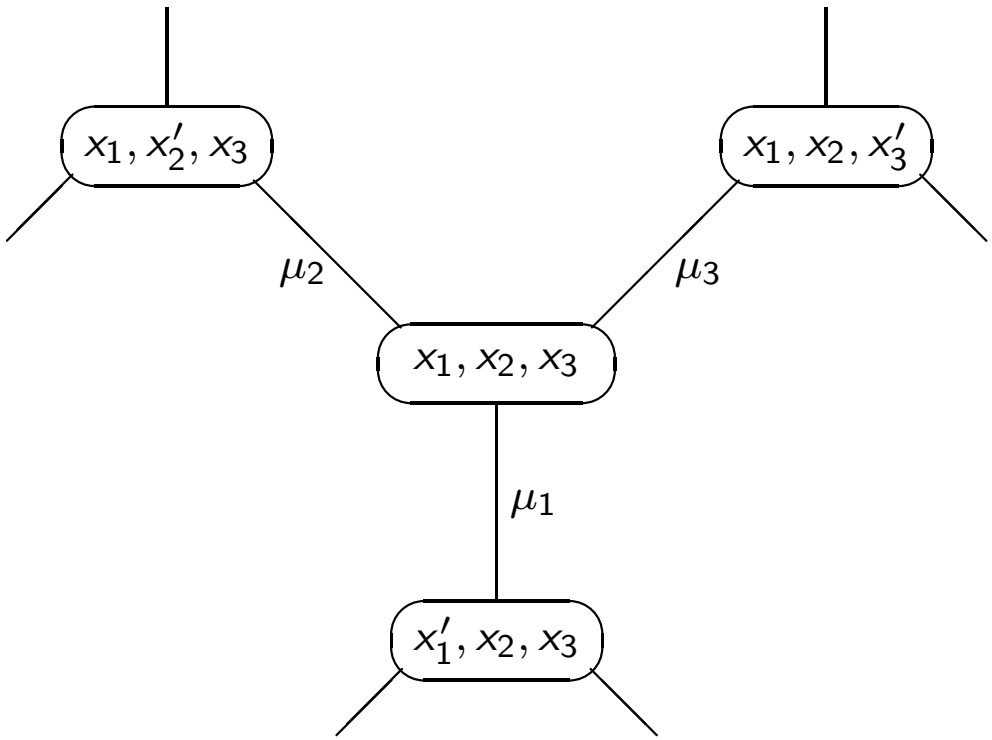
Mutation: an operation that takes a seed and gives a new seed.

- There are n "directions" for mutation.
- Mutation does two things:
  - switches out one cluster variable, replaces it with a new cluster variable to obtain a new cluster;
  - changes B (and some extra rows) by matrix mutation.

  The result is a new seed.
- Mutation is involutive.

(Principal coefficients) Cluster algebras (continued)

Do all possible sequences of mutations, and collect all the cluster variables which appear.



The cluster algebra $\mathcal{A}_\bullet(B)$ for the given initial seed is the subalgebra of $\mathcal{F}$ generated by all cluster variables.

Mutation

Write $[a]_+$ for $\max(a, 0)$. The mutation of B in direction k is the matrix $B' = \mu_k(B)$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}; \\ b_{ij} + \operatorname{sgn}(b_{kj})[b_{ik}b_{kj}]_+ & \text{otherwise.} \end{cases}$$

For principal coefficients, we replace B by $\left[\begin{smallmatrix} B \\ I \end{smallmatrix}\right]$ but we only mutate in directions $1, \ldots, n$.

We also introduce coefficients $y_1, \ldots, y_n$.

Mutating the cluster variables $x_1, \ldots, x_n$ in direction k means keeping x_i for $i \neq k$ and replacing x_k by x'_k according to the exchange relations

$$x_k x'_k = \prod_{i=1}^{n} x_i^{[b_{ik}]_+} y_i^{[b_{(n+i)k}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} y_i^{[-b_{(n+i)k}]_+}.$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\mu_1}{\longleftrightarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \overset{\mu_2}{\longleftrightarrow} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\updownarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{\mu_1}{\longleftrightarrow} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \overset{\mu_2}{\longleftrightarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\mu_1}{\longleftrightarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \overset{\mu_2}{\longleftrightarrow} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[x_1 \;\; x_2]$$

$$\updownarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{\mu_1}{\longleftrightarrow} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \overset{\mu_2}{\longleftrightarrow} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[x_1 \;\; x_2] \qquad \left[\frac{x_2+y_1}{x_1} \;\; x_2\right]$$

$$\updownarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[x_1 \;\; x_2] \qquad \left[\frac{x_2+y_1}{x_1} \;\; x_2\right] \qquad \left[\frac{x_2+y_1}{x_1} \;\; \frac{x_1y_1y_2+x_2+y_1}{x_1x_2}\right]$$

$$\updownarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[x_1 \;\; x_2] \qquad \left[\frac{x_2+y_1}{x_1} \;\; x_2\right] \qquad \left[\frac{x_2+y_1}{x_1} \;\; \frac{x_1y_1y_2+x_2+y_1}{x_1x_2}\right]$$

$$\updownarrow \mu_2$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\left[x_1 \;\; \frac{1+x_1y_2}{x_2}\right] \qquad \left[\frac{x_1y_1y_2+x_2+y_1}{x_1x_2} \;\; \frac{1+x_1y_2}{x_2}\right] \qquad \left[\frac{x_1y_1y_2+x_2+y_1}{x_1x_2} \;\; \frac{x_2+y_1}{x_1}\right]$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \longleftarrow$$

$$[x_1 \;\; x_2] \qquad \left[\frac{x_2+y_1}{x_1} \;\; x_2\right] \qquad \left[\frac{x_2+y_1}{x_1} \;\; \frac{x_1y_1y_2+x_2+y_1}{x_1x_2}\right]$$

$\updownarrow \mu_2$

identify these

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_1}{\longleftrightarrow} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \longleftarrow$$

$$\left[x_1 \;\; \frac{1+x_1y_2}{x_2}\right] \qquad \left[\frac{x_1y_1y_2+x_2+y_1}{x_1x_2} \;\; \frac{1+x_1y_2}{x_2}\right] \qquad \left[\frac{x_1y_1y_2+x_2+y_1}{x_1x_2} \;\; \frac{x_2+y_1}{x_1}\right]$$

Mutation example

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x_1 & x_2 \end{bmatrix} \qquad \overset{\mu_1}{\longleftrightarrow} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{x_2+y_1}{x_1} & x_2 \end{bmatrix} \qquad \overset{\mu_2}{\longleftrightarrow} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} \frac{x_2+y_1}{x_1} & \frac{x_1y_1y_2+x_2+y_1}{x_1x_2} \end{bmatrix} \longleftarrow$$

$$\updownarrow \mu_2$$

identify these

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} x_1 & \frac{1+x_1y_2}{x_2} \end{bmatrix} \qquad \overset{\mu_1}{\longleftrightarrow} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} \frac{x_1y_1y_2+x_2+y_1}{x_1x_2} & \frac{1+x_1y_2}{x_2} \end{bmatrix} \qquad \overset{\mu_2}{\longleftrightarrow} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{x_1y_1y_2+x_2+y_1}{x_1x_2} & \frac{x_2+y_1}{x_1} \end{bmatrix} \longleftarrow$$

This cluster algebra is of finite type (finite # of cluster variables).

The cluster variables are: $x_1,\ x_2,\ \frac{x_2+y_1}{x_1},\ \frac{x_1y_1y_2+x_2+y_1}{x_1x_2},\ \frac{1+x_1y_2}{x_2}.$

Cluster algebras of finite type

Theorem (FZ, 2001). For any exchange matrix B, every cluster variable is a Laurent polynomial in the x_i with coefficients (ordinary) polynomials in the y_i.

Thus, each cluster variable has a denominator vector (**d**-vector).

Example (continued):

Cluster variable:	x_1	x_2	$\frac{x_2+y_1}{x_1}$	$\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}$	$\frac{1+x_1y_2}{x_2}$
d-vector:	$[-1, 0]$	$[0, -1]$	$[1, 0]$	$[1, 1]$	$[0, 1]$

Theorem (FZ, 2003). The cluster algebra $\mathcal{A}_\bullet(B)$ is of finite type if and only if B is mutation-equivalent to an exchange matrix whose associated Cartan matrix A is of finite type. In this case, the **d**-vectors are the almost positive roots for A.

In the example, $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ (type A_2).

Cluster algebras of finite type (continued)

Fomin and Zelevinsky defined a notion of compatibility on the set of almost positive roots for A (of finite type).

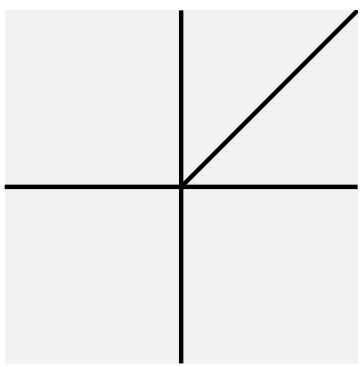
combinatorial clusters: maximal pairwise compatible sets of almost positive roots.

Theorem (FZ, 2003). For B of finite type, the map from cluster variables to denominator vectors takes clusters of cluster variables bijectively to combinatorial clusters of almost positive roots.

Theorem (FZ, 2003). For B of finite type, each combinatorial cluster spans a full-dimensional cone. These cones are the maximal cones of a complete fan (the **d**-vector fan).

Example (continued):

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

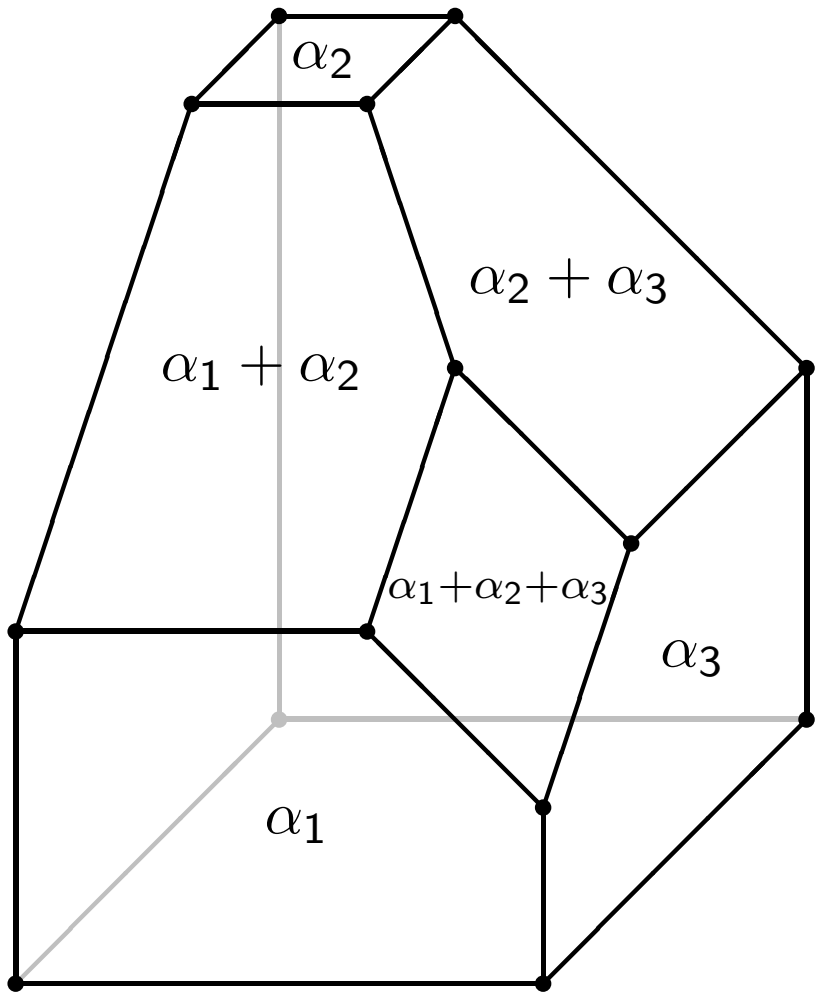


Generalized Associahedra

Theorem (Chapoton-FZ 2002). The **d**-vector fan is the normal fan of a polytope, the generalized associahedron.

Example:

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$



Section 2: $\mathbf{g}$-Vector fans

The **g**-vector fan

Define $\hat{y}_j = y_j \prod_{i=1}^{n} x^{b_{ij}}$.

Theorem (FZ 2007). Each cluster variable is a monomial in the x_i times a polynomial in the $\hat{y}_j$.

The **g**-vector of the cluster variable is the exponent sequence of that monomial.

Example (continued): $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

The **g**-vector fan

Define $\hat{y}_j = y_j \prod_{i=1}^{n} x^{b_{ij}}$.

Theorem (FZ 2007). Each cluster variable is a monomial in the x_i times a polynomial in the $\hat{y}_j$.

The **g**-vector of the cluster variable is the exponent sequence of that monomial.

Example (continued): $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\hat{y}_1 = y_1 x_2^{-1}$

The **g**-vector fan

Define $\hat{y}_j = y_j \prod_{i=1}^{n} x^{b_{ij}}$.

Theorem (FZ 2007). Each cluster variable is a monomial in the x_i times a polynomial in the $\hat{y}_j$.

The **g**-vector of the cluster variable is the exponent sequence of that monomial.

Example (continued): $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ $\hat{y}_1 = y_1 x_2^{-1}$ $\hat{y}_2 = y_2 x_1$.

The g-vector fan

Define $\hat{y}_j = y_j \prod_{i=1}^{n} x^{b_{ij}}$.

Theorem (FZ 2007). Each cluster variable is a monomial in the x_i times a polynomial in the $\hat{y}_j$.

The **g**-vector of the cluster variable is the exponent sequence of that monomial.

Example (continued): $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ $\hat{y}_1 = y_1 x_2^{-1}$ $\hat{y}_2 = y_2 x_1$.

The cluster variables:

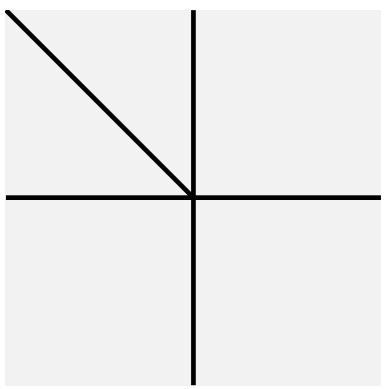
Cluster variable:	x_1	x_2	$\frac{x_2+y_1}{x_1}$	$\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}$	$\frac{1+x_1y_2}{x_2}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
	x_1	x_2	$\frac{x_2(1+\hat{y}_1)}{x_1}$	$\frac{1+\hat{y}_1+\hat{y}_1\hat{y}_2}{x_1}$	$\frac{1+\hat{y}_2}{x_2}$
g-vector:	$[1, 0]$	$[0, 1]$	$[-1, 1]$	$[-1, 0]$	$[0, -1]$

The $\mathbf{g}$-vector fan (continued)

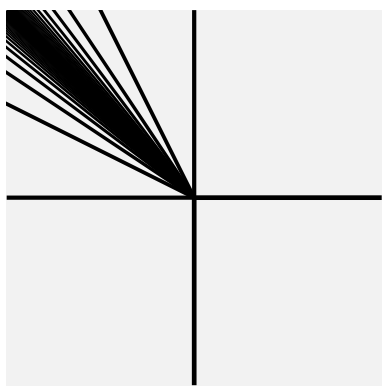
The $\mathbf{g}$-vectors of each cluster span a full-dimensional cone. These $\mathbf{g}$-vector cones are the maximal cones of a fan. (Conjectured: FZ 2007. Proved: GHKK 2014.)

The $\mathbf{g}$-vector fan is combinatorially isomorphic to the $\mathbf{d}$-vector fan.

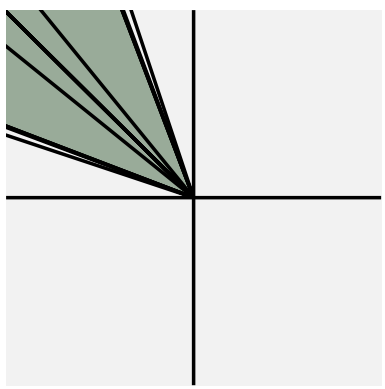
In finite type, the $\mathbf{g}$-vector fan is complete. Outside of finite type, it is not complete. (It does have nice properties: Maximal cones are full-dimensional, and each codimension-1 cone is a face of exactly two full-dimensional cones.)



$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

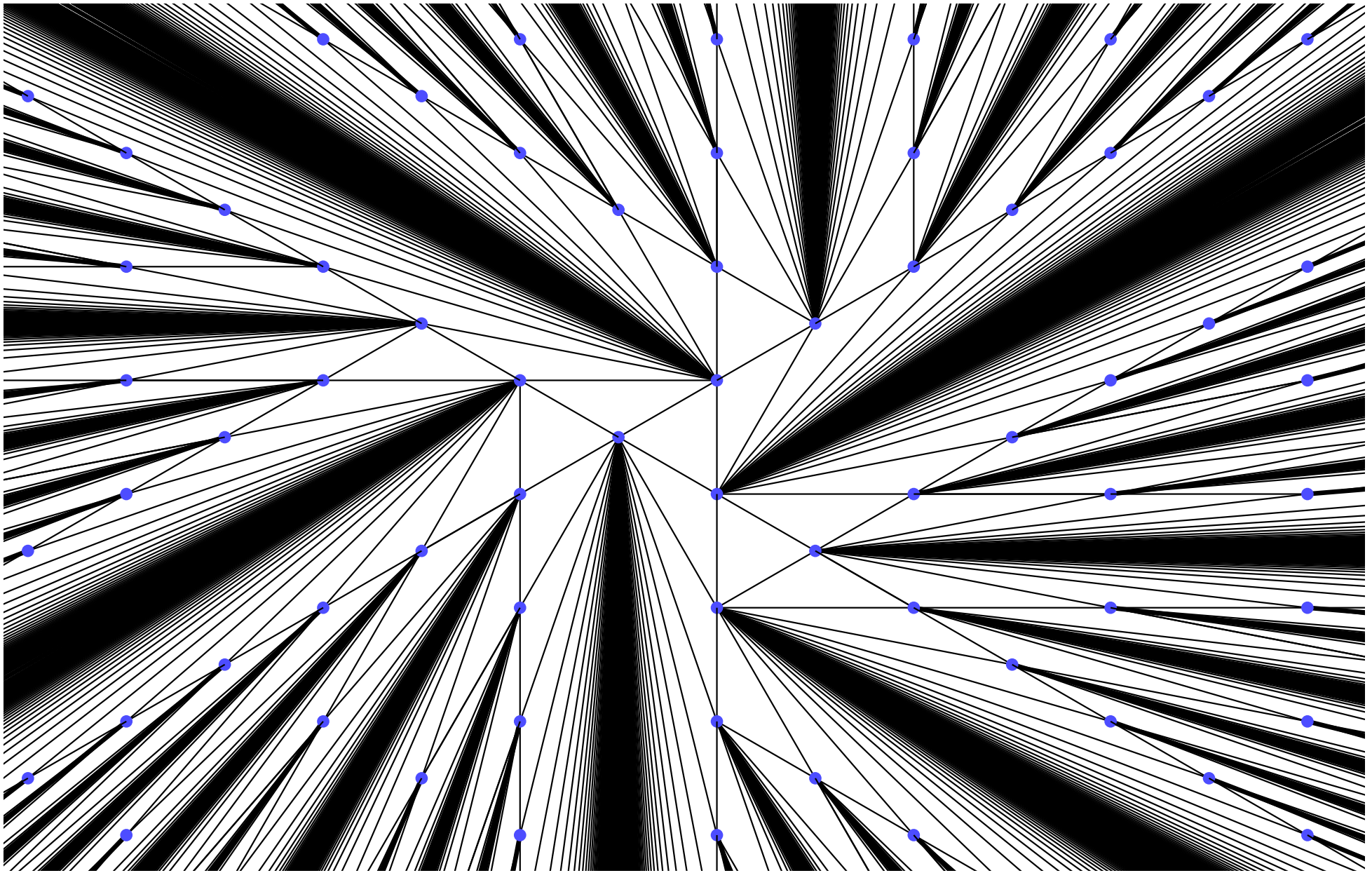


$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

Another infinite $\mathbf{g}$-vector fan example: $B = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$



Are $\mathbf{g}$-vector fans what we want?

The $\mathbf{g}$-vector fan is a fan. A fan for every cluster algebra!

(so, can we quit early and have an extra-long break?)

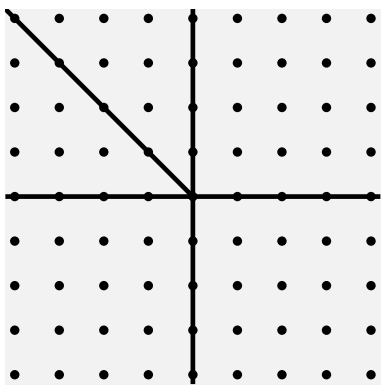
Are $\mathbf{g}$-vector fans what we want?

The $\mathbf{g}$-vector fan is a fan. A fan for every cluster algebra!
(so, can we quit early and have an extra-long break?)

The problem is, that the $\mathbf{g}$-vector fan is not complete if B is not of finite type. Why is that a problem?

Because we are interested in additive bases for the cluster algebra. (This was one of the original motivations. "Dual canonical basis".)

In finite type, you have a great additive basis: cluster monomials. A cluster monomial is a monomial in a set of cluster variables, all contained in one cluster. You have one basis element for each $\mathbf{g}$-vector.



Are $\mathbf{g}$-vector fans what we want?

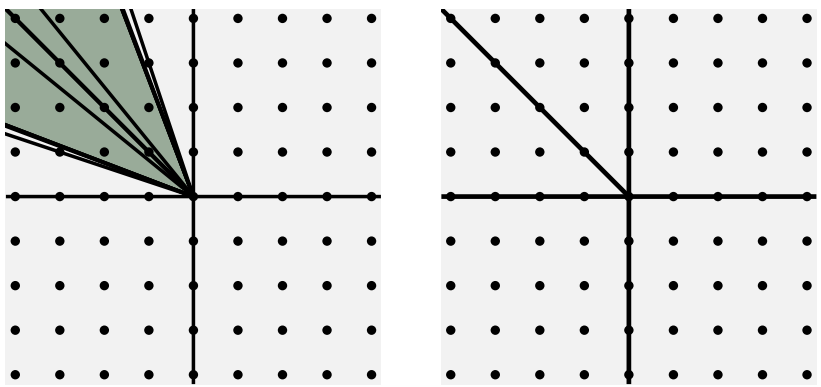
The $\mathbf{g}$-vector fan is a fan. A fan for every cluster algebra!
(so, can we quit early and have an extra-long break?)

The problem is, that the $\mathbf{g}$-vector fan is not complete if B is not of finite type. Why is that a problem?

Because we are interested in additive bases for the cluster algebra. (This was one of the original motivations. "Dual canonical basis".)

In finite type, you have a great additive basis: cluster monomials. A cluster monomial is a monomial in a set of cluster variables, all contained in one cluster. You have one basis element for each $\mathbf{g}$-vector.

In infinite type, you only have cluster monomials for $\mathbf{g}$-vectors in the $\mathbf{g}$-vector fan.



Are $\mathbf{g}$-vector fans what we want? (cont'd)

In infinite type, you only get one cluster monomial for each $\mathbf{g}$-vector in the $\mathbf{g}$-vector fan.

Work of various people (probably starting with Sherman-Zelevinsky's work on 2×2 exchange matrices) constructed bases with one element for each $\mathbf{g}$-vector in the space.

Are $\mathbf{g}$-vector fans what we want? (cont'd)

In infinite type, you only get one cluster monomial for each $\mathbf{g}$-vector in the $\mathbf{g}$-vector fan.

Work of various people (probably starting with Sherman-Zelevinsky's work on 2×2 exchange matrices) constructed bases with one element for each $\mathbf{g}$-vector in the space.

Upshot: We need to understand the space outside the $\mathbf{g}$-vector fan.

Are $\mathbf{g}$-vector fans what we want? (cont'd)

In infinite type, you only get one cluster monomial for each $\mathbf{g}$-vector in the $\mathbf{g}$-vector fan.

Work of various people (probably starting with Sherman-Zelevinsky's work on 2×2 exchange matrices) constructed bases with one element for each $\mathbf{g}$-vector in the space.

Upshot: We need to understand the space outside the $\mathbf{g}$-vector fan.

Topic for the next few slides: We want a complete fan for every exchange matrix B. In finite type, but not in infinite type, the $\mathbf{g}$-vector fan gives us what we want.

So, are there other constructions that give the same fan in finite type? And do they give complete fans in infinite type?

Section 3: Other fan constructions

Finite $\mathbf{g}$-vector fans are Cambrian fans*

The Cartan matrix A associated to B defines a root system Φ and a Weyl group W. The reflecting hyperplanes for W cut space into a fan (the Coxeter fan).

Sortable elements associated to B: Certain elements of W defined by admitting a particular form of reduced word.

Each sortable element defines a cone via the combinatorics of reduced words, with inequalities described in terms of roots. These are the $\mathbf{g}$-vector cones. In particular, the $\mathbf{g}$-vector fan refines the Coxeter fan.

*When B is acyclic.

Finite $\mathbf{g}$-vector fans are Cambrian fans*

The Cartan matrix A associated to B defines a root system Φ and a Weyl group W. The reflecting hyperplanes for W cut space into a fan (the Coxeter fan).

Sortable elements associated to B: Certain elements of W defined by admitting a particular form of reduced word.

Each sortable element defines a cone via the combinatorics of reduced words, with inequalities described in terms of roots. These are the $\mathbf{g}$-vector cones. In particular, the $\mathbf{g}$-vector fan refines the Coxeter fan.

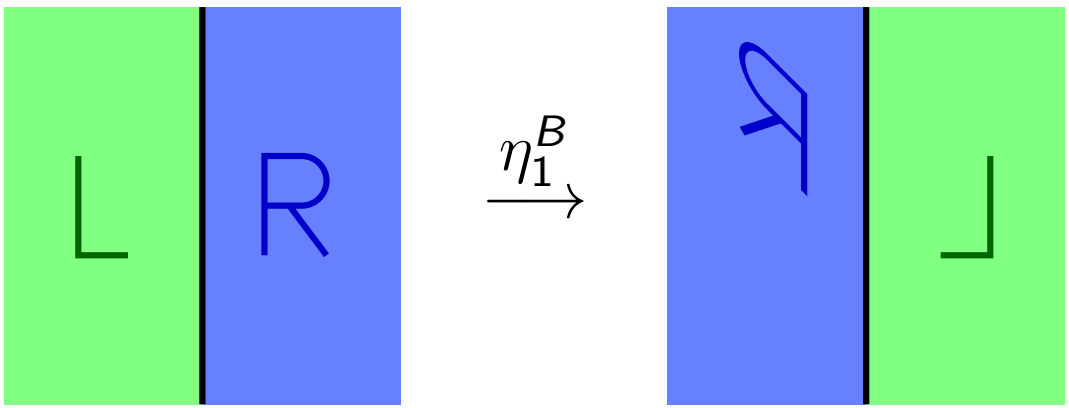
Does this generalize nicely to infinite type? Many nice things happen, but infinite Cambrian fans are not complete, and in fact are subfans of the $\mathbf{g}$-vector fan. (The issue: Cambrian fans can't "know" much about space outside of the Tits cone.)

*When B is acyclic.

The mutation fan

Let $\tilde{B}$ be $\left[\begin{smallmatrix} B \\ \mathbf{a} \end{smallmatrix}\right]$ (i.e. B with an extra row $\mathbf{a} \in \mathbb{R}^n$).
For $\mathbf{k} = k_q, k_{q-1}, \ldots, k_1$, define $\eta_{\mathbf{k}}^B(\mathbf{a})$ to be the last row of $\mu_{\mathbf{k}}(\tilde{B})$.

Example: $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$

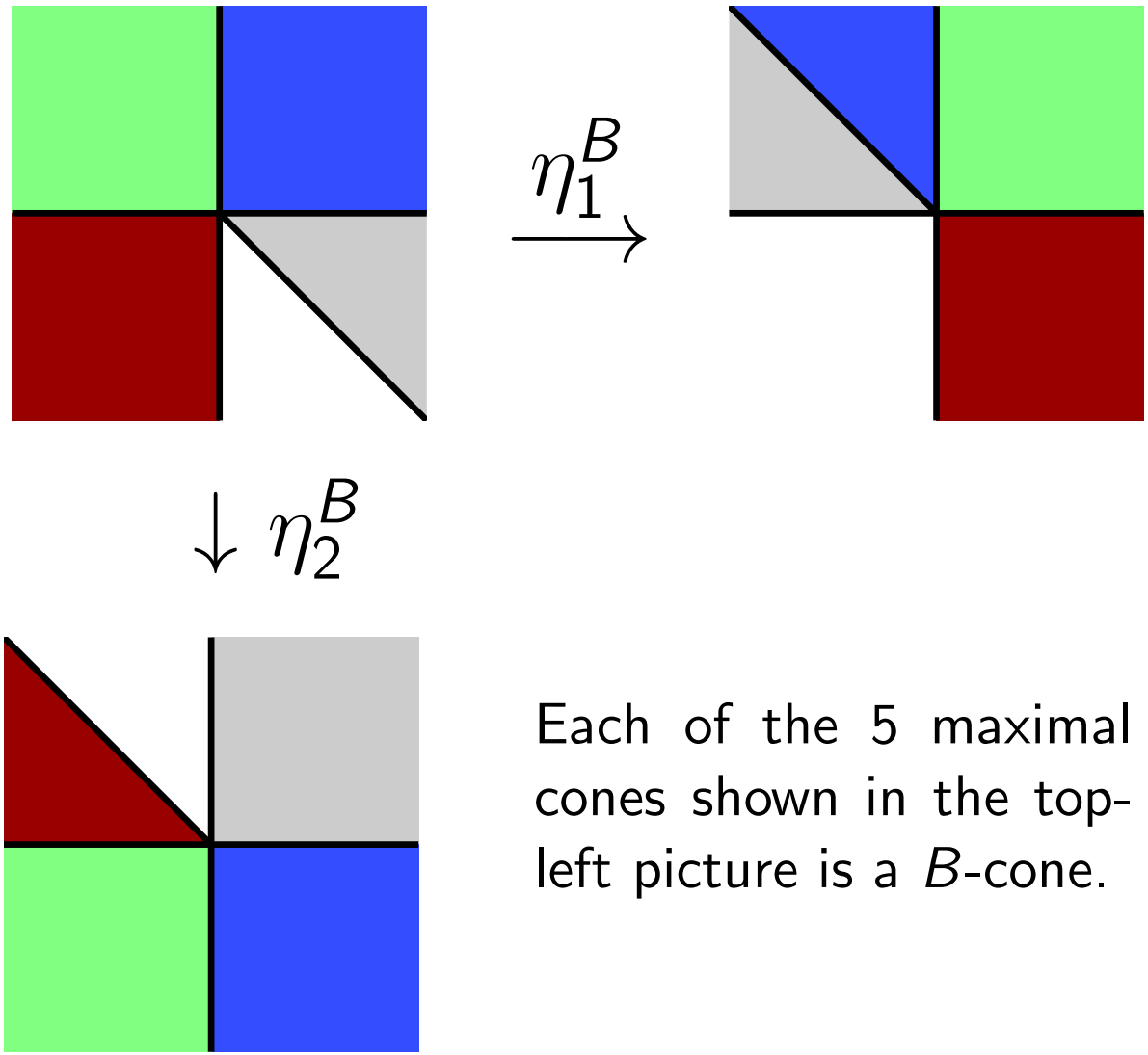


B-cones: common domains of linearity of all mutation maps.*
Mutation fan: the set of all B-cones and all faces of B-cones.

Theorem (R., 2011). $\mathcal{F}_B$ is a complete fan (possibly with infinitely many cones).

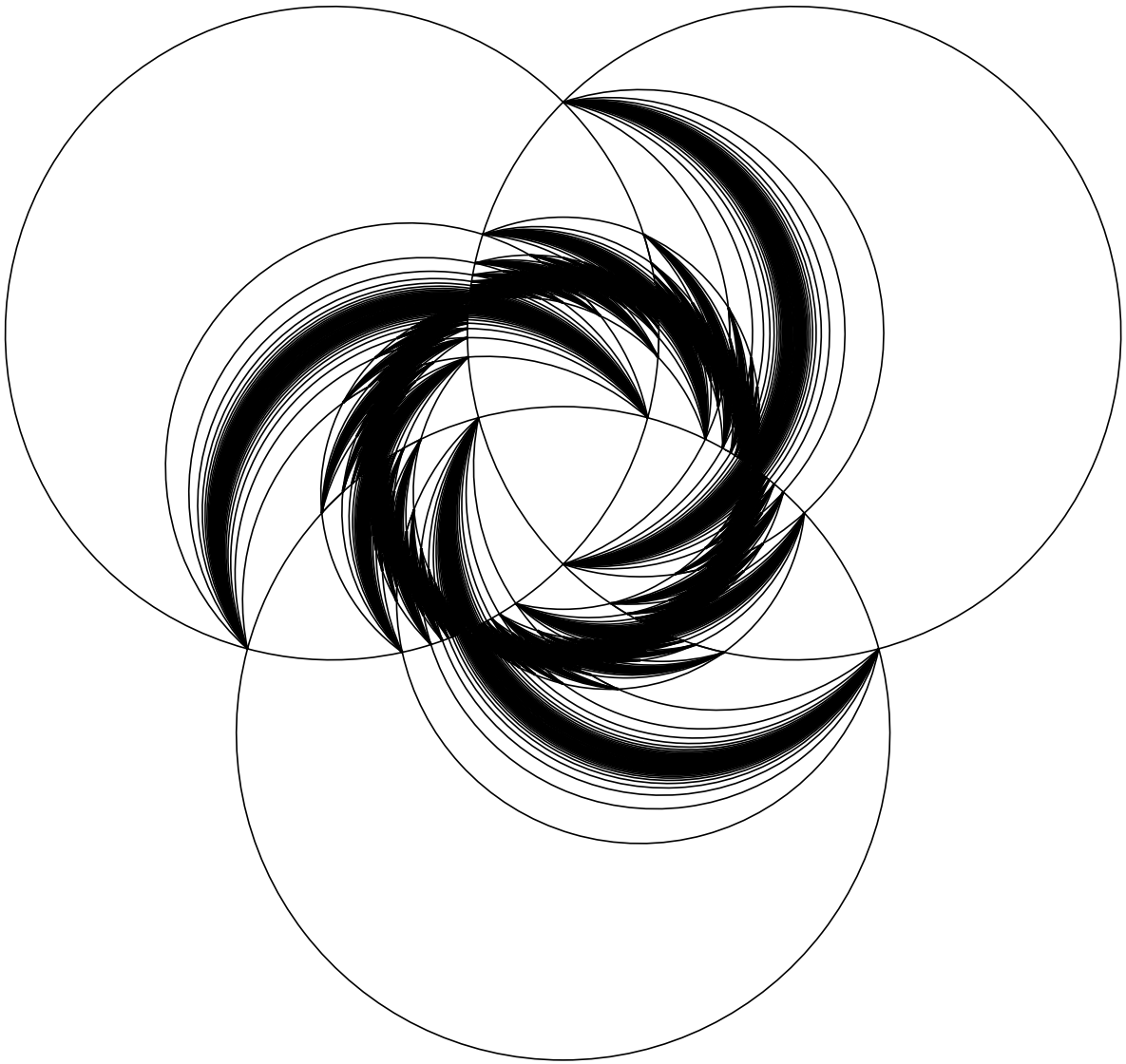
($\mathcal{F}_B$ is related to a notion of "universal" cluster algebras.)

Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



Each of the 5 maximal cones shown in the top-left picture is a B-cone.

Example: $B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ (Markov quiver)



Section 4: Scattering diagrams/fans

Scattering diagrams

Scattering diagrams arose from mirror symmetry, Donaldson-Thomas theory (string theory), integrable systems. There are also connections to stability conditions (representation theory).

Gross, Hacking, Keel, and Kontsevich recently applied scattering diagrams to prove longstanding conjectures about cluster algebras.

Scattering diagrams

Scattering diagrams arose from mirror symmetry, Donaldson-Thomas theory (string theory), integrable systems. There are also connections to stability conditions (representation theory).

Gross, Hacking, Keel, and Kontsevich recently applied scattering diagrams to prove longstanding conjectures about cluster algebras.

(Personal thanks to organizers of a recent MSRI Hot Topics workshop and to Mandy Cheung.)

Scattering diagram setup

Summary: skew-symmetric matrix, vector space and its dual, integer points $\leftrightarrow$ Laurent monomials.

Scattering diagram setup

Summary: skew-symmetric matrix, vector space and its dual, integer points $\leftrightarrow$ Laurent monomials.

Scattering diagram setup

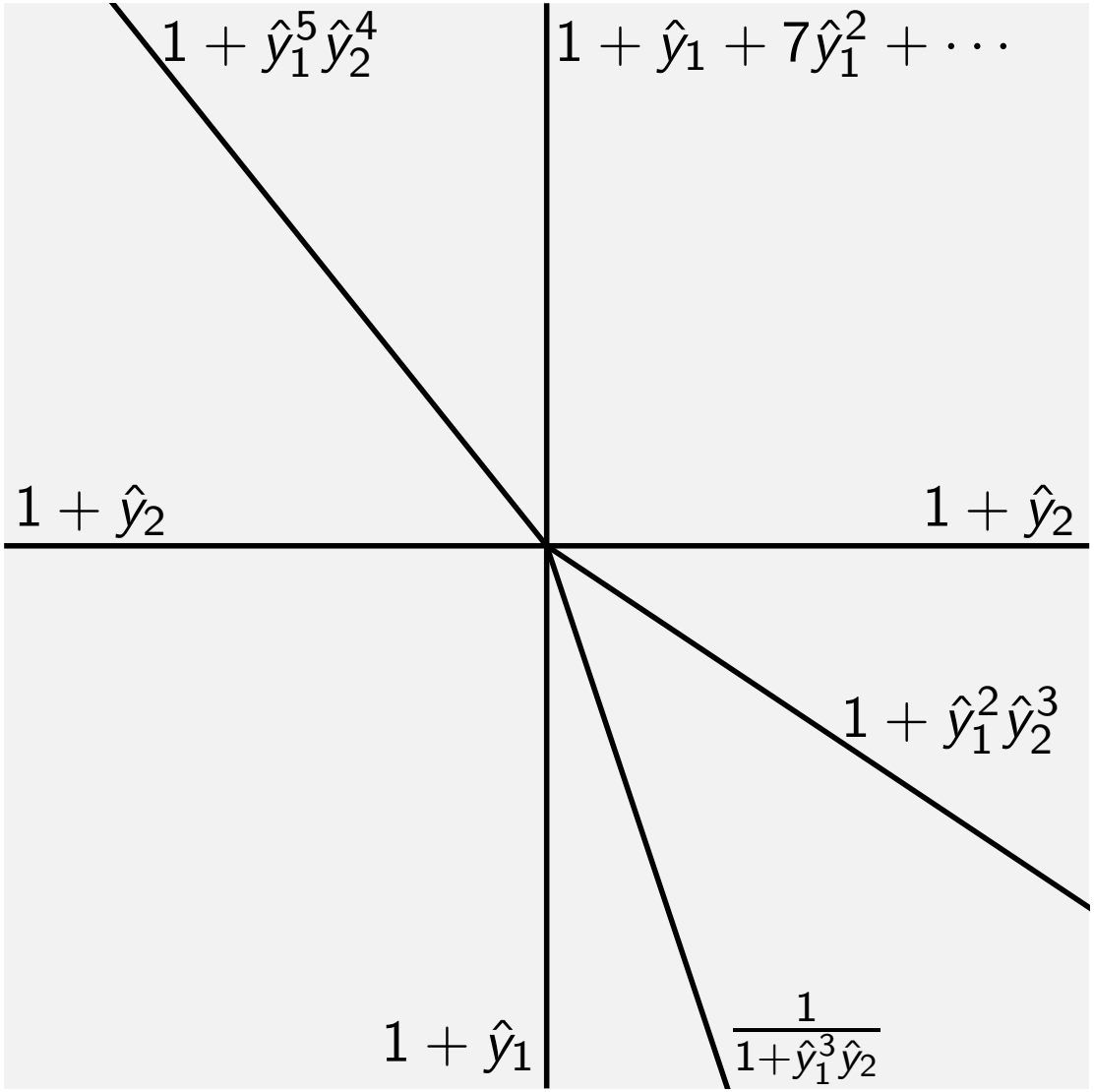
Summary: skew-symmetric matrix, vector space and its dual, integer points $\leftrightarrow$ Laurent monomials.

Details:

- B is an $n \times n$ skew-symmetric integer matrix
- V real vector space, basis $\alpha_1, \ldots, \alpha_n$
- V^* its dual space, basis $\rho_1, \ldots, \rho_n$
- $\langle \rho_i, \alpha_j \rangle = \delta_{ij}$ (Kronecker delta)
- integer points in V^*: $\lambda = \sum_{i=1}^n c_i \rho_i \quad \leftrightarrow \quad x^\lambda = x_1^{c_1} \cdots x_n^{c_n}$
- integer points in V: $\beta = \sum_{i=1}^n d_i \alpha_i \quad \leftrightarrow \quad \hat{y}^\beta = \hat{y}_1^{d_1} \cdots \hat{y}_n^{d_n}$
- $\omega : V \times V \to \mathbb{R}$ skew-symmetric, bilinear. In the α_i basis, its matrix is B.

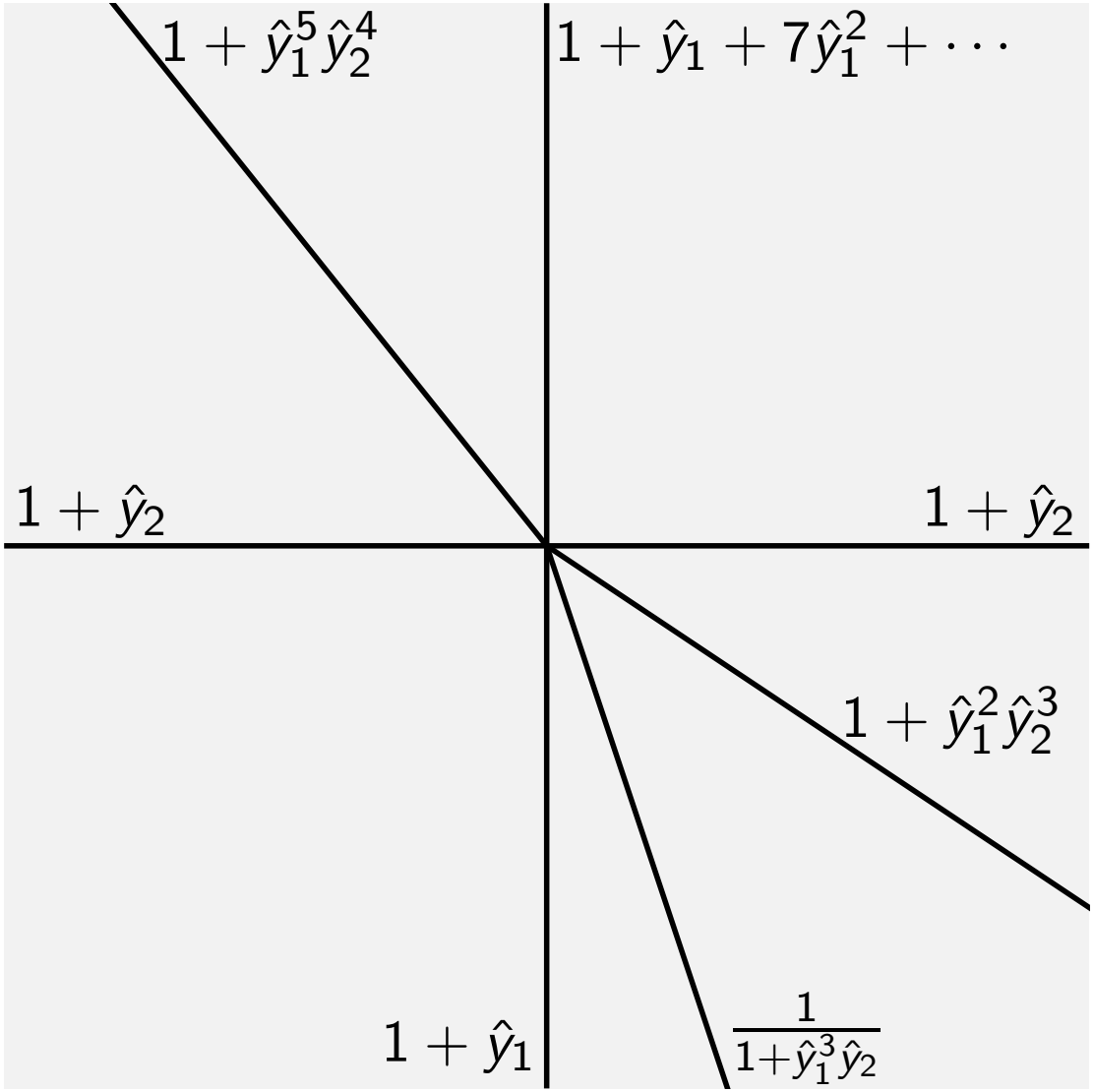
There is a global transpose $B \leftrightarrow B^T$ relating this setup to GHKK.

Scattering diagrams



A scattering diagram is a set of walls. Each wall is a codimension-1 cone in V^*, decorated with a scattering term—a formal power series in the $\hat{y}_i$.

Scattering diagrams

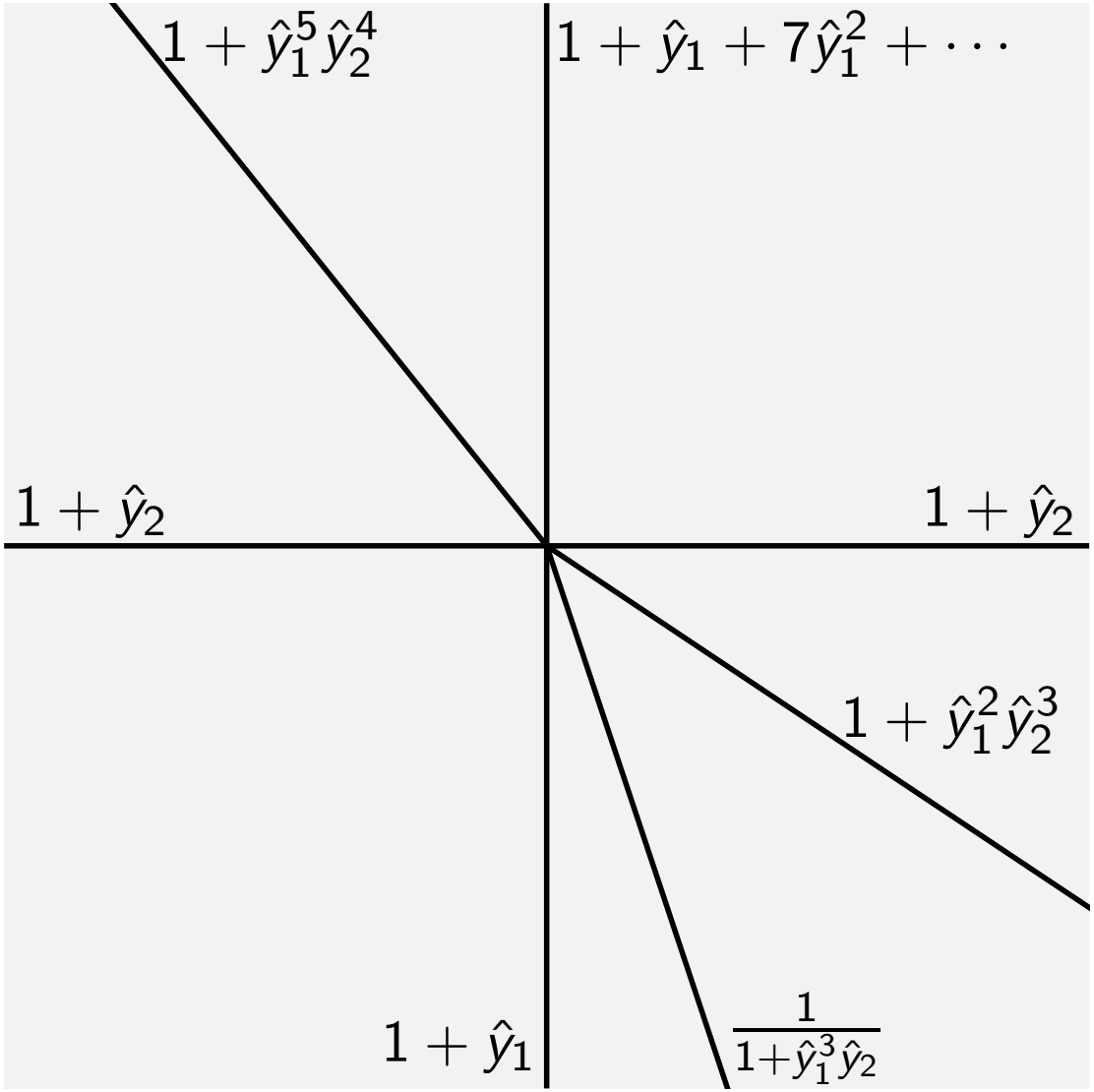


A scattering diagram is a set of walls. Each wall is a codimension-1 cone in V^*, decorated with a scattering term—a formal power series in the $\hat{y}_i$.

Details:

- Each wall is normal to a primitive, positive integer vector β. (That is, $\beta = \sum c_i \alpha_i$ with $c_i \geq 0$, $\sum c_i > 0$, $\gcd(c_i) = 1$.)

Scattering diagrams

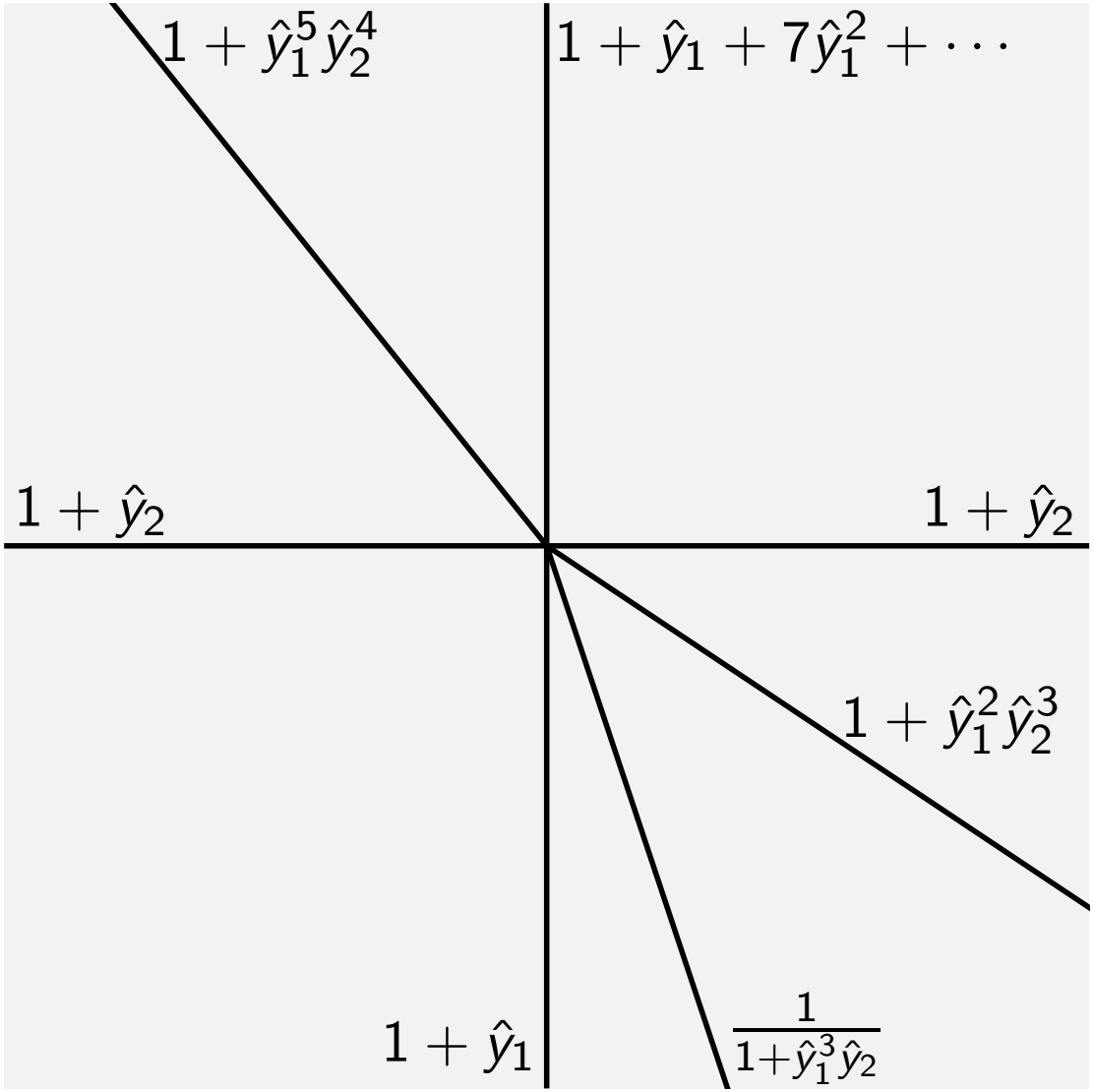


A scattering diagram is a set of walls. Each wall is a codimension-1 cone in V^*, decorated with a scattering term—a formal power series in the $\hat{y}_i$.

Details:

- Each wall is normal to a primitive, positive integer vector β. (That is, $\beta = \sum c_i \alpha_i$ with $c_i \geq 0$, $\sum c_i > 0$, $\gcd(c_i) = 1$.)
- The scattering term is a univariate FPS in $\hat{y}^\beta$ with constant term 1.

Scattering diagrams



A scattering diagram is a set of walls. Each wall is a codimension-1 cone in V^*, decorated with a scattering term—a formal power series in the $\hat{y}_i$.

Details:

- Each wall is normal to a primitive, positive integer vector β. (That is, $\beta = \sum c_i \alpha_i$ with $c_i \geq 0$, $\sum c_i > 0$, $\gcd(c_i) = 1$.)
- The scattering term is a univariate FPS in $\hat{y}^\beta$ with constant term 1.
- A finiteness condition

Wall-crossing homomorphisms and path-ordered products

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$ acts on polynomials (or FPS):

$$x^{\lambda} \mapsto x^{\lambda} f_{\mathfrak{d}}^{\langle \lambda, \pm\beta \rangle}$$
$$\hat{y}^{\phi} \mapsto \hat{y}^{\phi} f_{\mathfrak{d}}^{\omega(\pm\beta, \phi)}$$

Take "$-$" if crossing with β or "$+$" if crossing against β.

Path-ordered product $\mathfrak{p}_{\gamma}$: compose these along a (generic) path γ.

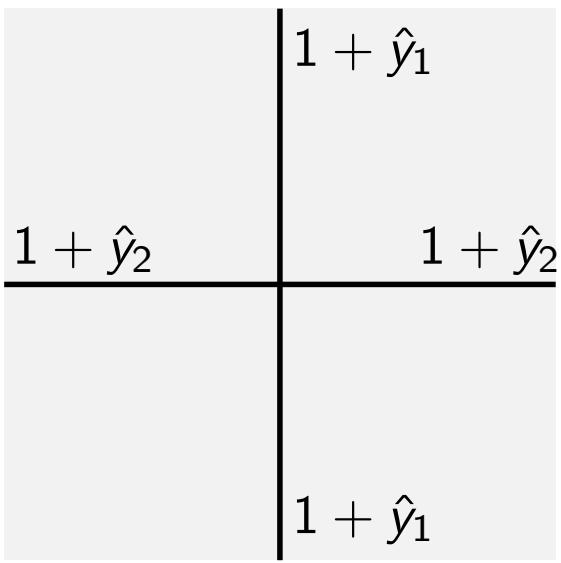
Wall-crossing homomorphisms and path-ordered products

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$ acts on polynomials (or FPS):

$$x^{\lambda} \mapsto x^{\lambda} f_{\mathfrak{d}}^{\langle \lambda, \pm\beta \rangle}$$
$$\hat{y}^{\phi} \mapsto \hat{y}^{\phi} f_{\mathfrak{d}}^{\omega(\pm\beta, \phi)}$$

Take "$-$" if crossing with β or "$+$" if crossing against β.

Path-ordered product $\mathfrak{p}_{\gamma}$: compose these along a (generic) path γ.



Let's try this in an example ($B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$):

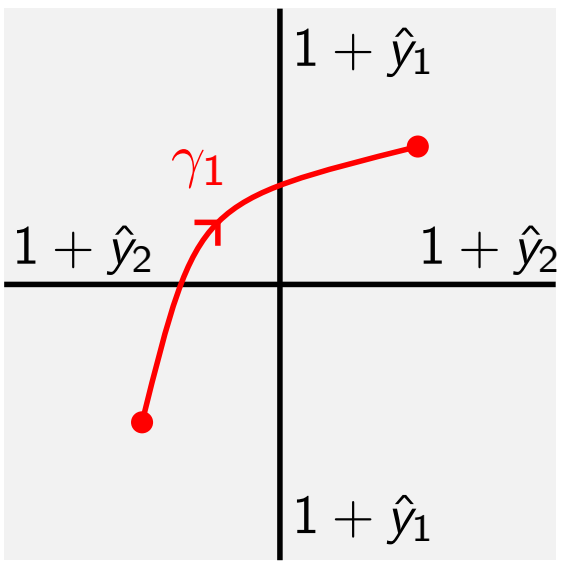
Wall-crossing homomorphisms and path-ordered products

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$ acts on polynomials (or FPS):

$$x^{\lambda} \mapsto x^{\lambda} f_{\mathfrak{d}}^{\langle \lambda, \pm\beta \rangle}$$
$$\hat{y}^{\phi} \mapsto \hat{y}^{\phi} f_{\mathfrak{d}}^{\omega(\pm\beta, \phi)}$$

Take "$-$" if crossing with β or "$+$" if crossing against β.

Path-ordered product $\mathfrak{p}_{\gamma}$: compose these along a (generic) path γ.



Let's try this in an example ($B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$):

$$\mathfrak{p}_{\gamma_1} : x_1^{-1} \mapsto x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1)$$

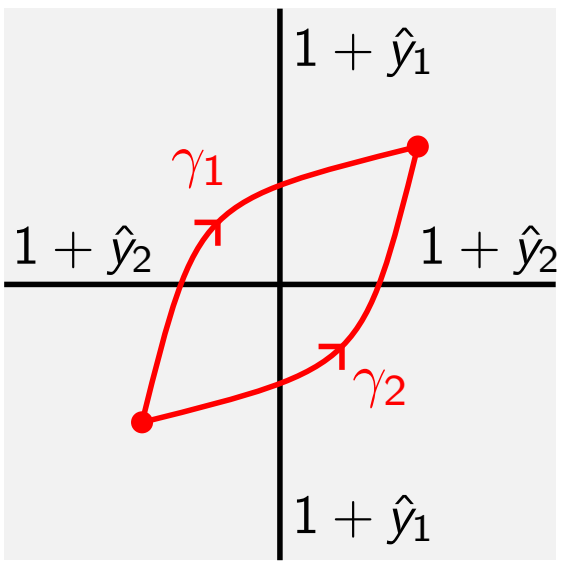
Wall-crossing homomorphisms and path-ordered products

Crossing a wall $(\mathfrak{d}, f_{\mathfrak{d}}(\hat{y}^{\beta}))$ acts on polynomials (or FPS):

$$x^{\lambda} \mapsto x^{\lambda} f_{\mathfrak{d}}^{\langle \lambda, \pm\beta \rangle}$$
$$\hat{y}^{\phi} \mapsto \hat{y}^{\phi} f_{\mathfrak{d}}^{\omega(\pm\beta, \phi)}$$

Take "$-$" if crossing with β or "$+$" if crossing against β.

Path-ordered product $\mathfrak{p}_{\gamma}$: compose these along a (generic) path γ.



Let's try this in an example ($B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$):

$$\mathfrak{p}_{\gamma_1} : x_1^{-1} \mapsto x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1)$$

$$\mathfrak{p}_{\gamma_2} : x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1) \mapsto x_1^{-1}(1+\hat{y}_1(1+\hat{y}_2))$$

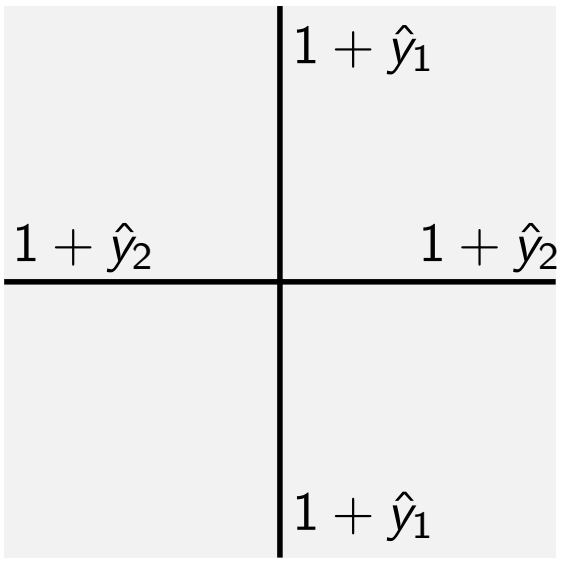
Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

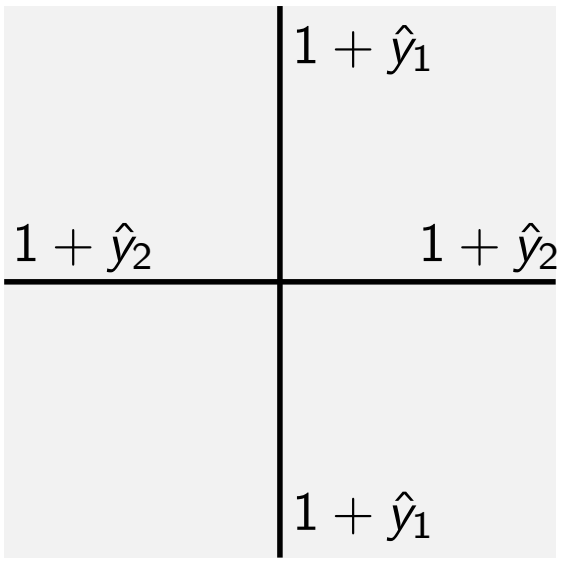


Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

A scattering diagram is consistent if path-ordered products depend only on the endpoints of the path.



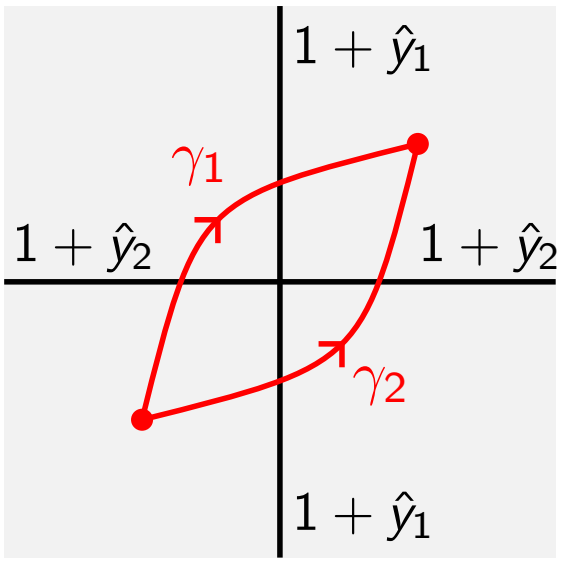
Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

A scattering diagram is consistent if path-ordered products depend only on the endpoints of the path.

Example. As we saw, this scattering diagram is not consistent



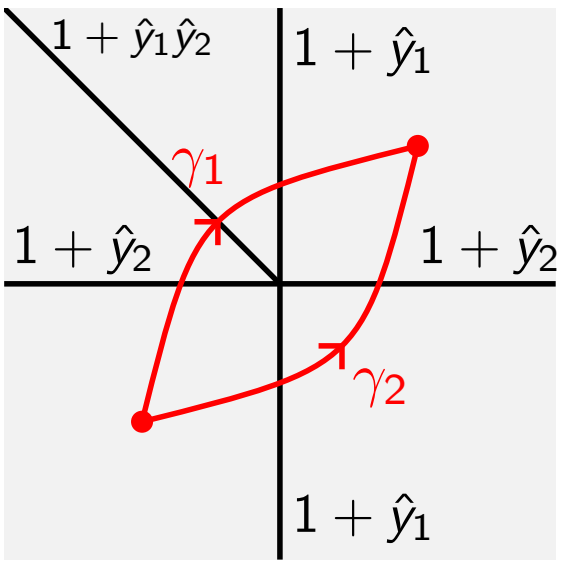
Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

A scattering diagram is consistent if path-ordered products depend only on the endpoints of the path.

Example. As we saw, this scattering diagram is not consistent



We can make it consistent by adding one wall.

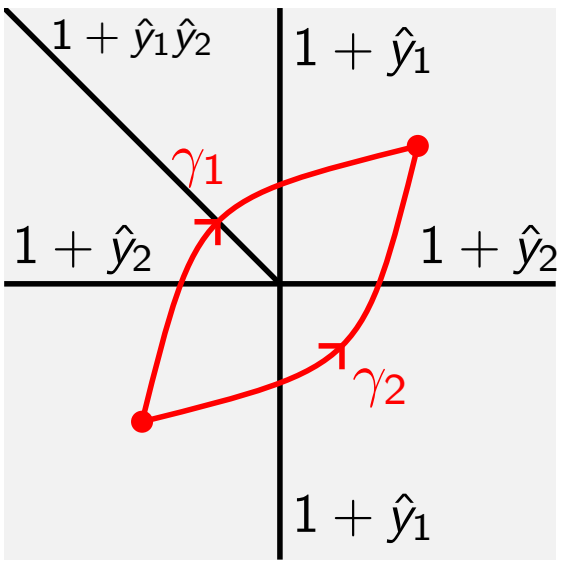
Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

A scattering diagram is consistent if path-ordered products depend only on the endpoints of the path.

Example. As we saw, this scattering diagram is not consistent



We can make it consistent by adding one wall.

$$\mathfrak{p}_{\gamma_1} : x_1^{-1} \mapsto x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1\hat{y}_2) \mapsto x_1^{-1}(1+\hat{y}_1)(1+\hat{y}_1\hat{y}_2(1+\hat{y}_1)^{-1})$$

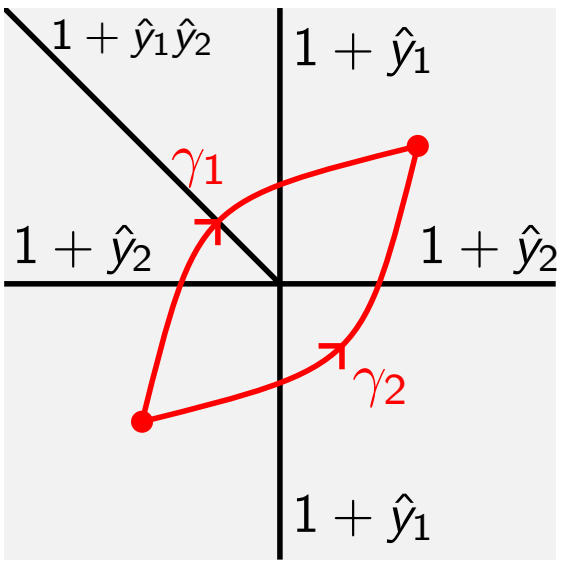
Equivalence and consistency

Two scattering diagrams are equivalent if they give the same path-ordered products (for generic paths).

Example. Does the diagram below have 2 walls or 4?

A scattering diagram is consistent if path-ordered products depend only on the endpoints of the path.

Example. As we saw, this scattering diagram is not consistent



We can make it consistent by adding one wall.

$$\mathfrak{p}_{\gamma_1} : x_1^{-1} \mapsto x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1\hat{y}_2) \mapsto x_1^{-1}(1+\hat{y}_1)(1+\hat{y}_1\hat{y}_2(1+\hat{y}_1)^{-1})$$

$$\mathfrak{p}_{\gamma_2} : x_1^{-1} \mapsto x_1^{-1}(1+\hat{y}_1) \mapsto x_1^{-1}(1+\hat{y}_1(1+\hat{y}_2))$$

Scattering fans

Theorem (R., 2017). A consistent scattering diagram with minimal support cuts V^* into a complete fan.

Decoding this statement:

- Support of a scattering diagram $\mathfrak{D}$ is the union of its walls. Equivalent scattering diagrams can have different supports.
- The rampart associated to a positive integer vector β is the union of all the walls contained in $\beta^\perp$.
- $\mathrm{Ram}_{\mathfrak{D}}(p)$: the set of ramparts of containing a given point p
- Declare $p, q \in V^*$ to be $\mathfrak{D}$-equivalent if and only if there is a path γ from p to q on which $\mathrm{Ram}_{\mathfrak{D}}(\,\cdot\,)$ is constant.
- $\mathfrak{D}$-cone: The closure of a $\mathfrak{D}$-equivalence class.
- The scattering fan is the set of all $\mathfrak{D}$-cones and all faces of $\mathfrak{D}$-cones.

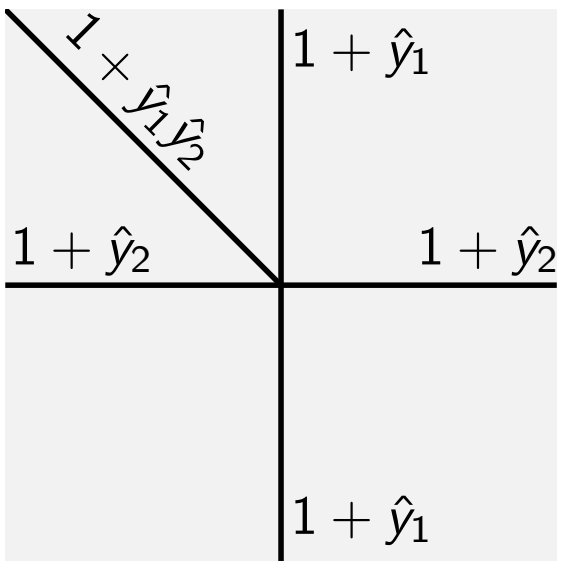
Cluster scattering diagrams

Theorem/Definition (Gross, Hacking, Keel, Kontsevich, 2014). Given a skew-symmetric integer matrix B, the cluster scattering diagram is the unique (up to equivalence) consistent scattering diagram $\mathfrak{D}$ such that

- $\mathfrak{D}$ contains the walls $(\alpha_i^\perp, 1 + \hat{y}_i)$.
- All other walls are outgoing. (That is, each wall $(\mathfrak{d}, f_\mathfrak{d}(\hat{y}^\beta))$ does not contain $\omega(\,\cdot\,, \beta)$.)

Example. The cluster scattering diagram for $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

One can check that the wall we added is outgoing.



We get basis elements from cluster scattering diagram for any **g**-vector. (The theta basis—"broken lines".)

Cluster scattering diagrams (cont'd)

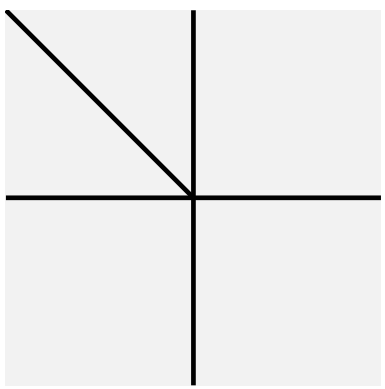
Theorem (GHKK, 2014). The $\mathbf{g}$=vector fan is a subfan of the cluster scattering fan. Scattering terms in these walls are $1 + \hat{y}^{\beta}$.

Example: $B = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

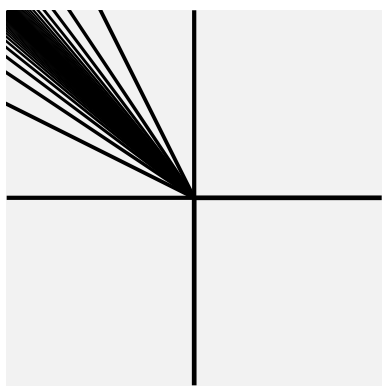
Finite case ($a < 2$): Scattering fan = $\mathbf{g}$-vector fan. All scattering terms easy.

Affine case ($a = 2$): Scattering term on one wall is not easy.

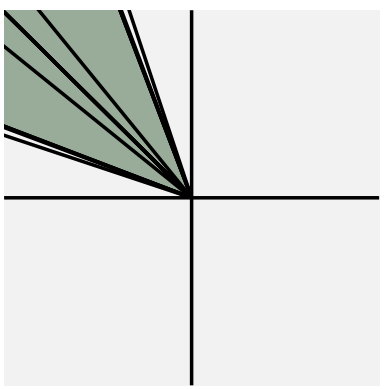
Wild case ($a > 2$): $\exists$ region where scattering terms are unknown.



$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$



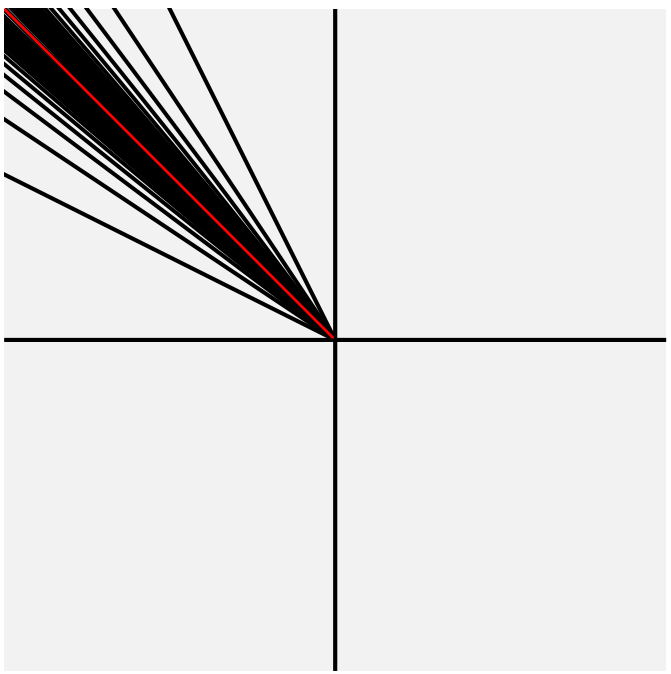
$\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

Affine rank-2 cluster scattering diagrams

Scattering terms are $1 + \hat{y}^\beta$ except on the limiting ray.

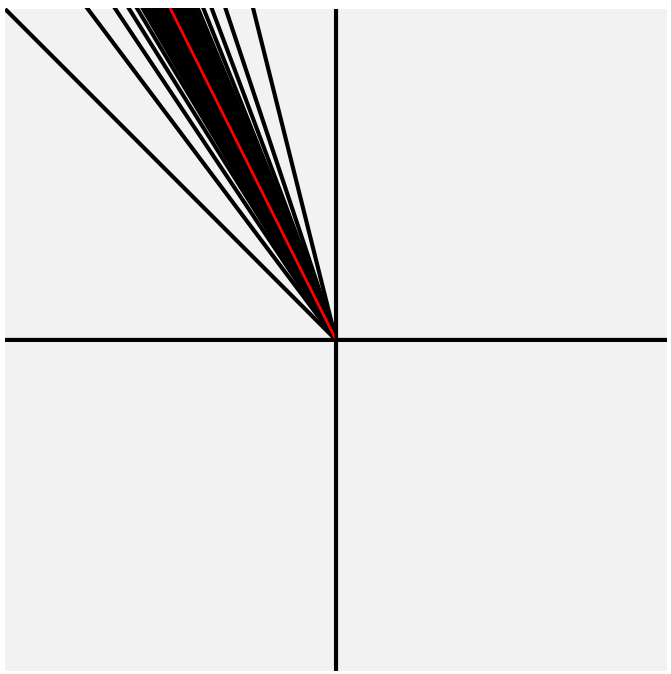
Theorem (Reineke, 2011 for $\left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$, R. 2017 for $\left[\begin{smallmatrix} 0 & 1 \\ -4 & 0 \end{smallmatrix}\right]$?).
The scattering term on the limiting walls are:

$\frac{1}{(1-\hat{y}_1\hat{y}_2)^2} = 1 + 2\hat{y}_1^1\hat{y}_2^1 + 3\hat{y}_1^2\hat{y}_2^2 + \cdots$



$\left[\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right]$

$\frac{1+\hat{y}_1\hat{y}_2^2}{(1-\hat{y}_1\hat{y}_2^2)^2} = 1 + 3\hat{y}_1\hat{y}_2^2 + 5\hat{y}_1^2\hat{y}_2^4 + \cdots$



$\left[\begin{smallmatrix} 0 & 1 \\ -4 & 0 \end{smallmatrix}\right]$

Scattering fans and mutation fans

Theorem (R., 2017). For all B, the scattering fan $\mathrm{ScatFan}(B)$ refines the mutation fan $\mathcal{F}_B$.

Conjecture (R., 2017). For rank ≥ 3, they coincide iff B mutation-finite.

Scattering fans and mutation fans

Theorem (R., 2017). For all B, the scattering fan ScatFan(B) refines the mutation fan $\mathcal{F}_B$.

Conjecture (R., 2017). For rank ≥ 3, they coincide iff B mutation-finite.

Work is in progress to construct scattering diagrams combinatorially in two cases: Affine type and the surfaces/orbifolds case.

(Both of these cases are mutation finite.)

Section 5: Combinatorial models

Cluster scattering diagrams of acyclic affine type

(joint with Salvatore Stella)

Theorem (Felikson-Shapiro-Thomas-Tumarkin 2012). When B is $n \times n$ and $n \geq 3$, the cluster algebra $\mathcal{A}_\bullet(B)$ has linear growth if and only if B is mutation-equivalent to an acyclic exchange matrix whose associated Cartan matrix is of affine type.

Accordingly: Say B is of affine type iff $\mathcal{A}_\bullet(B)$ has linear growth.

Stella and I have constructed cluster scattering diagrams for acyclic affine B, building on earlier work with Speyer and with Stella.

We are working on proving that the scattering fan coincides with the mutation fan in this case.

(Closely connected with that: We also hope to prove a conjecture on universal geometric cluster algebras of affine type.)

Cluster scattering diagrams for surfaces/orbifolds

(joint with Greg Muller and Shira Viel)

Fomin, Shapiro, and Thurston: a model for (certain) cluster algebras based on triangulated (oriented) surfaces. (B is the signed adjacency matrix of the triangulation.)

Felikson, Shapiro, and Tumarkin: modeled a larger class of cluster algebras by triangulated orbifolds.

Muller, Viel and I are working on constructing scattering diagrams for surfaces. (Orbifolds are the next step.) We have combinatorial gadgets in the surface that specify walls explicitly, and we think that we have constructed the scattering diagram in general, but there are some details to check.

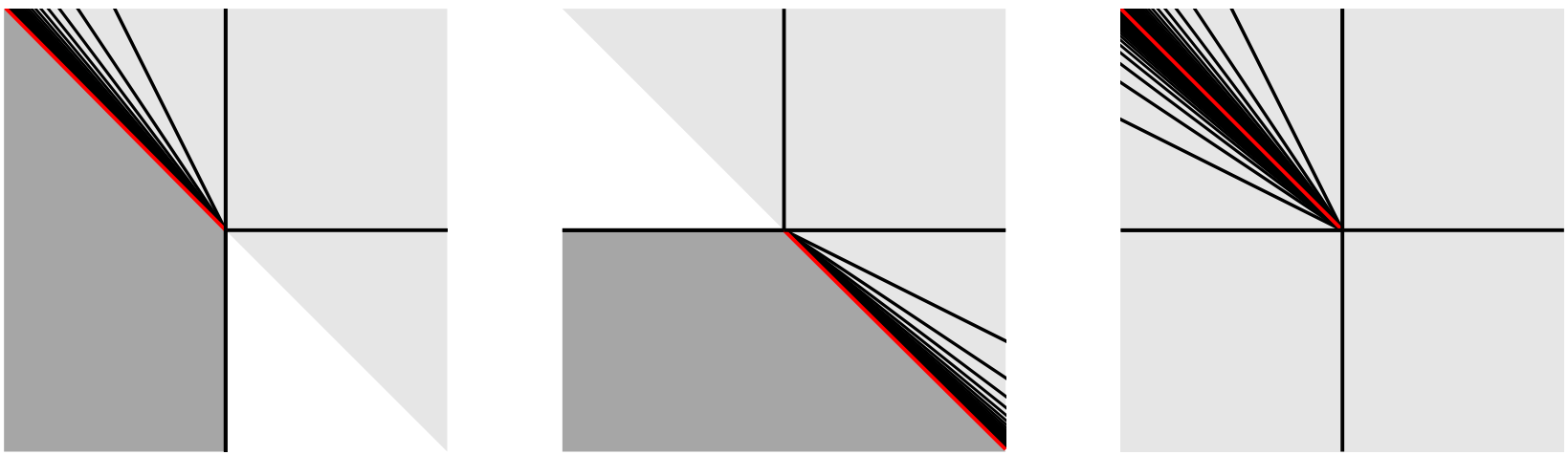
We also expect to prove that scattering fans coincide with mutation fans in the surfaces case.

The doubled Cambrian fan

Speyer and I built the $\mathbf{g}$-vector fan of acyclic affine type as a doubled Cambrian fan—the Cambrian fan for B union the antipodal image of the Cambrian fan for $-B$.

We characterized the space outside the $\mathbf{g}$-vector fan as a certain codimension-1 cone given by explicit inequalities (but no information about how the space outside decomposes into cones).

Example: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



An affine almost-positive roots model

Stella and I defined the affine-type almost positive Schur roots and showed that they are precisely the **d**-vectors of cluster variables.

We defined a notion of compatibility. (Real) combinatorial clusters are maximal sets of pairwise compatible real almost positive Schur roots.

Each combinatorial cluster spans a full-dimensional cone. These cones (for real clusters) are the maximal cones of the **d**-vector fan.

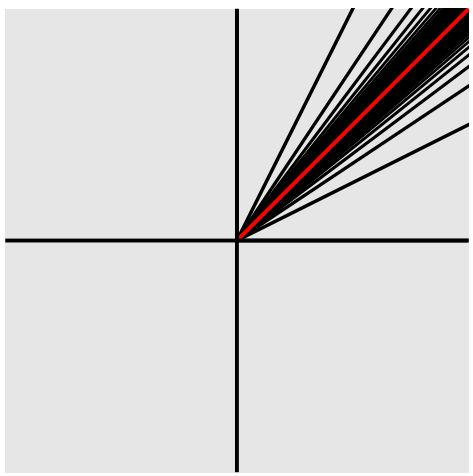
An affine almost-positive roots model

Stella and I defined the affine-type almost positive Schur roots and showed that they are precisely the **d**-vectors of cluster variables.

We defined a notion of compatibility. (Real) combinatorial clusters are maximal sets of pairwise compatible real almost positive Schur roots.

Each combinatorial cluster spans a full-dimensional cone. These cones (for real clusters) are the maximal cones of the **d**-vector fan.

There is a unique imaginary Schur root δ. Extending the definition of compatibility in a natural way, we get a complete fan having the **d**-vector fan as a subfan.



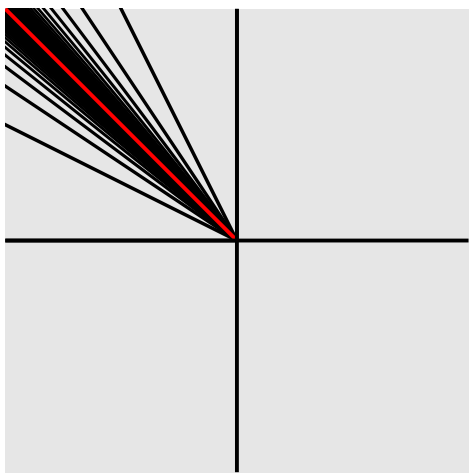
An affine almost-positive roots model

Stella and I defined the affine-type almost positive Schur roots and showed that they are precisely the **d**-vectors of cluster variables.

We defined a notion of compatibility. (Real) combinatorial clusters are maximal sets of pairwise compatible real almost positive Schur roots.

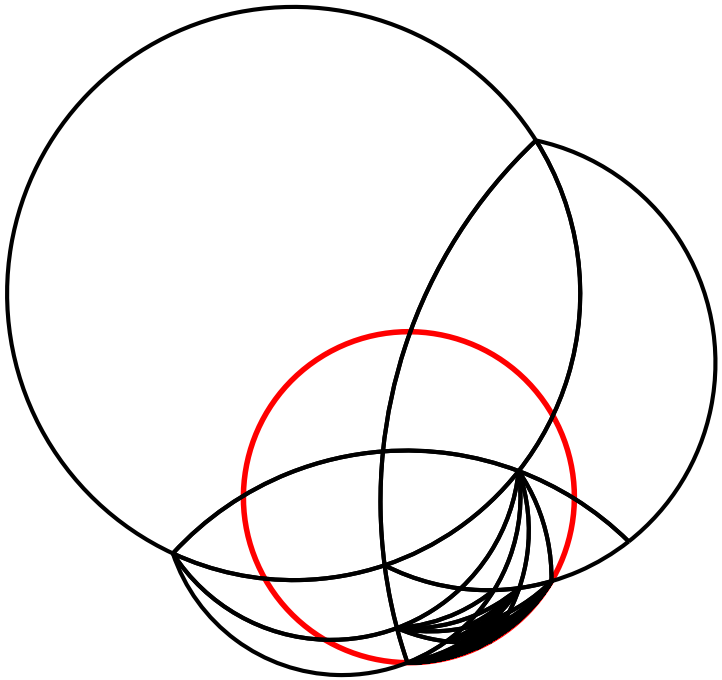
Each combinatorial cluster spans a full-dimensional cone. These cones (for real clusters) are the maximal cones of the **d**-vector fan.

There is a unique imaginary Schur root δ. Extending the definition of compatibility in a natural way, we get a complete fan having the **d**-vector fan as a subfan.

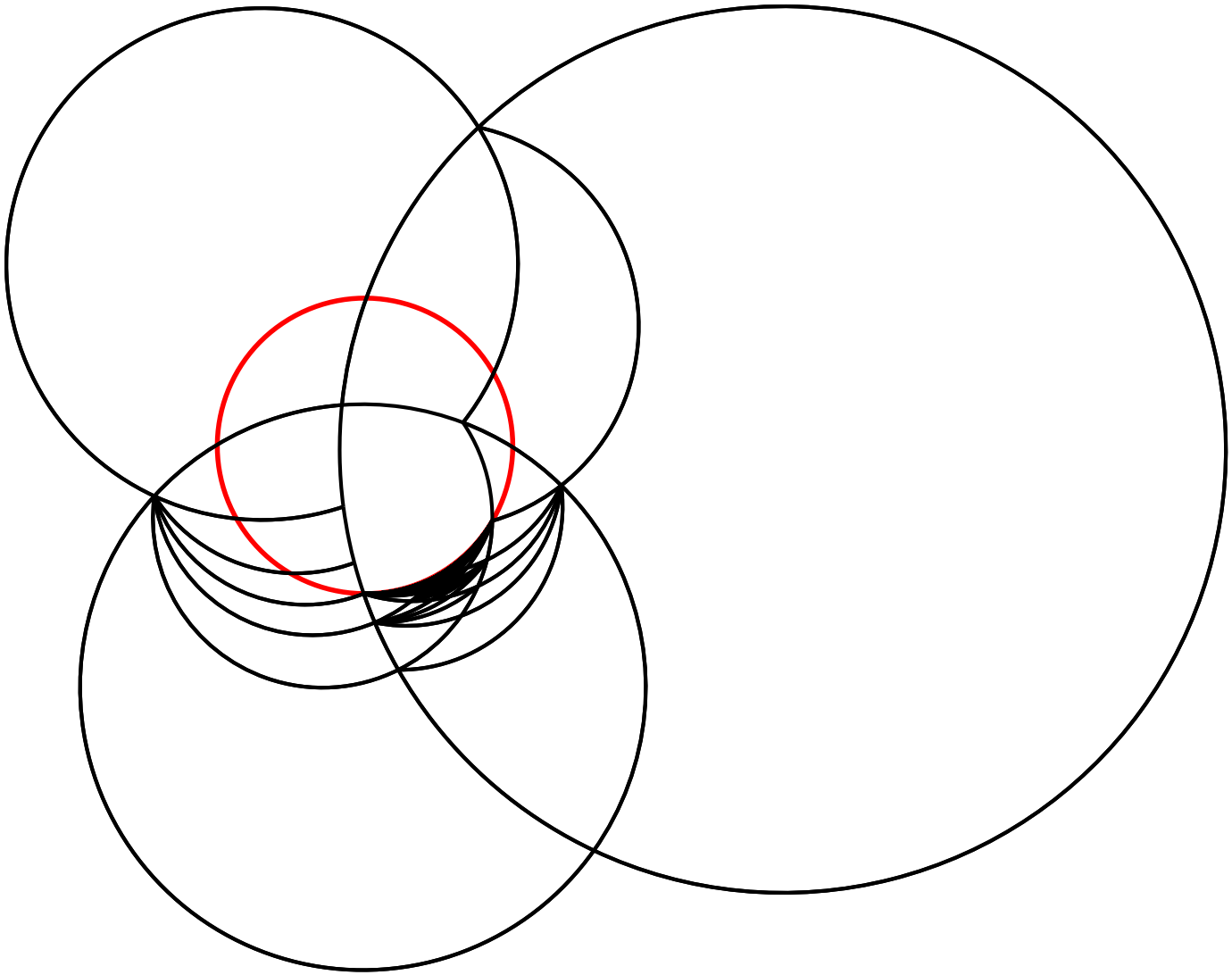


By a piecewise-linear map, the fan of almost positive Schur roots becomes a completion of the **g**-vector fan.

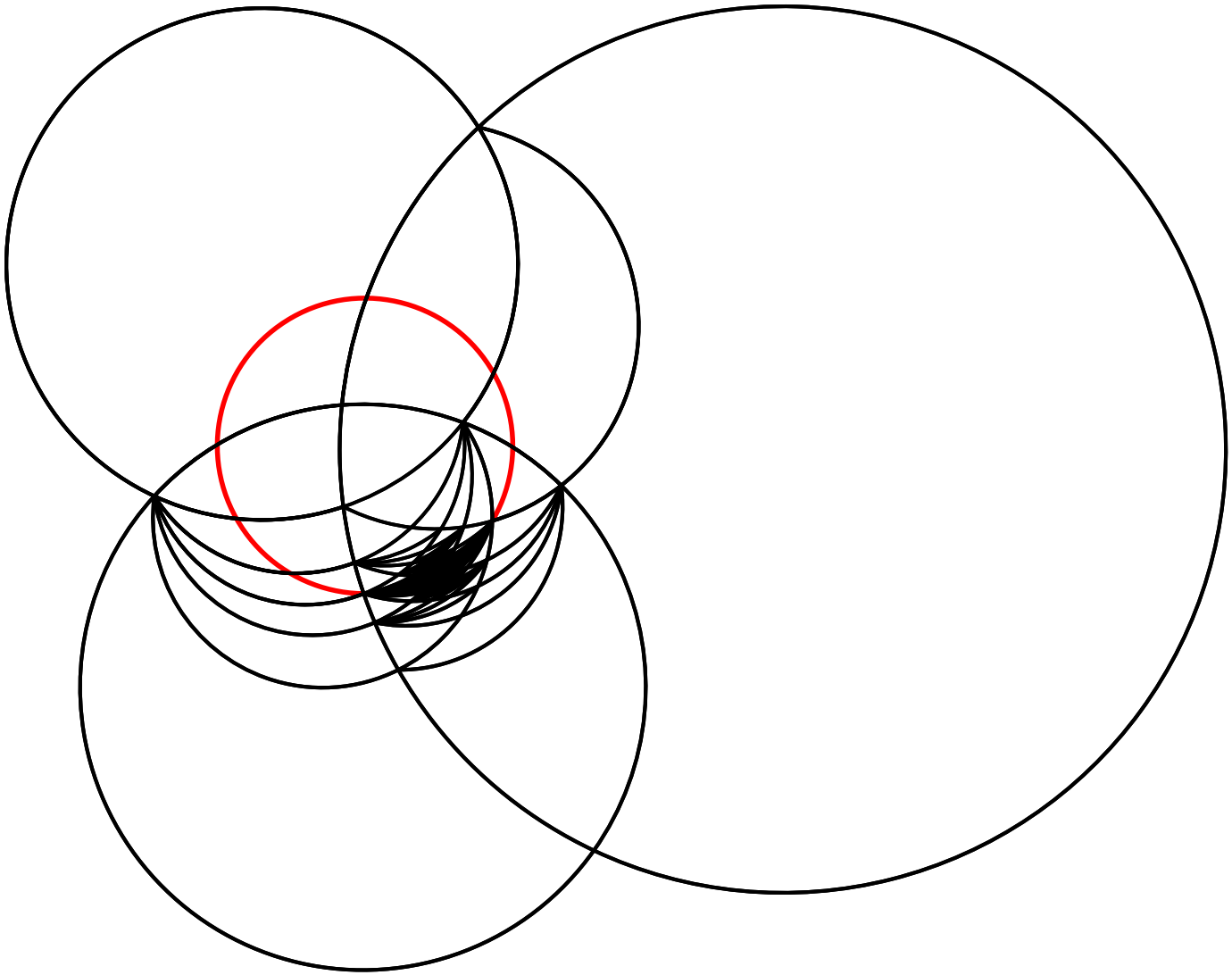
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



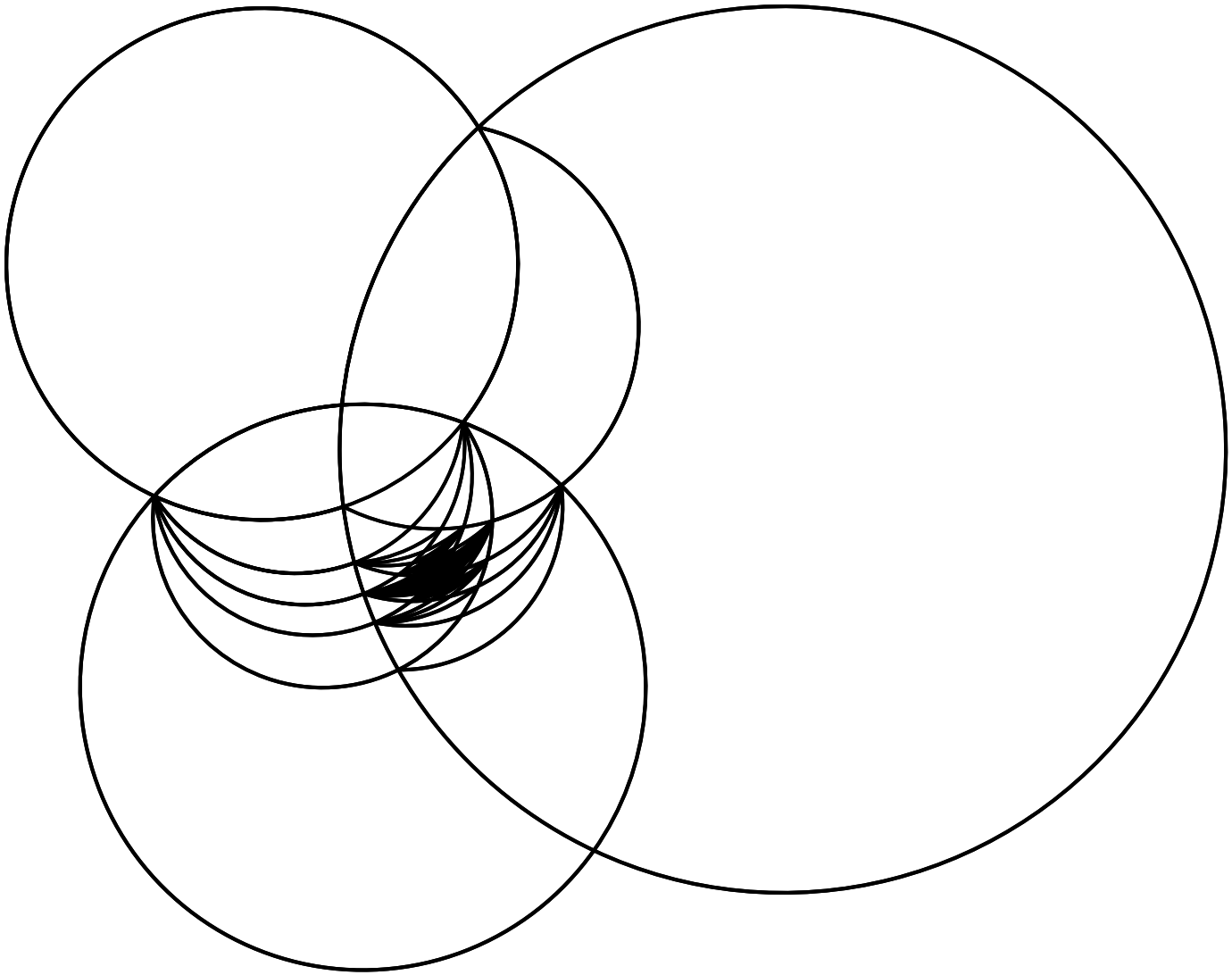
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



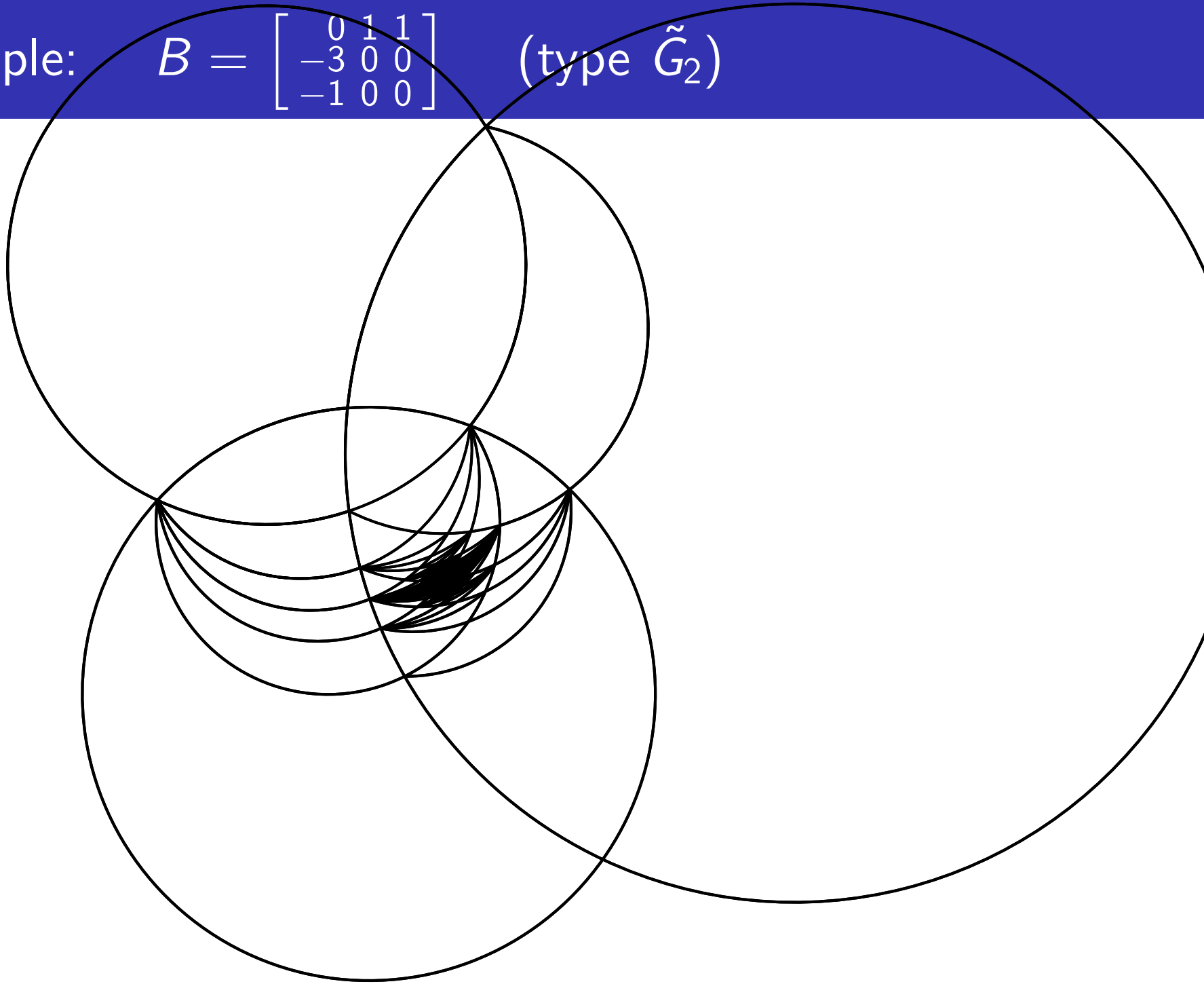
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



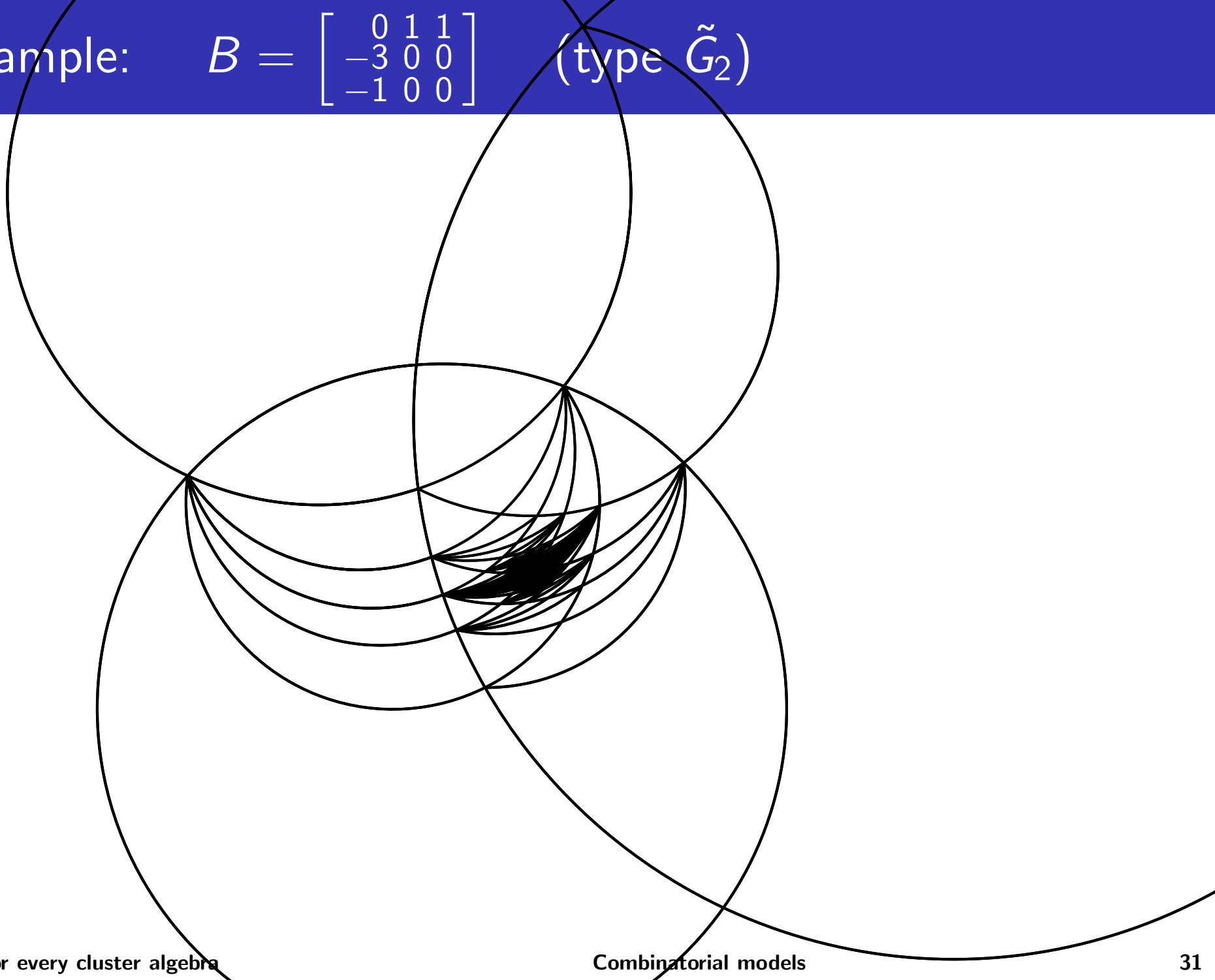
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



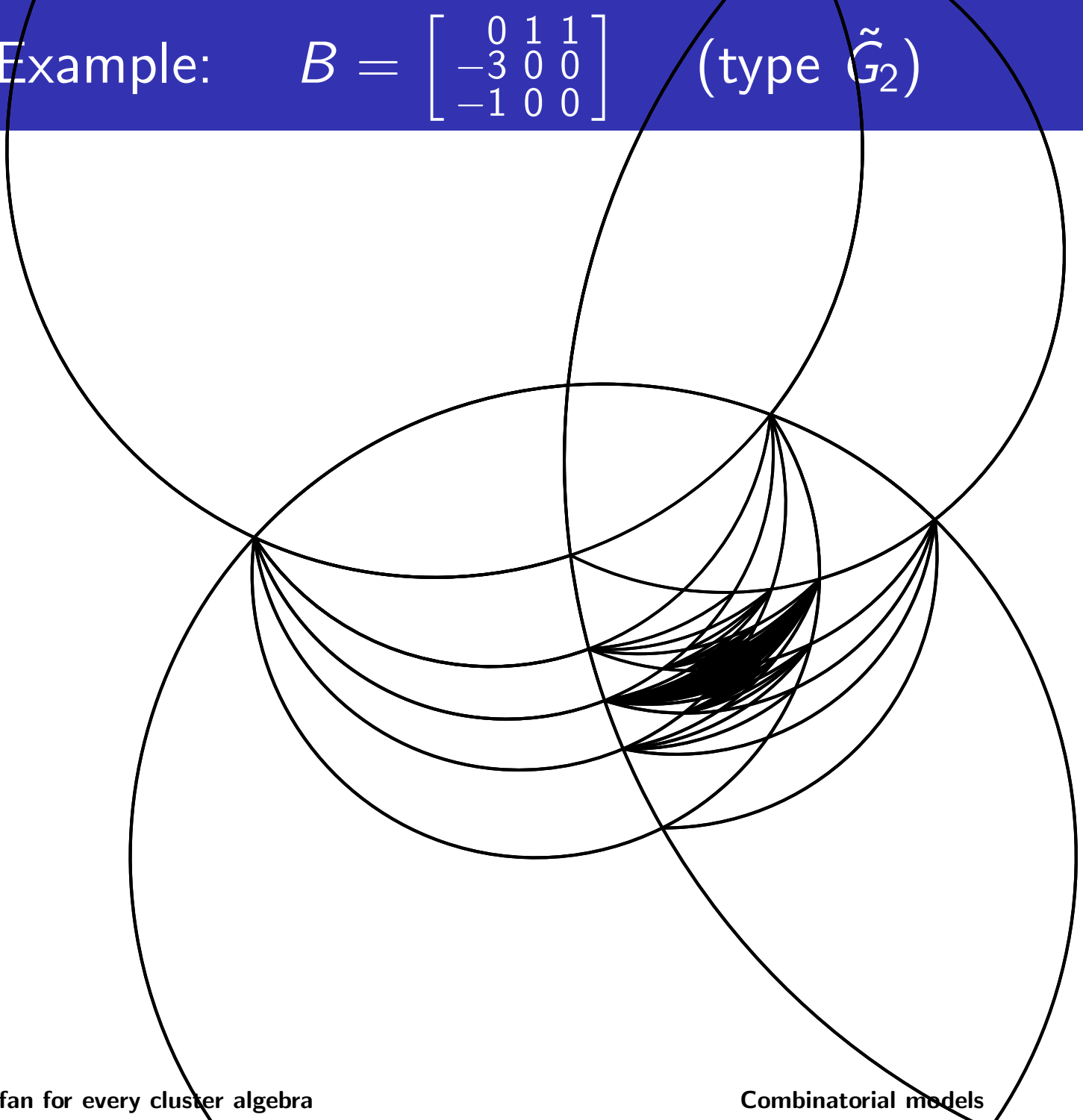
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



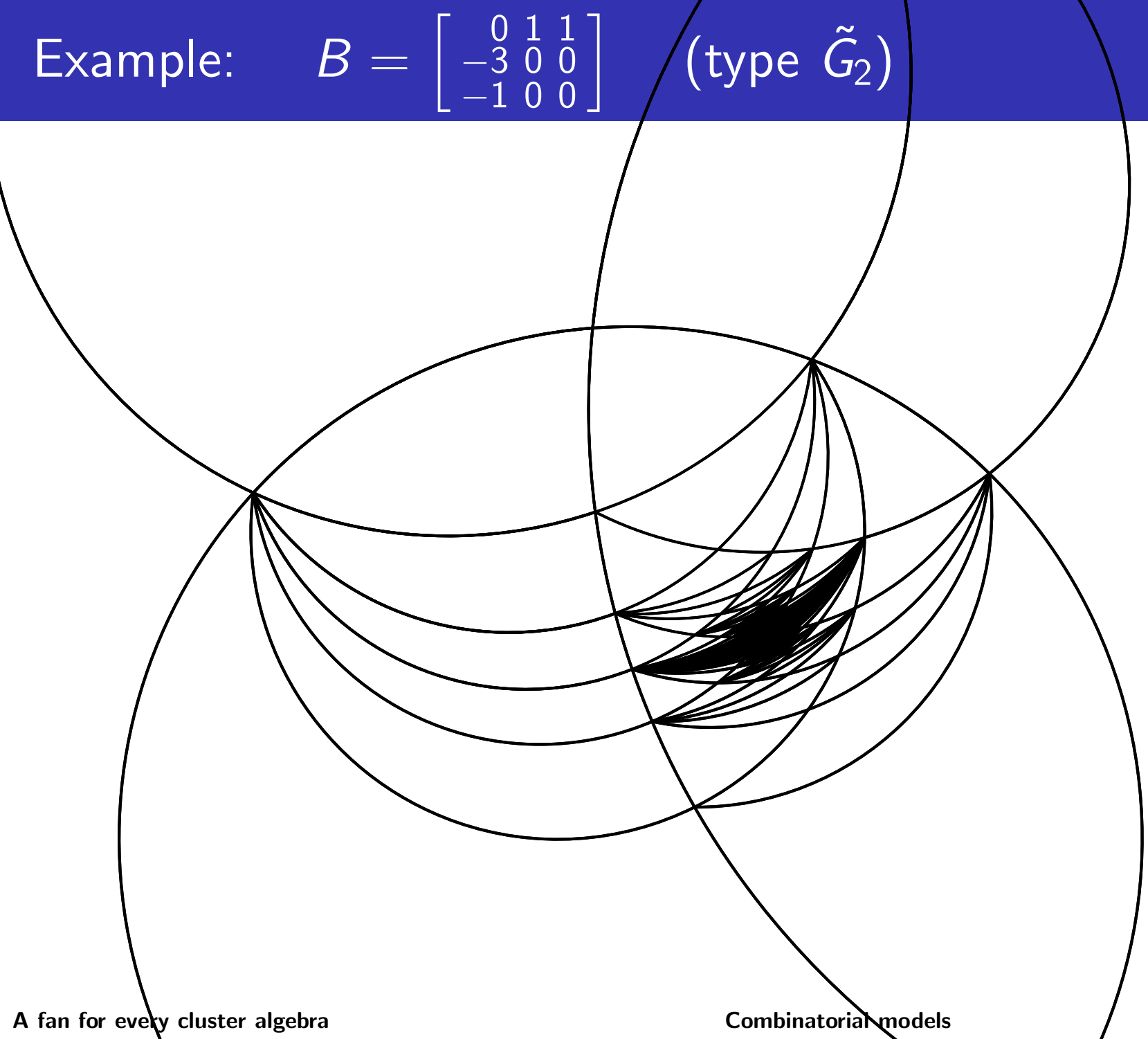
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



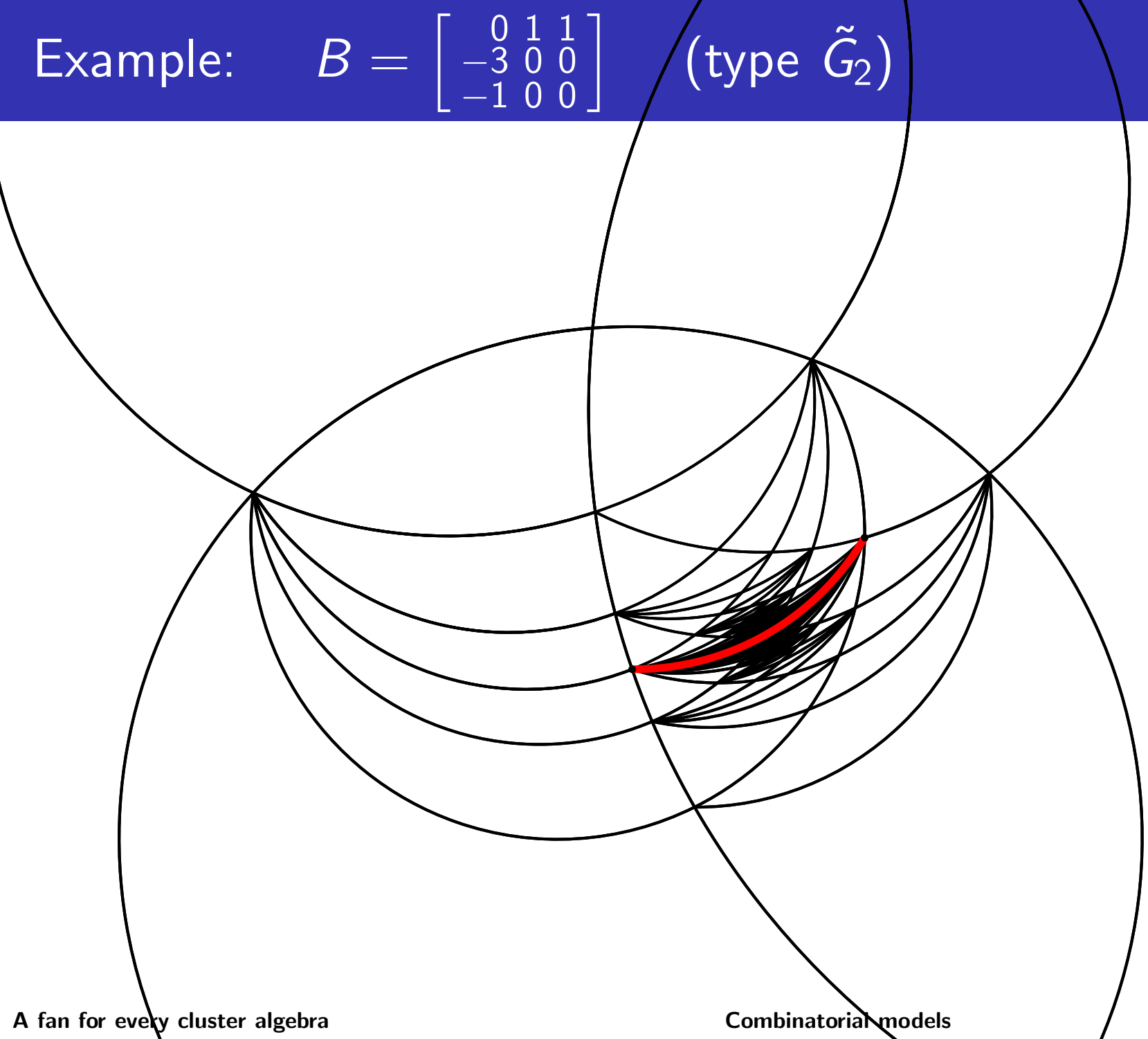
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



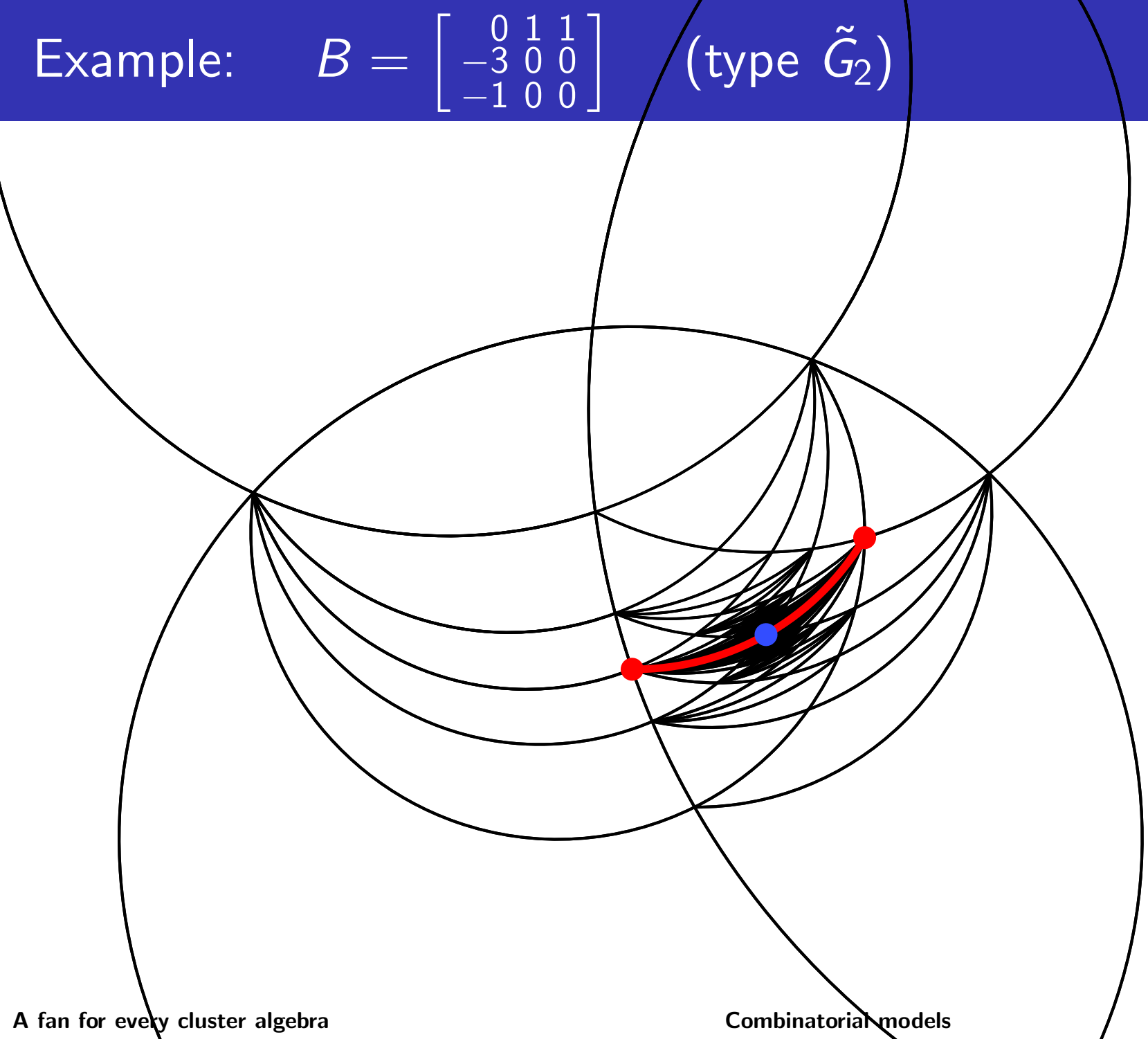
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (type $\tilde{G}_2$)



An affine almost-positive roots model (continued)

The link of the imaginary Schur root is combinatorially isomorphic to a join of boundary complexes of type-B simplicial associahedra (graph associahedra of cycles).

Example: If the rank-3 example we just did, the link was a 0-sphere (the boundary complex of a 1-cyclohedron).

Example: If the rank-4 examples there are two possibilities for the link:

- A quadrilateral (the join of two 0-spheres), or
- A hexagon (the boundary complex of a 2-cyclohedron).

Making the cluster scattering diagram/fan

We would like to show that the complete fan defined by (real and imaginary!) clusters of almost positive Schur roots is the cluster scattering fan. So far, we can construct the cluster scattering diagram:

For each almost positive Schur root β, there is either

- a B-sortable join-irreducible element whose unique cover reflection corresponds to β, or
- a B-sortable join-irreducible element whose unique cover reflection corresponds to β, or (only finitely often) both.

From each join-irreducible element, we make a wall (a "shard"), and in the "both" case, we get the same wall either way. Each of these walls gets scattering term $1 + \hat{y}^{\beta}$.

Finally, we have an "imaginary wall": Precisely the space outside the $\mathbf{g}$-vector fan. The scattering term is a FPS in $\hat{y}^{\delta}$, given by a formula like in rank 2.

Making the cluster scattering diagram/fan (cont'd)

...We have the cluster scattering diagram. It has exactly one wall orthogonal to each positive Schur root.

We are currently trying to show that it coincides with both the mutation fan and the fan defined from almost positive Schur roots.

The mutation fan connection would prove a case of the conjecture on when the mutation fan and cluster scattering fan coincide.

It would also lead to the proof of a conjecture on universal coefficients.

Thank you for listening.

Cambrian frameworks for cluster algebras of affine type (with Speyer). Trans. AMS 2018

An affine almost positive roots model (with Stella). arXiv:1707.00340

Scattering fans. arXiv:1712.06968 (to appear in IMRN)

A combinatorial approach to scattering diagrams. arXiv.1806.05094