Noncrossing partitions of an annulus

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(reporting on joint work, much of which appeared in Laura Brestensky's thesis)

The noncrossing partition poset

Let W be a Coxeter group with simple reflections S and reflections T.

Coxeter element: a product $c = s_1 s_2 \cdots s_n$ of the elements of S in any order.

Absolute order $u \leq_T w$ is prefix/suffix/subword order relative to the alphabet T.

The noncrossing partition poset: the interval $[1, c]_T$.

Noncrossing partition poset (lattice) prototypical example

W: the symmetric group S_{n+1}. (This is "Type A".)

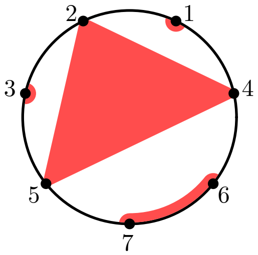
S: adjacent transpositions $s_i = (i \;\; i+1)$ for $i = 1, \ldots, n$.

T: arbitrary transpositions $(i \;\; j)$.

c is an $(n+1)$-cycle.

$[1, c]_T$ is modeled by noncrossing partitions of that cycle.

Example: $c = s_3 s_5 s_2 s_1 s_6 s_4 = (1 \;\; 4 \;\; 6 \;\; 7 \;\; 5 \;\; 3 \;\; 2)$

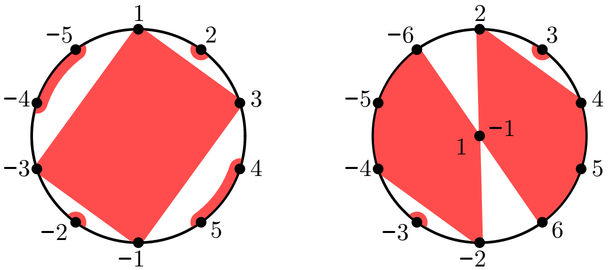


$\longleftrightarrow$ $(1)\,(2 \;\; 4 \;\; 5)\,(3)\,(6 \;\; 7)$

Other finite types

Type B: W is the group of signed permutations and $[1, c]_T$ is modeled by centrally symmetric noncrossing partitions.

Type D: W is even-signed permutations and $[1, c]_T$ is centrally symmetric noncrossing partitions with a double point at the center.



Type B Type D

All three constructions (A, B, D) can be understood in terms of "projecting a small orbit to the Coxeter plane".

Connections to Artin groups

The defining presentation of a Coxeter group has

- braid relations of the form $stst = tsts$ for $s, t \in S$, and
- $s^2 = 1$ for each $s \in S$.

The Artin group associated to W has only the braid relations.

When W is finite, $[1, c]_T$ serves as a Garside structure for the corresponding Artin group. This gives a dual presentation of the Artin group, generated by T and proving desirable properties

Crucial: $[1, c]_T$ is a lattice when W is finite.

Outside of finite type, the interval $[1, c]_T$ need not be a lattice.

Work of McCammond (variously with Brady and Sulway) extends the affine Coxeter group W to a larger group, thus extending $[1, c]_T$ to a lattice. The lattice is a Garside structure for a supergroup of the Artin group, which inherits desirable properties from the supergroup.

Noncrossing partitions of classical affine types

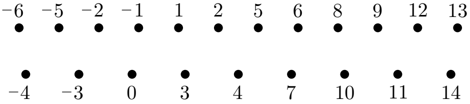
Goal: Planar diagrams for $[1, c]_T$ and the larger lattice.

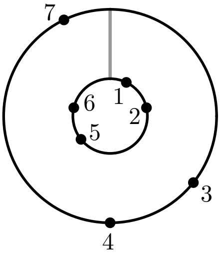
Where to start: Project a "small" orbit to the "Coxeter plane". Mod out by some or all of the symmetries in the Coxeter plane.

In every case, the orbit is a collection of vectors $\mathbf{e}_i$ for integers i and the projection is an infinite strip with translational symmetry. This becomes an annulus.

Example: Affine type $\widetilde{A}_6$

$c = s_6 s_5 s_2 s_1 s_3 s_4 s_7$





Type $\widetilde{A}$: Affine permutations and periodic permutations

Type $\widetilde{A}_{n-1}$ affine Coxeter group $\widetilde{S}_n$ is affine permutations π of $\mathbb{Z}$:

- $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$
- $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$.

Larger group $S_{\mathbb{Z}}(\mathrm{mod}\ n)$: $\pi(i+n) = \pi(i) + n \quad \forall i$.

Cycle notation:

$(a_1\ \ a_2\ \cdots\ a_k)_n$ means $\prod_{q\in\mathbb{Z}}(a_1 + qn\ \ a_2 + qn\ \cdots\ a_k + qn)$.

Infinite cycles are $(\cdots\ a_1\ \ a_2\ \cdots\ a_k\ \ a_i + qn\ \cdots)$, $\quad q \neq 0$.

Reflections: $T = \{(i\ \ j)_n : i < j, i \not\equiv j \pmod{n}\}$.

Loops: $\ell_i = (\cdots\ i\ \ \ i+n\ \cdots) \qquad L = \left\{\ell_i^{\pm 1} : i \in 1, \ldots, n\right\}$

Generators: $\widetilde{S}_n$ generated by T. $\quad S_{\mathbb{Z}}(\mathrm{mod}\ n)$ generated by $T \cup L$.

Affine type $\widetilde{A}$: Coxeter elements

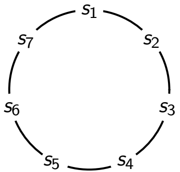
The Coxeter diagram for $\widetilde{S}_n$ is an n-cycle.

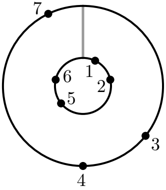
Choosing a Coxeter element means choosing, for each i, whether s_i is before or after s_{i-1}. Record by placing of numbers on the annulus:

Place $1, \ldots, n$ in clockwise order.

- i on the outer boundary iff s_{i-1} is before s_i.
- i on the inner boundary iff s_{i-1} is after s_i.

Example: $n = 6$ $\quad c = s_6 s_5 s_2 s_1 s_3 s_4 s_7$





Noncrossing partitions of an annulus

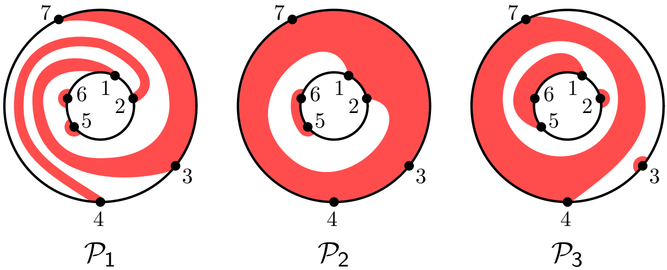
Start with an annulus with inner points and outer points.

An embedded block is

- a disk block, a closed disk containing at least one numbered point. (It may be a degenerate disk, i.e. just one point or a curve connecting two points.)
- a dangling annular block, a closed annulus with one boundary component containing numbered points, the other "free".
- a nondangling annular block, a closed annulus with each component of its boundary containing numbered points.

Noncrossing partitions: Set partitions plus additional topology.
$\mathcal{P} = \{E_1, \ldots, E_k\}$ disjoint embedded blocks, every numbered point is in some E_i, at most one annular block. Considered up to isotopy.

Noncrossing partitions of an annulus (continued)

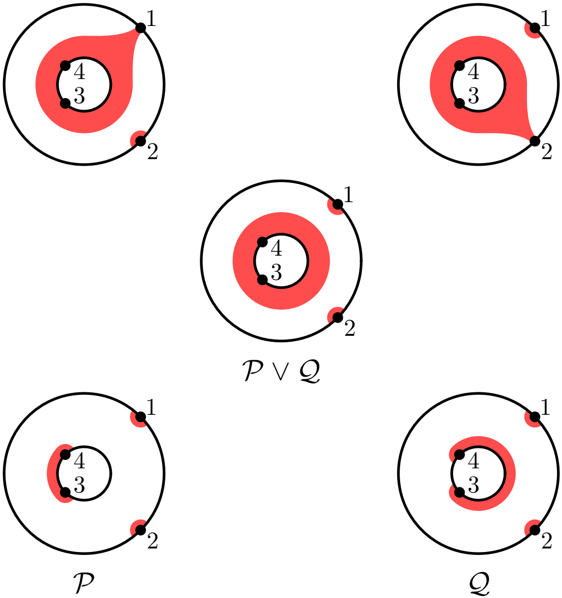


Noncrossing partition lattice $\widetilde{NC}_c^A$: $\mathcal{P} \leq \mathcal{Q}$ iff there are embeddings of $\mathcal{P}$ and $\mathcal{Q}$ with every block of $\mathcal{P}$ contained in a block of $\mathcal{Q}$.

Theorem. $\widetilde{NC}_c^A$ is a graded lattice, with rank function given by n minus the number of non-annular blocks.

Proof idea. Show that the partial order is containment of *curve sets*. Given $\mathcal{P}$ and $\mathcal{Q}$, explicitly construct a noncrossing partition whose curve set is $\text{curve}(\mathcal{P}) \cap \text{curve}(\mathcal{Q})$.

The lattice property needs dangling annular blocks



Define a map perm : $\widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$:
Read each component of each block as a cycle, keeping the interior of the block on the right.

Date line: Radial segment between 1 and n. Add n each time it is crossed clockwise, $-n$ when it is crossed counterclockwise.

Disks give finite cycles, annuli give infinite cycles.

$$\mathrm{perm}(\mathcal{P}_1) = (1\ \ -7\ \ -4)_7\,(2\ \ -3)_7\,(5)_7\,(6)_7$$
$$\mathrm{perm}(\mathcal{P}_2) = (\cdots\ 1\ \ -5\ \ -6\ \cdots)\,(\cdots\ 3\ \ 4\ \ 7\ \ 10\ \cdots)\,(5\ \ 6)_7$$
$$\mathrm{perm}(\mathcal{P}_3) = (1\ \ -1\ \ -2)_7\,(2)_7\,(3)_7\,(\cdots\ 4\ \ 7\ \ 11\ \cdots)$$

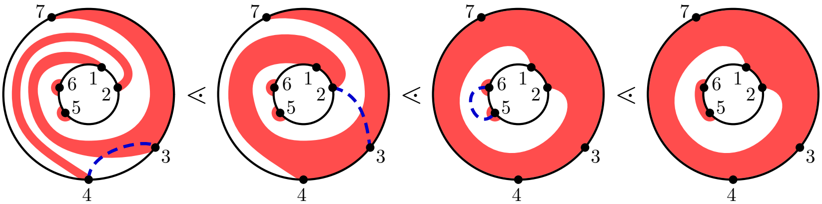
Theorem. The map perm : $\widetilde{NC}_c^A \to S_{\mathbb{Z}}(\mathrm{mod}\ n)$ is an isomorphism from $\widetilde{NC}_c^A$ to the interval $[1, c]_{T\cup L}$ in $S_{\mathbb{Z}}(\mathrm{mod}\ n)$. It restricts to an isomorphism from $\widetilde{NC}_c^{A,\circ}$ to the interval $[1, c]_T$ in $\widetilde{S}_n$.

$\widetilde{NC}_c^{A,\circ}$: Noncrossing partitions with no dangling annular blocks.

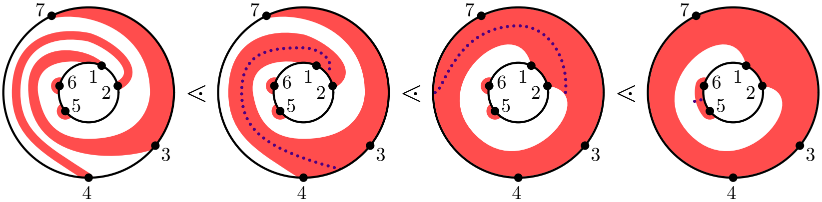
Cover relations in $\widetilde{NC}_c^A$

Proving the isomorphisms involves understanding cover relations in $\widetilde{NC}_c^A$ and $[1, c]_{T \cup L}$.

Covers in $\widetilde{NC}_c^A$ are described by simple connectors

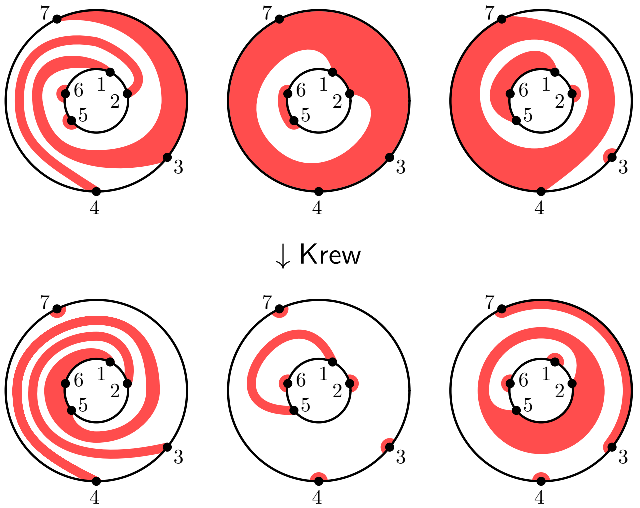


or by cutting curves.



Kreweras complements

Kreweras complementation is an antiautomorphism of $\widetilde{NC}_c^A$ that restricts to an antiautomorphism of $\widetilde{NC}_c^{A,\circ}$.



Factored translations and dangling annular blocks

McCammond and Sulway build their larger interval (in their larger group) by factoring the translations in $[1, c]_T$.

Recall $\widetilde{NC}_c^{A,\circ}$ is noncrossing partitions, no dangling annular blocks. We know $[1, c]_T \cong \widetilde{NC}_c^{A,\circ}$.

The translations in $[1, c]_T$ are $(\cdots\ i \quad i+n\ \cdots)\,(\cdots\ j \quad j-n\ \cdots)$ for i outer and j inner. These correspond to the noncrossing partitions with only one nontrivial block—an annulus with one numbered point on each boundary component.

What is the obvious way to factor a translation? As $\ell_i \cdot \ell_j^{-1}$.

ℓ_i corresponds to the dangling annular block containing only i.

ℓ_j^{-1} corresponds to the dangling annular block containing only j.

Noncrossing partitions of a marked surface

Planar models for in types A and $\widetilde{A}$ generalize to noncrossing partitions of a marked surface $(\mathbf{S}, \mathbf{M})$ with no punctures, in the sense of the marked surfaces model for cluster algebras.

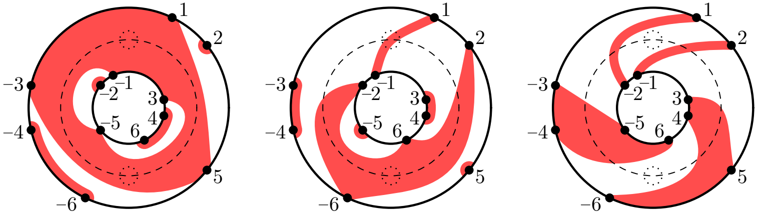
Theorem. $\mathrm{NC}(\mathbf{S}, \mathbf{M})$ is a graded lattice.

The rank function is simple and topological.

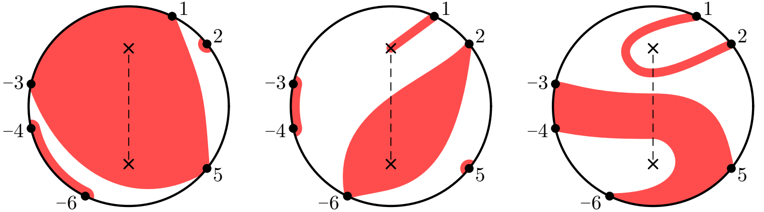
Punctures don't seem to work well, but are replaced by symmetry and "double points".

Affine type $\widetilde{C}$: affine signed permutations

The lattice $[1, c]_T$ is modeled by symmetric noncrossing partitions of an annulus

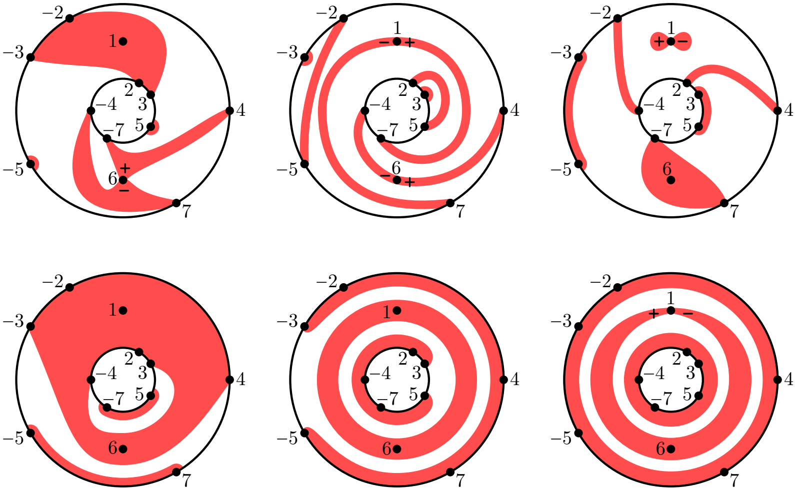


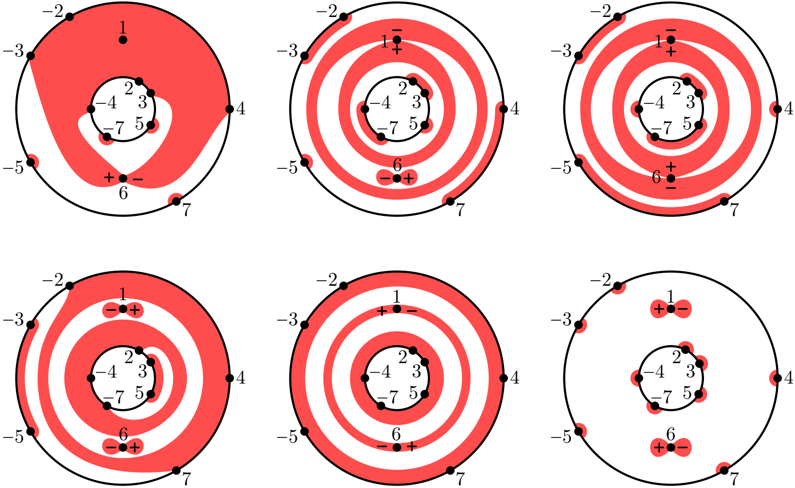
or noncrossing partitions of a disk with 2 orbifold points.



Affine type $\widetilde{D}$

Symmetric n.c. partitions of an annulus with two double points

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