Clusters, Coxeter-sortable elements and noncrossing partitions

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Basic objects

W	a finite Coxeter group
S	the simple generators (reflections)
n	the rank $\|S\|$ of W
T	the reflections in W
Φ	a root system for W
c	a Coxeter element $c = s_1 \cdots s_n$ for $S = \{s_1, \ldots, s_n\}$
h	the Coxeter number (order of c)
e_i	exponents (eigenvalues of c are ζ^{e_i})

W-Catalan numbers

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}$$

Generalizes usual Catalan number ($W = S_n$).

W-Catalan combinatorics

Various constructions involving (W, c)—or involving only W—yield sets of objects counted by the W-Catalan number. Connections are not fully understood.

Clusters (Fomin, Marsh, Reineke, Zelevinsky)

Almost positive roots: the set $\Phi_{\geq -1}$ of positive roots Φ together with negative simple roots $(-\Pi)$.

For $s \in S$, define $\sigma_s : \Phi_{\geq -1} \to \Phi_{\geq -1}$:

$$\sigma_s(\alpha) := \begin{cases} \alpha & \text{if } \alpha \in (-\Pi) \text{ and } \alpha \neq -\alpha_s, \text{ or} \\ s(\alpha) & \text{otherwise.} \end{cases}$$

c-*Compatibility* relation $\|_c$ on $\Phi_{\geq -1}$:

(i) For any $s \in S$,

$$-\alpha_s \|_c \beta \text{ if and only if } \beta \in (\Phi_{\langle s \rangle})_{\geq -1}.$$

(ii) For s initial in c,

$$\alpha_1 \|_c \alpha_2 \text{ if and only if } \sigma_s(\alpha_1) \|_{scs} \sigma_s(\alpha_2).$$

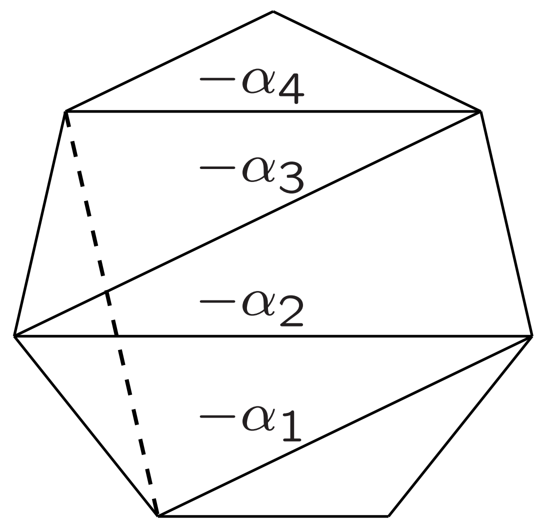
Clusters: maximal sets of pairwise c-compatible roots in $\Phi_{\geq -1}$.

Counted by $\mathrm{Cat}(W)$.

Example ($W = S_n$, special choice of c)

Simple roots are $\alpha_1, \cdots, \alpha_{n-1}$ and positive roots are $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots \alpha_j$.

$n = 5$:



Almost positive root $\leftrightarrow$ diagonal of (n+2)-gon.
Negative simple roots $\leftrightarrow$ diagonals forming "snake."
Positive root $\alpha_{ij} \leftrightarrow$ diagonal crossing $-\alpha_i, \ldots, -\alpha_j$ but no other negative simple. (α_{23} shown.)

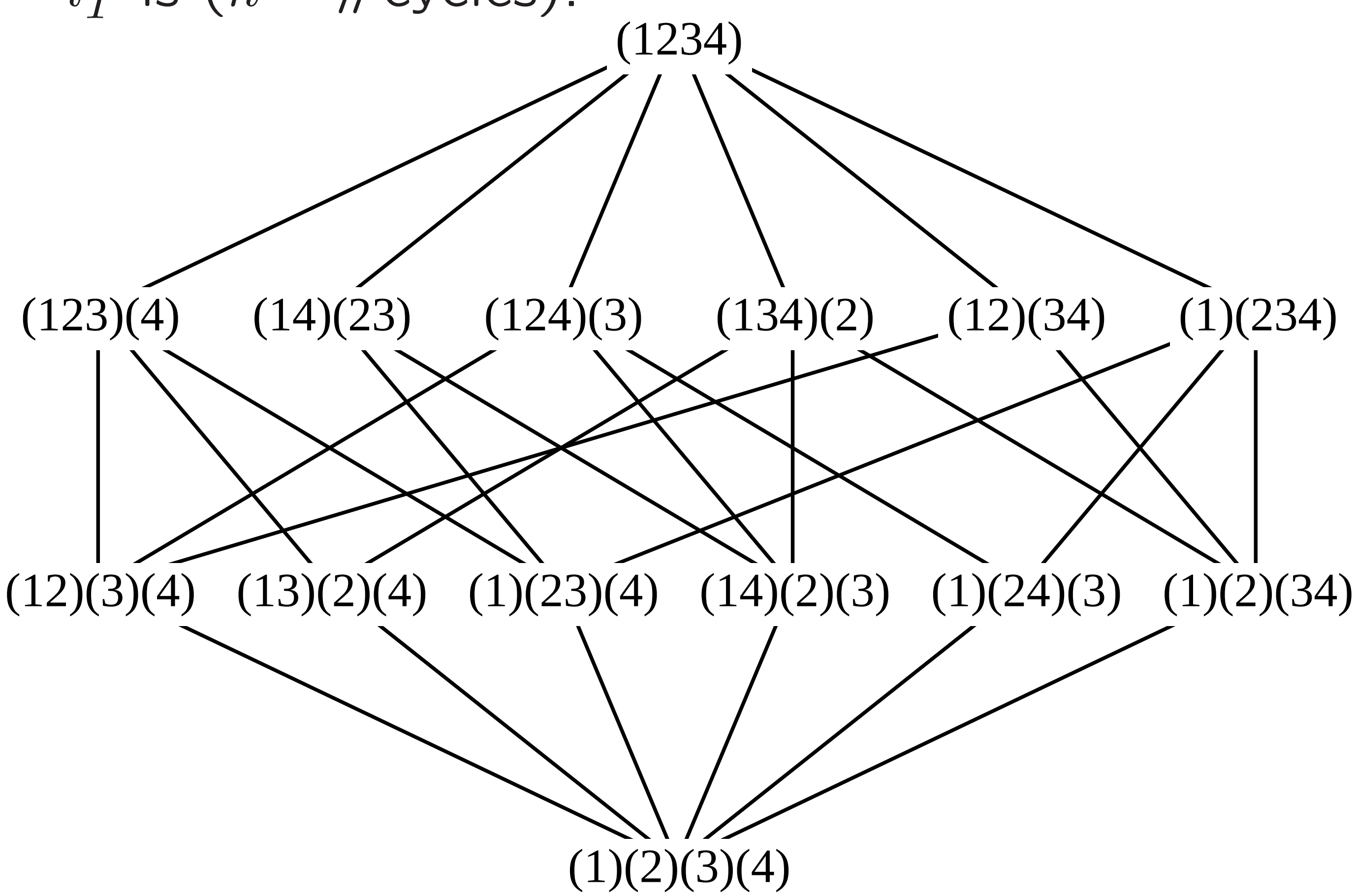
$\sigma_{s_1} \cdots \sigma_{s_n} \leftrightarrow$ rotation through 1/7 of a turn.
Compatible $\leftrightarrow$ noncrossing.
Clusters $\leftrightarrow$ triangulations.

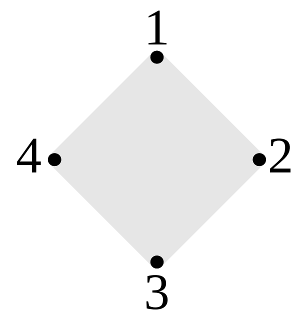
Noncrossing partitions

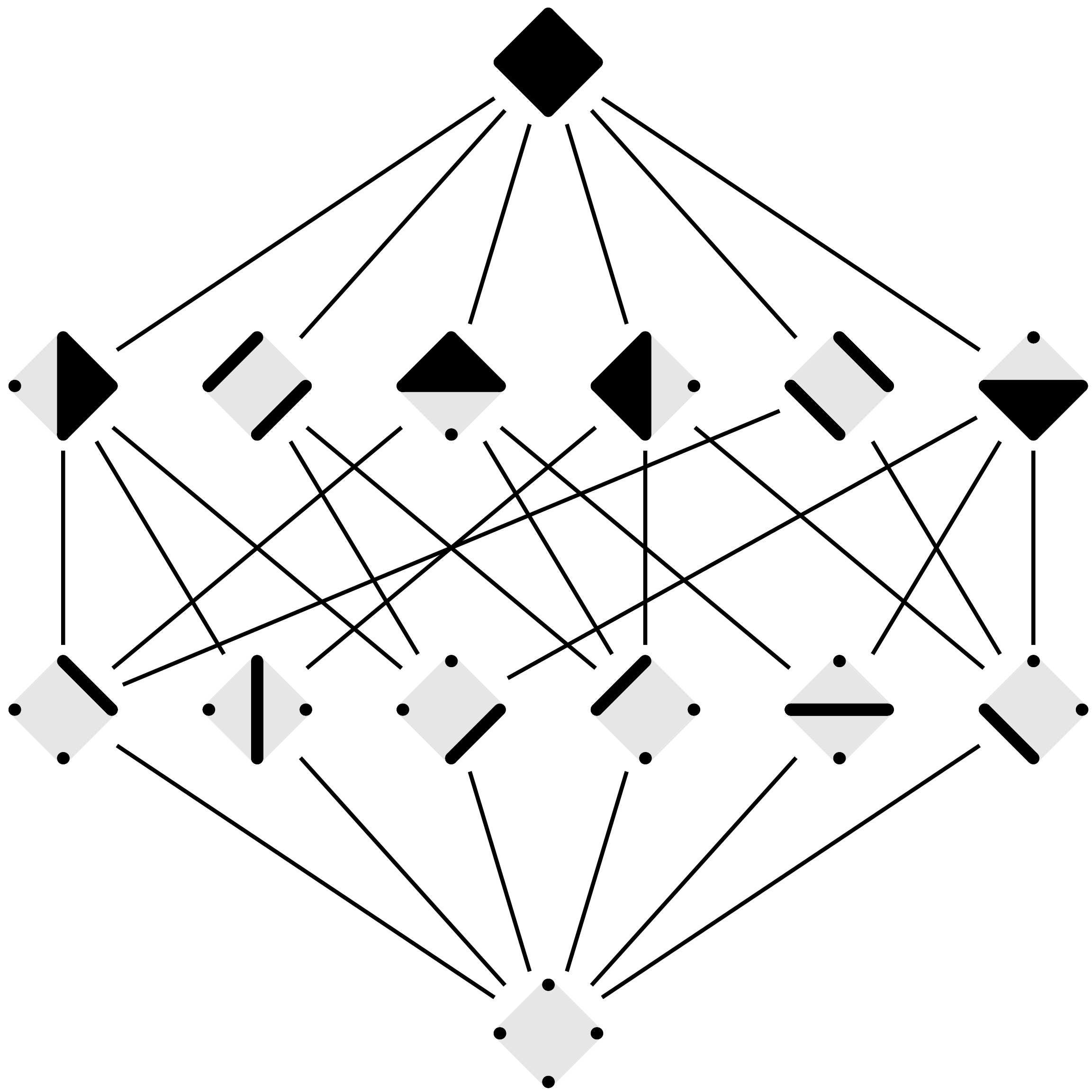
(Athanasiadis, Bessis, Biane, Brady-Watt, Picantin, Reiner)

$l_T(w)$: length of a shortest factorization of w into reflections.
Set $w \preceq wt$ when $l_T(wt) = l_T(w) + 1$. $(t \in T)$
The *noncrossing partition lattice* is $[1, c]_{\preceq}$.
This lattice has Cat(W) elements.

Example: $W = S_n$, $c = (1\ 2 \cdots n)$
l_T is ($n - $#cycles).




1
4
2
3



Sorting words

Fix (some reduced word for) a Coxeter element c. Form an infinite word

$$c^\infty = c|c|c|c|\cdots$$

c-Sorting word for w: the lex. leftmost subword of c^∞ which is a reduced word for w.

Example: $W = S_5$, $c = s_1s_2s_3s_4$,

$$c^\infty = s_1s_2s_3s_4|s_1s_2s_3s_4|s_1s_2s_3s_4|\cdots$$

The c-sorting word for 42351 is $s_1s_2s_3s_4|s_2|s_1$.

Step	c-Sorting word	Permutation
0		42351
1	s_1	41352
2	s_1s_2	41253
3	$s_1s_2s_3$	31254
4	$s_1s_2s_3s_4\vert$	31245
5	$s_1s_2s_3s_4\vert$	31245
6	$s_1s_2s_3s_4\vert s_2$	21345
7	$s_1s_2s_3s_4\vert s_2$	21345
8	$s_1s_2s_3s_4\vert s_2$	21345
9	$s_1s_2s_3s_4\vert s_2\vert s_1$	12345

Sortable elements

A sorting word can be interpreted as a sequence of sets (letters between *dividers* "$|$"). If the sequence is nested then w is c-*sortable*.

Example: w with c-sorting word $s_1s_2s_3s_4|s_2|s_1$ is not c-sortable because $\{s_1\} \not\subseteq \{s_2\}$.

The c-sortable elements are counted by Cat(W).

Example: $W = S_3, \quad c = s_1s_2.$

c-sortable:		not c-sortable:	
1	123	$s_2\|s_1$	312
s_1	213		
s_1s_2	231		
$s_1s_2\|s_1$	321		
s_2	132		

Example: $W = S_n$

For one choice of c, the c-sortable elements are the "231-avoiding" or "stack-sortable" permutations.

For another c, "c-sortable" = "312-avoiding".

Proof methods

$l(w)$ is the length of a shortest word for w in the alphabet S. ($l(w)$ and $l_T(w)$ typically differ.)

$l(sw) < l(w)$ means w has a reduced word beginning with $s \in S$. Otherwise $l(sw) > l(w)$.

Two lemmas are immediate from the definition, because we can take $s_1 = s$ in
$c^\infty = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | \ldots$

Lemma: Let s be an initial letter of c and let $w \in W$ have $l(sw) > l(w)$. Then w is c-sortable if and only if it is an sc-sortable element of $W_{\langle s \rangle}$.

Lemma: Let s be an initial letter of c and let $w \in W$ have $l(sw) < l(w)$. Then w is c-sortable if and only if sw is scs-sortable.

These lemmas allow proofs by induction on rank of W and on length of w.

Sortable $\leftrightarrow$ NC

The *cover reflections* of w are the reflections of the form wsw^{-1} for $s \in S$ such that w has a reduced word ending in s. (In this case $w \gtrdot ws$ is a cover in weak order.)

Let w have c-sorting word $s_1 s_2 \cdots s_k$.

Each cover reflection t has a unique $i \in [k]$ with $tw = s_1 s_2 \cdots \widehat{s}_i \cdots s_k$.

Let $\mathrm{nc}_c(w)$ be the product of the cover reflections of w, ordered by increasing i.

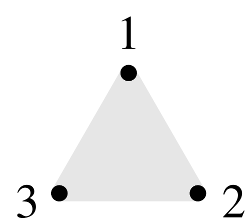
The map nc_c is a bijection between c-sortable elements and noncrossing partitions (i.e. elements of $[1, c]_T$).

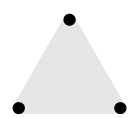
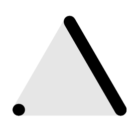
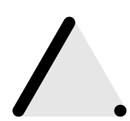
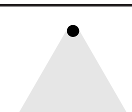
Example: $W = S_n$

Cover reflections of π are transpositions $(i\ j)$ with $i > j$ and i immediately before j in π. (Related to "descents" of π.)

For $W = S_n$, ignore the order of multiplying transpositions. Interpret the transposition $(i\ j)$ as "i and j in the same block." This gives a noncrossing partition of the cycle c.

In S_3 with $c = s_1 s_2 = (1\ 2\ 3)$, every permutation except 312 is c-sortable.



π	$\mathrm{nc}_c(\pi)$
123	
213	
231	
321	
132	

Sortable $\leftrightarrow$ Clusters

Let w have c-sorting word $a_1a_2\cdots a_k$ and let $s \in S$.

The *last root* for s in w is the positive root for the reflection $a_1a_2\cdots a_{i-1}a_ia_{i-1}\cdots a_2a_1$, where a_i is the rightmost instance of s in $a_1a_2\cdots a_k$.

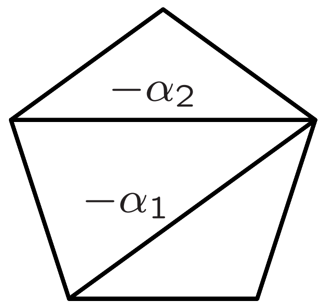
If s doesn't occur in $a_1a_2\cdots a_k$ then the last root for s in w is $-\alpha_s$, where α_s is the simple root associated to s.

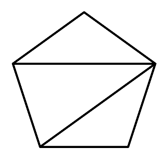
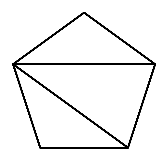
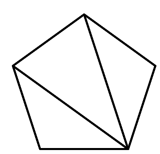
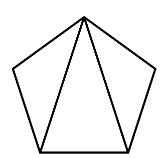
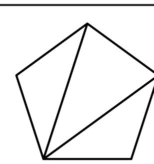
Define $\mathrm{cl}_c(w) = \{\text{last roots of } w\}$

The map cl_c is a bijection between c-sortable elements and c-clusters.

Example:

For $W = S_3$ and $c = s_1 s_2$ the map is as follows:



w	$\mathsf{nc}_c(\pi)$	
1	$-\alpha_1, -\alpha_2$	
s_1	$\alpha_1, -\alpha_2$	
$s_1 s_2$	α_1, α_{12}	
$s_1 s_2 \vert s_1$	α_2, α_{12}	
s_2	$-\alpha_1, \alpha_2$	

Lattices and fans

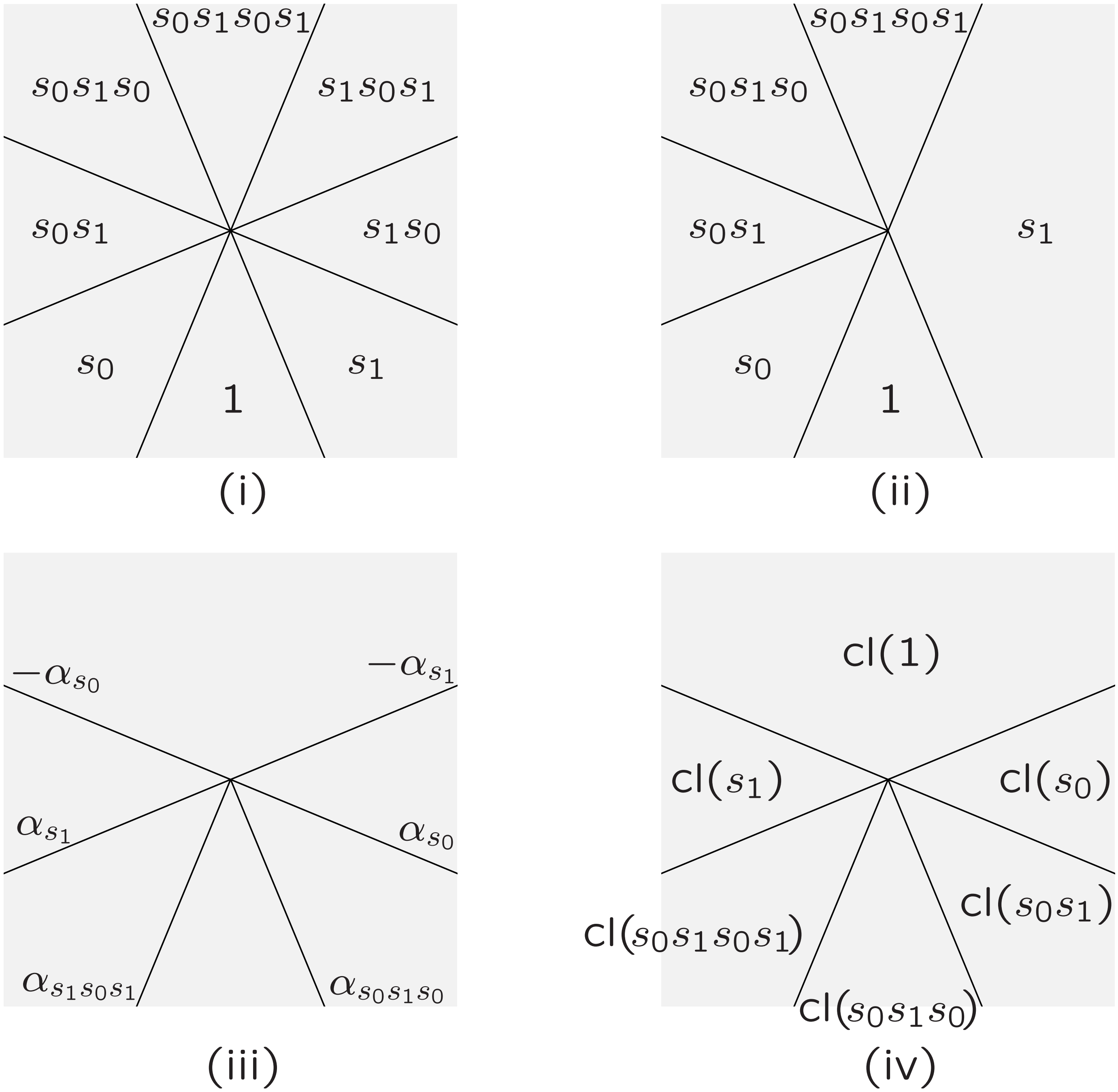
Define $x \equiv y$ if the maximal c-sortable element below x (in weak order) equals the maximal c-sortable element below y. This is a lattice congruence on the weak order on W—in fact, the "Cambrian" congruence.

Any lattice congruence on weak order defines a complete fan. Maximal cones are unions (over congruence classes) of maximal cones of the fan defined by the reflecting hyperplanes of W.

The clusters also define a complete fan. Each cluster defines a maximal cone: the positive linear span of the roots in the cluster.

Joint with D. Speyer: The map cl_c induces a combinatorial isomorphism between the Cambrian fan and the cluster fan. For special ("bipartite") choices of c, there is also a linear isomorphism.

Example ($W = B_2$, $c = s_0 s_1$)



(i) The fan defined by reflecting hyperplanes
(ii) The c-Cambrian fan
(iii) The c-cluster fan
(iv) The map cl_c from (ii) to (iii)

Clusters $\leftrightarrow$ NC partitions

(with D. Speyer)

For a special ("bipartite") choice of c, there is a completely geometric bijection between clusters and noncrossing partitions.

Choose a certain vector v. For each maximal cone C of the cluster fan, v chooses a "bottom" face $B(C)$ in a way that can be made precise. The map taking a maximal cone to $L(B(C))$ is a bijection between clusters and noncrossing partitions.

(L is some specific linear map. Noncrossing partitions are represented here by their fixed subspaces.)

Example

In picture (ii) on the previous slide, v can be chosen to point left and L is the reflection fixing the lower left and upper right corners of the shaded square.

Remarks

• All definitions in the theory of sortable elements are given without relying on the classification of finite Coxeter groups. However, a few key lemmas about sortable elements currently only have "case-by-case" proofs.

• Several cluster algebra constructions have interpretations in terms of the Cambrian fan. For example, the "g-vector" of a cluster variable is the coordinates, in the basis of fundamental weights, of the corresponding Cambrian ray. (Joint with D. Speyer.) Work is in progress to extend these results beyond finite type.

• The nonnesting partitions (antichains in the root poset) are also counted by $\mathrm{Cat}(W)$. No bijection is known between nonnesting partitions and any other object discussed in this talk.