Theta functions in acyclic affine type

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Overview

Cluster algebras arose in the study of total positivity and found surprising connections to many algebraic and geometric areas. More recently: a connection with scattering amplitudes in physics.

Scattering diagrams arose from mirror symmetry. They were applied to prove longstanding conjectures about cluster algebras.
Cluster scattering diagrams $\rightarrow$
theta functions $=$ cluster monomials.

The definition of cluster scattering diagrams is, in principle, constructive. In practice, it is an existence theorem.

To make theta functions (and their structure constants): "Find all (pairs of) broken lines. . . ". But finding broken lines and ruling out possible broken lines is hard.
We need combinatorial models.

This talk: Results on theta functions in affine type and the models/tools needed to prove them.

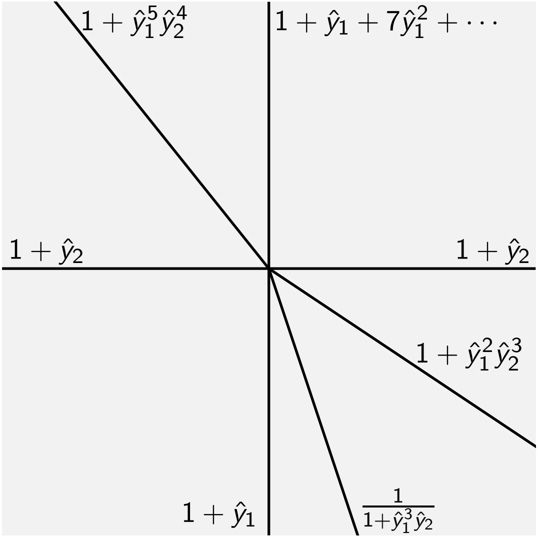
Section 1: Scattering diagrams

Basic setup

- B is an $n \times n$ skew-symmetrizable integer "exchange matrix"
- n is the "rank"
- V real vector space, basis $\alpha_1, \ldots, \alpha_n$
- V^* its dual space, basis $\rho_1, \ldots, \rho_n$, dual to $\alpha_1^\vee, \ldots, \alpha_n^\vee$
- $\lambda = \sum_{i=1}^n c_i \rho_i \quad \leftrightarrow \quad x^\lambda = x_1^{c_1} \cdots x_n^{c_n}$
- $\beta = \sum_{i=1}^n d_i \alpha_i \quad \leftrightarrow \quad \hat{y}^\beta = \hat{y}_1^{d_1} \cdots \hat{y}_n^{d_n}$

But let your eyes unfocus on this slide:
We have Laurent monomials in $x_1, \ldots, x_n$ specified by integer vectors λ and similarly Laurent monomials in $\hat{y}_1, \ldots, \hat{y}_n$ specified by integer vectors β.

Scattering diagrams



A scattering diagram is a set of walls $(\mathfrak{d}, f_{\mathfrak{d}})$:

$\mathfrak{d}$ is a codim.-1 cone.

The scattering term $f_{\mathfrak{d}}$ is a FPS in the $\hat{y}_i$.

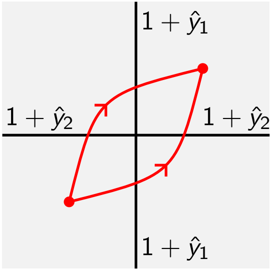
- $\mathfrak{d}$ is normal to a primitive, positive integer vector β.
- $f_{\mathfrak{d}}$ is a univariate FPS in $\hat{y}^{\beta}$ with constant term 1.
- A finiteness condition

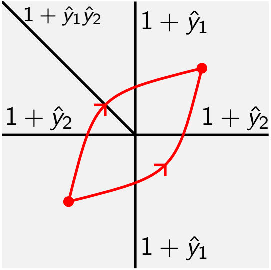
Consistent scattering diagrams

Crossing a wall acts on Laurent polynomials (or series). Each monomial is modified in a way that depends on its exponent vector, the normal to the wall, the scattering term, and B.

Consistent: The action of crossing walls from one point to another depends only on the starting point and the ending point.

Example ($B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$). The scattering diagram on the left is not consistent. We can make it consistent by adding one wall.





Cluster scattering diagrams

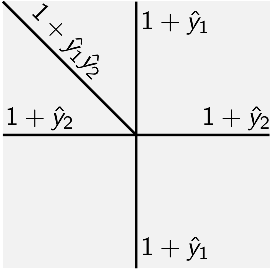
Theorem (Gross, Hacking, Keel, Kontsevich, 2014). Given an exchange matrix B, there is unique consistent scattering diagram (the cluster scattering diagram) that

- contains the walls $(\alpha_i^\perp, 1+\hat{y}_i)$, and
- otherwise only contains walls that are outgoing.

Outgoing: $\mathfrak{d}$ does not contain $B\beta$, where β is the positive normal.

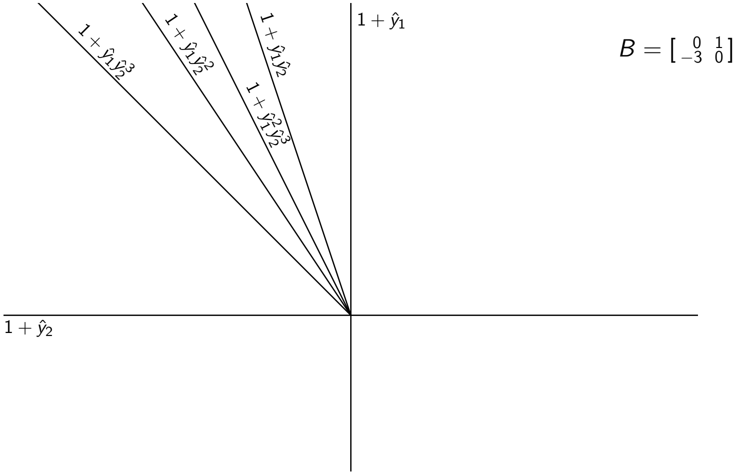
Example. The cluster scattering diagram for $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

The point: The wall we added is outgoing.



Theta functions in the cluster scattering diagram

Take an integer vector $\lambda \in V^*$. The theta function ϑ_λ is a sum over monomials labeling broken lines.



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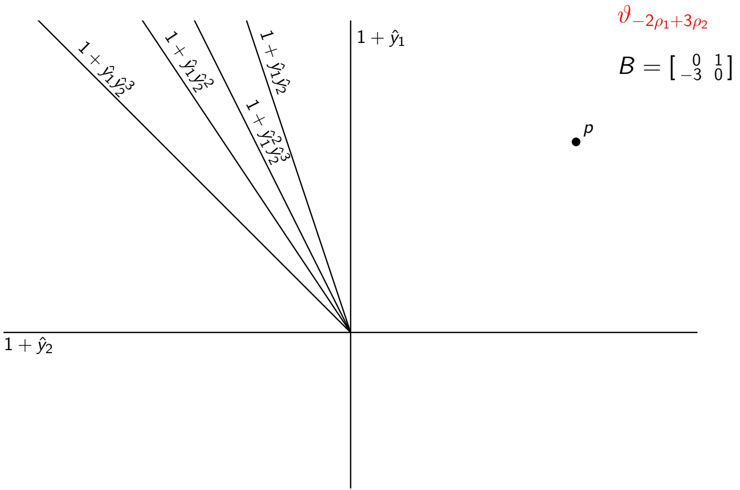

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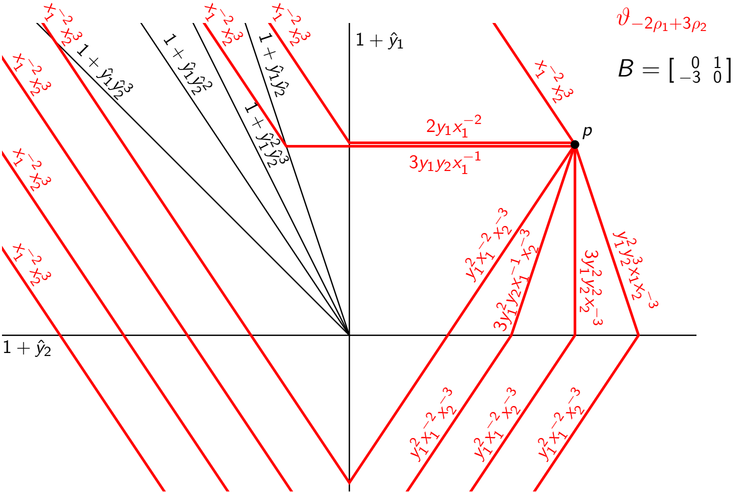

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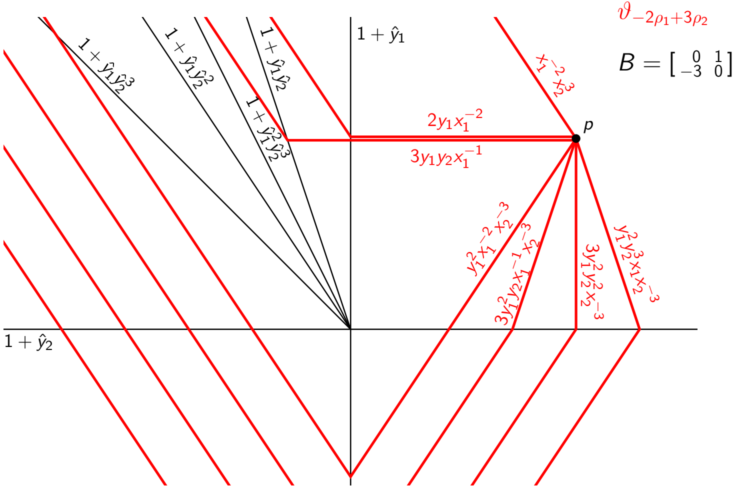


Multiplying theta functions (GHKK, 2014)

Given vectors $\lambda, \nu \in V^*$, the product $\vartheta_\lambda \cdot \vartheta_\nu$ expands as a (possibly infinite) linear combination of theta functions.

Structure constants:

To find the coefficient of a given theta function in $\vartheta_\lambda \cdot \vartheta_\nu$, we find all pairs of broken lines satisfying certain conditions.

Recap of Section 1: Scattering diagrams

A scattering diagram is a collection of walls $(\mathfrak{d}, f_{\mathfrak{d}})$

- $\mathfrak{d}$ is a codimension-1 cone.
- $f_{\mathfrak{d}}$ is the scattering term.

Consistent scattering diagram: The result of wall-crossing along a path depends only on the starting point and the ending point.

Cluster scattering diagram: Start with coordinate hyperplanes. $\exists!$ way to add "outgoing" walls to get a consistent scattering diagram.

Theta functions: ϑ_λ for each integer vector λ. Sum over "broken lines".

Multiplying theta functions: Sum over pairs of broken lines.

To do this in practice, we need a combinatorial model.

Questions?

Section 2: Cluster algebras

(Principal coefficients) Cluster algebras

Start with an initial seed consisting of initial cluster variables $x_1, \ldots x_n$ and a skew-symmetric integer matrix B.

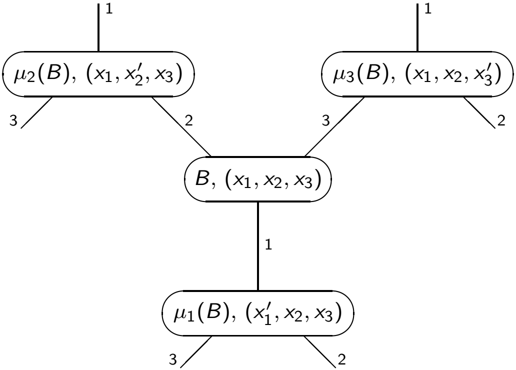
Mutation: an operation that takes a seed and gives a new seed.

- There are n "directions" for mutation.
- Mutation
  - switches out one cluster variable, replaces it with a new one;
  - changes B (and some extra rows) by matrix mutation.

  The result is a new seed.
- Mutation is involutive.

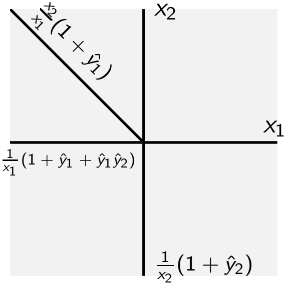
For those in the know: In this talk, I'll only state results carefully enough for principal coefficients, but essentially everything I talk about works with arbitrary coefficients of geometric type.

Do all possible sequences of mutations, and collect all the cluster variables that appear.



The cluster algebra for the given initial seed is the ring generated by all cluster variables.

Theorem (Fomin-Zelevinsky, 2007). Each cluster variable v is a Laurent monomial $x^{\mathbf{g}(v)}$ times a polynomial in the $\hat{y}_i$.
(The vector $\mathbf{g}(v)$ is the g-vector of v.)



Theorem (GHKK, 2014).
Part of the cluster scattering diagram cuts out the g-vector fan.
Rays: Spanned by $\mathbf{g}$-vectors of cluster variables
Cones: Spanned by the $\mathbf{g}$-vectors of clusters.

Cluster monomial: monomial in the clus. variables in some cluster.
$\exists$ cluster monomial for every integer vector in the $\mathbf{g}$-vector fan.

Theorem (GHKK, 2014). Any cluster monomial v is $\vartheta_{\mathbf{g}(v)}$.

Theorem (GHKK, 2014). Theta functions are a basis for a "canonical algebra" that often agrees with the cluster algebra.

Theta functions outside the $\mathbf{g}$-vector fan

From the previous slide:

Theta functions ϑ_λ for λ in $\mathbf{g}$-vector fan are cluster monomials.

A big question: What is ϑ_λ for λ outside the $\mathbf{g}$-vector fan?

(Why? Because we want to know the whole theta-basis.)

Shameless tangential plug for old and new work

Define the return function ret_λ: start at λ and apply wall crossings to x^λ along a path to the positive cone.

Cluster monomials $\leftrightarrow$ return functions ret_λ for λ in $\mathbf{g}$-vector fan.

ϑ_λ and ret_λ are well-defined even for λ outside the $\mathbf{g}$-vector fan.

Theta functions are on everyone's "radar screen" (if they think about cluster scattering diagrams). Return functions ret_λ are not.

Old: ret_λ for λ in limiting ray in rank-2 affine type is a limit of ratios of cluster variables and is a g.f. for Narayana numbers.

New: Analogous results in affine type, all ranks.
Also, ret_λ and ϑ_λ are related by a quadratic equation.

New: Some return functions in affine/surface type resemble irrational letters in symbols for scattering amplitudes.

New: In the surfaces model, return functions of closed curves are laminated lambda lengths. Same quadratic relation $\mathrm{ret}_\lambda \leftrightarrow \vartheta_\lambda$.

Shameless tangential plug for old and new work

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c.f. Çanakçı–Schiffler

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c.f. Henderschee

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c.f. Dani Kaufman

Section 3: Theta functions in affine type

Scattering diagrams in finite type

Finite type means finitely many different cluster variables.

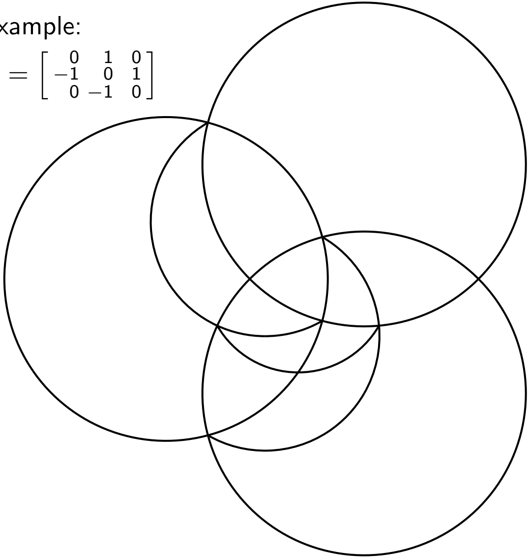
In this case, finitely many walls in the cluster scattering diagram.

All walls have scattering term of the form $1+\hat{y}^{\beta}$.

All theta functions are cluster monomials.

Example:

$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$



What *is* affine type?

One answer:

A cluster algebra of rank ≥ 3 is of affine type if and only if it is infinite and has linear growth. (Combine results of Seven and Felikson-Shapiro-Thomas-Tumarkin.)

A more constructive answer:

Start with a Cartan matrix A of affine type, put zeros on the diagonal, sign it to make it skew-symmetrizable and acyclic, and you get an exchange matrix B (and cluster algebra) of affine type.

The root system associated to A is crucially important.

There is a special imaginary root δ.

There is also an associated Coxeter group W. The sign information in c determines a special element of W called a Coxeter element.

Affine cluster (scattering) fans

In affine type, the $\mathbf{g}$-vector fan covers all of space except a codimension-1 cone (R., Speyer) called the imaginary wall.

The decomposition of this cone into cones in the cluster scattering fan is, combinatorially, a product of simplicial cyclohedra, AKA graph associahedra of cycles or type-B associahedra (R., Stella).

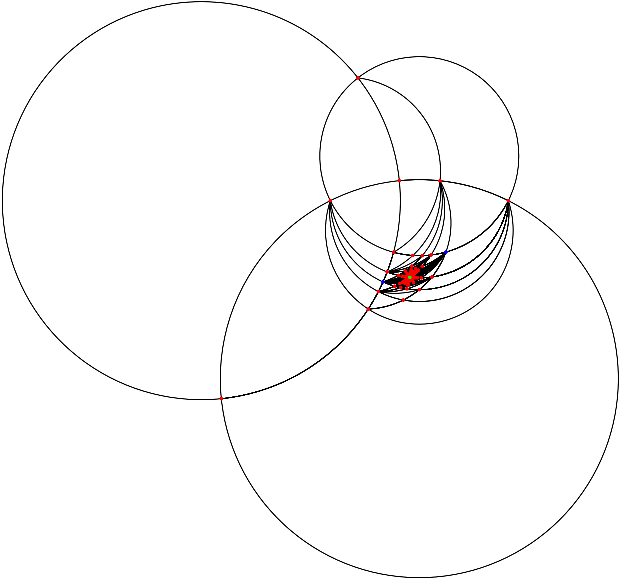
These imaginary cones are the star of a single imaginary ray. It is spanned by $\nu_c(\delta)$, where δ is the imaginary root.

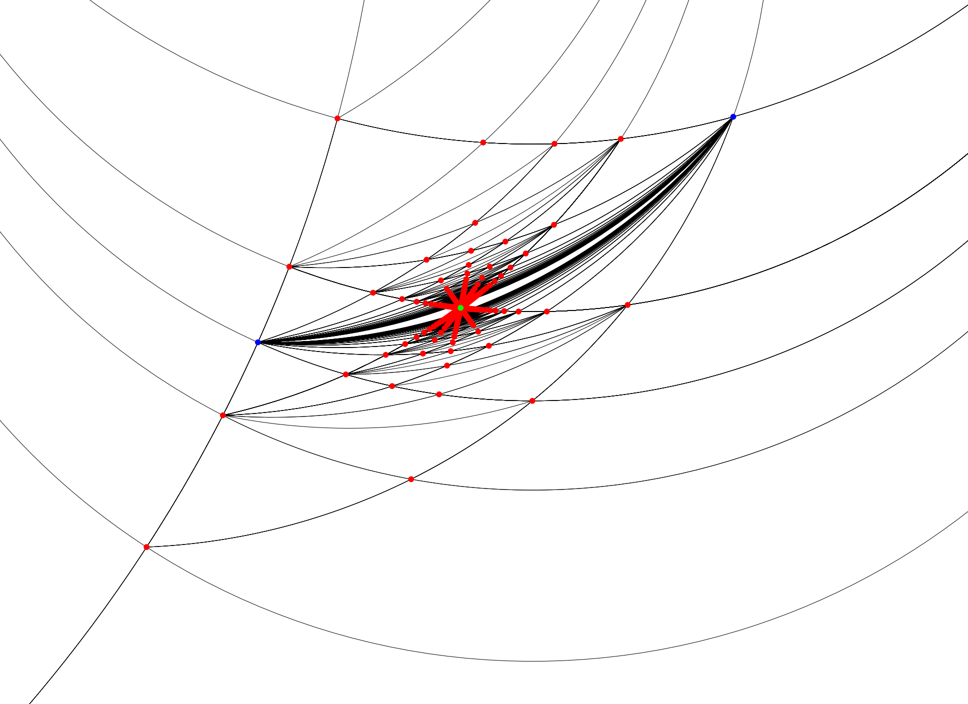
All other rays in the imaginary wall (and in the whole fan) are spanned by $\mathbf{g}$-vectors of cluster variables.
(These $\mathbf{g}$-vectors are $\nu_c(\beta)$ for roots β.)

ν_c??

It's some piecewise linear map from roots to $\mathbf{g}$-vectors that I don't have time to define.

Rank-3 example ($\widetilde{G}_2$)

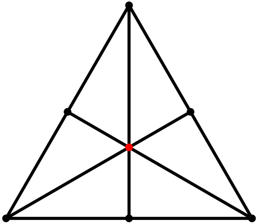




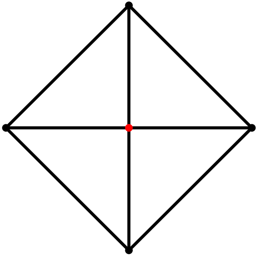
The imaginary wall in rank 4

The scattering fan is 4-dimensional, so the imaginary wall is 3-dimensional. Two possibilities:

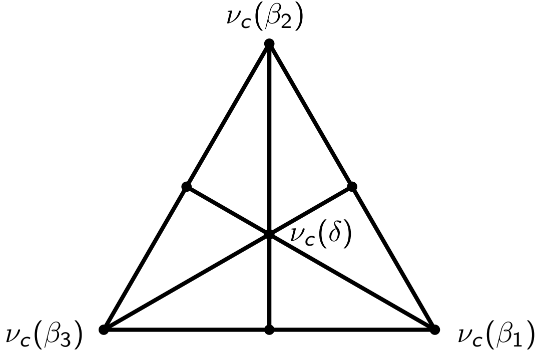
One 2-cyclohedron:



Two 1-cyclohedra:



More about the imaginary wall



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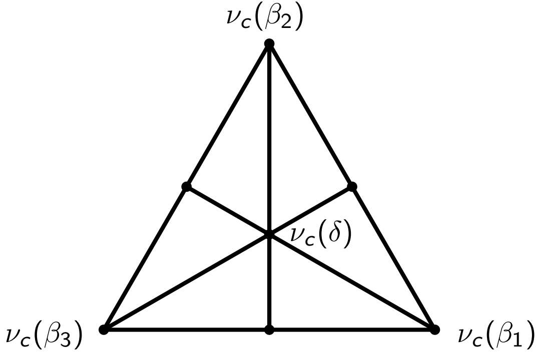
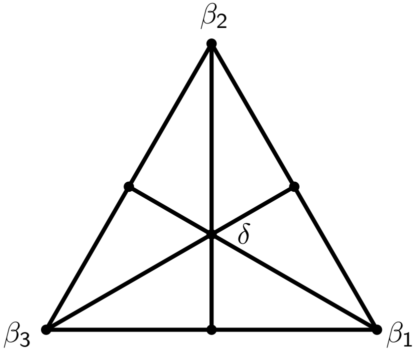


May I leave out all the ν_c's?

More about the imaginary wall



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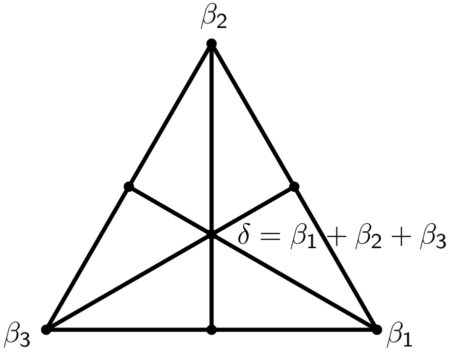

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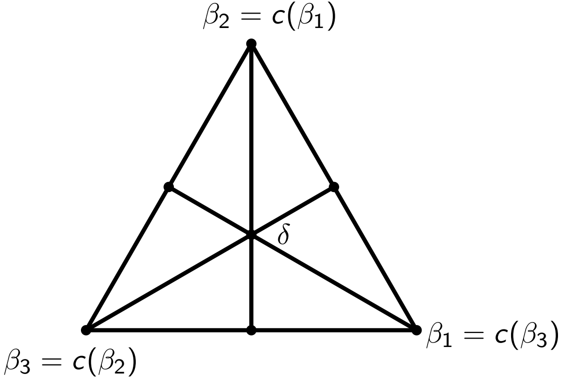

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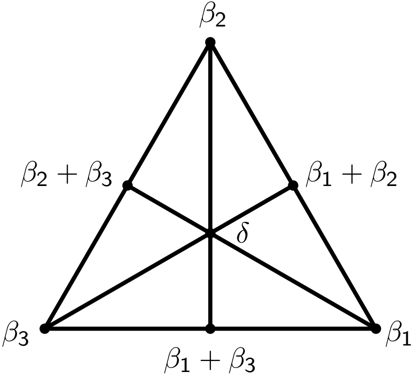

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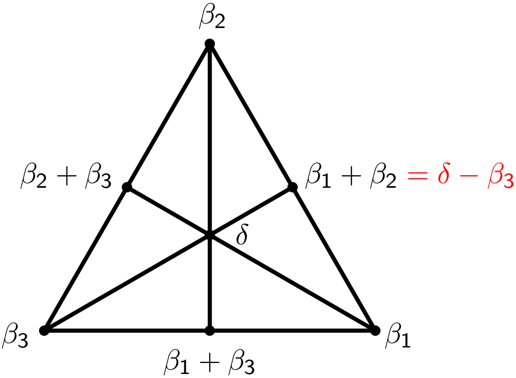

Determining the theta functions

Philosophy: The theta functions that are cluster variables are "known". (This is computationally reasonable in affine type.)

Theorem (R., Stella, 2026). For any of these β,

$$\vartheta_{\nu_c(\delta)} = \vartheta_{\nu_c(\beta)} \cdot \vartheta_{\nu_c(\delta-\beta)} - y^{\beta}\vartheta_{\nu_c(\delta-\beta-c^{-1}\beta)} - y^{c\beta}\vartheta_{\nu_c(\delta-\beta-c\beta)}.$$

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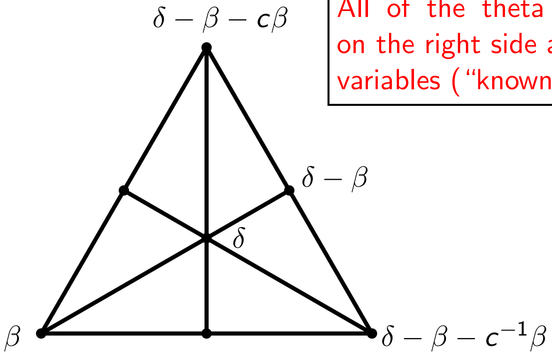

All of the theta functions on the right side are cluster variables ("known").

Determining the theta functions

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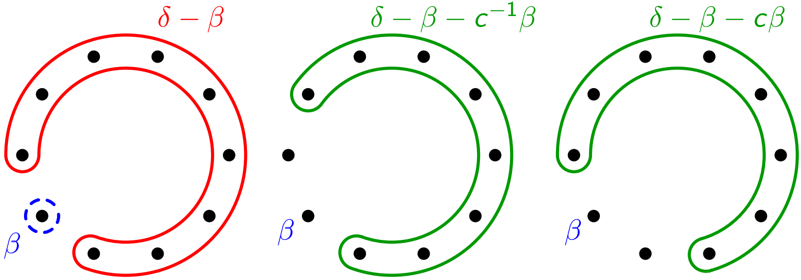

Determining the theta functions (continued)

Theorem (R., Stella, 2026).

$$\vartheta_{2\nu_c(\delta)} = (\vartheta_{\nu_c(\delta)})^2 - 2y^{\delta} \qquad \boxed{\text{Chebyshev}}$$

and, for $k \geq 3$,

$$\vartheta_{k\nu_c(\delta)} = \vartheta_{(k-1)\nu_c(\delta)} \cdot \vartheta_{\nu_c(\delta)} - y^{\delta}\vartheta_{(k-2)\nu_c(\delta)}.$$

Determining the theta functions (continued)

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Where affine type overlaps with surface type, this was already known: Musiker-Schiffler-Williams

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This is really a formula for $\vartheta_{\nu_c(\delta)} \cdot \vartheta_{(k-1)\nu_c(\delta)}$. More generally:

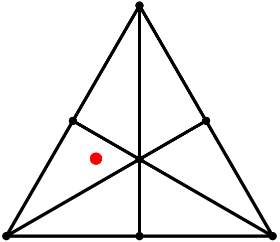
Theorem (R., Stella, 2026).

$$\begin{aligned} (\vartheta_{k\nu_c(\delta)})^2 &= \vartheta_{2k\nu_c(\delta)} + 2y^{k\delta} && \text{for } k \geq 1, \text{ and} \\ \vartheta_{k\nu_c(\delta)} \cdot \vartheta_{\ell\nu_c(\delta)} &= \vartheta_{(k+\ell)\nu_c(\delta)} + y^{\ell\delta}\vartheta_{(k-\ell)\nu_c(\delta)} && \text{for } k > \ell \geq 1. \end{aligned}$$

Determining the theta functions (continued)

Theorem (R., Stella, 2026). If λ is in an imaginary cone, write λ (uniquely!) as a sum $\sum \lambda_i$ of vectors in distinct extreme rays of that cone. Then

$$\vartheta_\lambda = \prod \vartheta_{\lambda_i}.$$



This completes the description of all theta functions in affine type.

Imaginary exchangeability

Some cluster variables are exchangeable in the cluster scattering fan even though they are not exchangeable in the $\mathbf{g}$-vector fan. Call these imaginary exchangeable.

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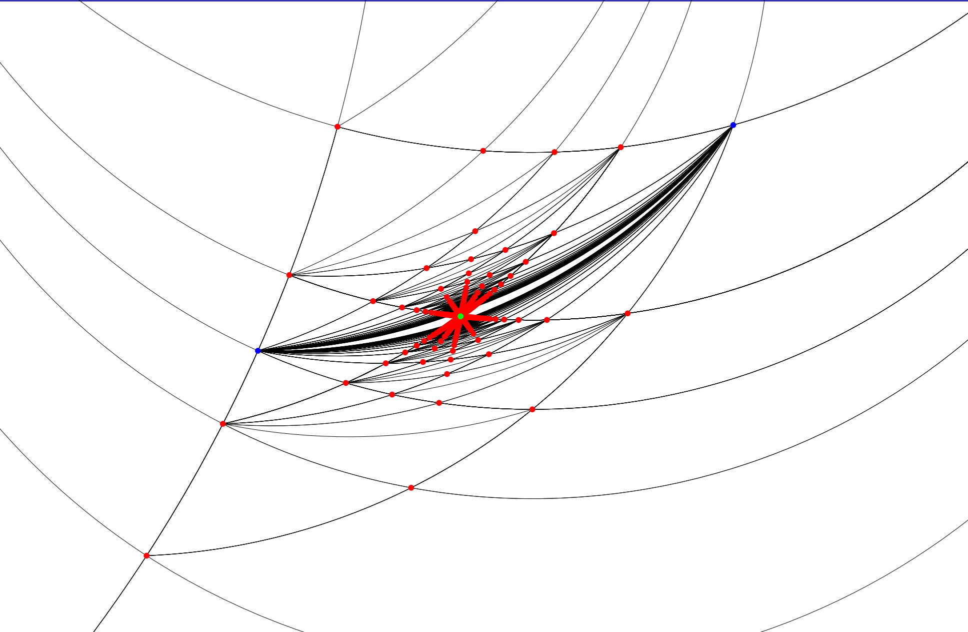
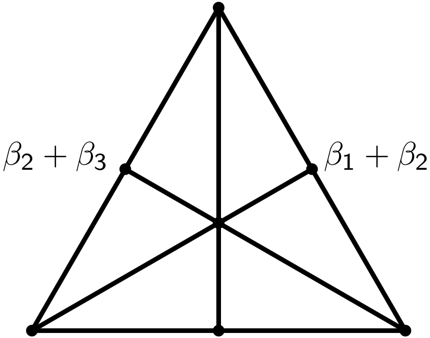
"Exchangeable" ???

Let's pause for examples.

Imaginary exchangeability example



Another imaginary exchangeability example



Imaginary exchangeability

Some cluster variables are exchangeable in the cluster scattering fan even though they are not exchangeable in the **g**-vector fan. Call these imaginary exchangeable.

Theorem (R., Stella, 2020). The pairs that are imaginary exchangeable but not exchangeable are $\nu_c(\delta - \beta), \nu_c(\delta - \beta')$ for β, β' in the same c-orbit.

Suppose $\beta' = c^\ell \beta$ and $\beta = c^m \beta'$. Define

$$\phi = \sum_{i=1}^{\ell-1} c^i \beta \quad \text{and} \quad \phi' = \sum_{i=1}^{m-1} c^i \beta'.$$

Theorem (R., Stella, 2026).

$$\vartheta_{\nu_c(\delta-\beta)} \cdot \vartheta_{\nu_c(\delta-\beta')} = \vartheta_{\nu_c(\delta)} \vartheta_{\nu_c(\phi)} \vartheta_{\nu_c(\phi')} + y^{\phi'+\beta} (\vartheta_{\nu_c(\phi)})^2 + y^{\phi+\beta'} (\vartheta_{\nu_c(\phi')})^2.$$

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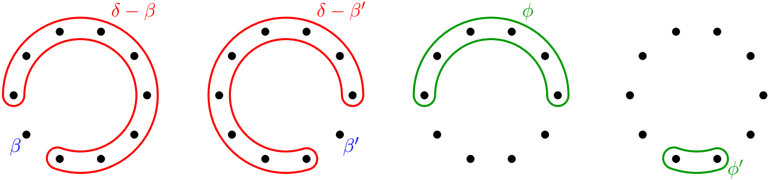


The imaginary subalgebra

Theorem (R., Stella, 2026). The theta functions for vectors in the imaginary wall span a subalgebra of the cluster algebra.

Call this the imaginary subalgebra.

Combinatorially, the imaginary wall is a product of cyclohedra.

Theorem (R., Stella, 2026). The imaginary subalgebra is a generalized cluster algebra of finite type (a product of type-C generalized cluster algebras).

Details:

- Using the Chekhov-Shapiro definition.

- All of the exchange relations are "ordinary", except that the imaginary exchange relations involve polynomials of degree 2.

Section 4: Tools

Tools needed

Construction of the cluster scattering diagrams uses:

Doubled Cambrian fans (R., Speyer) and
The affine almost positive roots model (R., Stella).
Shards

Proofs of theta function results need some additional tools.

The point: To compute a product of theta functions, we sum over all pairs of broken lines satisfying a certain description. Each pair contributes a coefficient times a theta functions.

We need tools especially to rule out possible theta functions and to rule out possible broken lines:

Neighboring seeds (and finite type-C combinatorics).
Mutation symmetry.
Mutating to one side of the imaginary wall.
Dominance regions (more or less in the sense of Fan Qin).

Finite type-C combinatorics in affine type

Observation: Combinatorics involving a Coxeter element in affine type always has a "piece" that is really finite type-C (or B).

Arguably, the first appearance was "tubes" in AR-quivers of affine type: The combinatorics at the bottoms of tubes is the simplicial cyclohedron (later known as the "type-B associahedron").

More recently:

McCammond and Sulway completed the affine noncrossing partition poset to a lattice by adjoining a product of finite-type-B noncrossing partition lattices.

Type-B associahedra (cyclohedra) appear in the affine almost positive roots model.

Neighboring seeds are a further development along these lines.

Tool: Neighboring seeds

Some cones in the $\mathbf{g}$-vector fan share as many rays as possible with the imaginary wall.

Call the seed associated to such a seed a neighboring seed.

There is a precise characterization of the exchange matrices of neighboring seeds (R., Rupel, Stella, 2025). Also:

Theorem (R., Rupel, Stella, 2025). If we take a neighboring seed as the initial seed, the imaginary cone becomes metrically a product of type-C $\mathbf{g}$-vector fans.

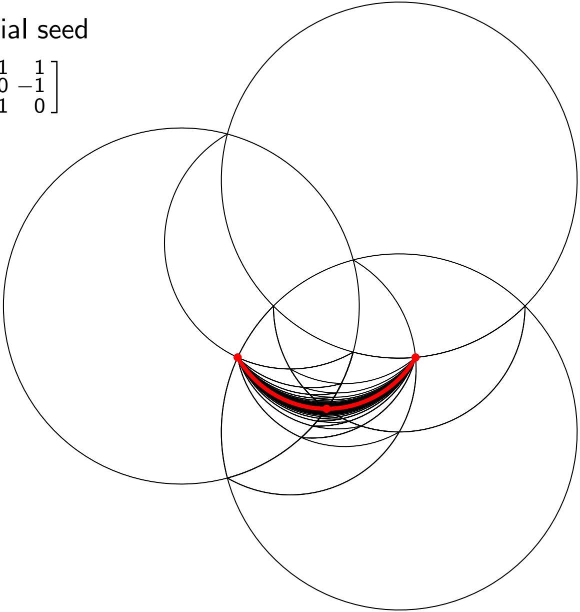
This allows certain computations in affine type to be restricted to a finite-type computation.

For example, it reduces the affine "dominance regions" computation to finite type C.

Depressingly low-dimensional neighboring seed example

Acyclic initial seed

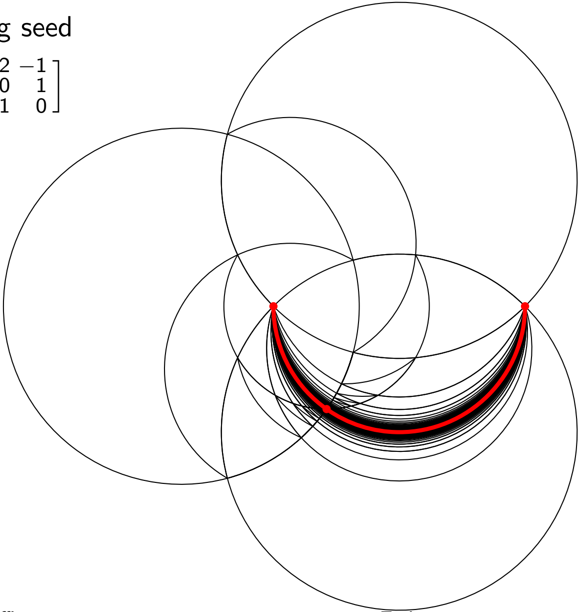
$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$



Depressingly low-dimensional neighboring seed example

Neighboring seed

$$B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$



Tool: Mutation symmetry

When we do a mutation $\mu_{\mathbf{k}}$ on the exchange matrix, we change the cluster scattering fan by a piecewise linear mutation map $\eta_{\mathbf{k}}^{B^T}$.
($\exists$ easy, explicit definition, but not today.)

Mutation symmetry: a sequence $\mathbf{k}$ of indices such that $\mu_{\mathbf{k}}(B) = B$.

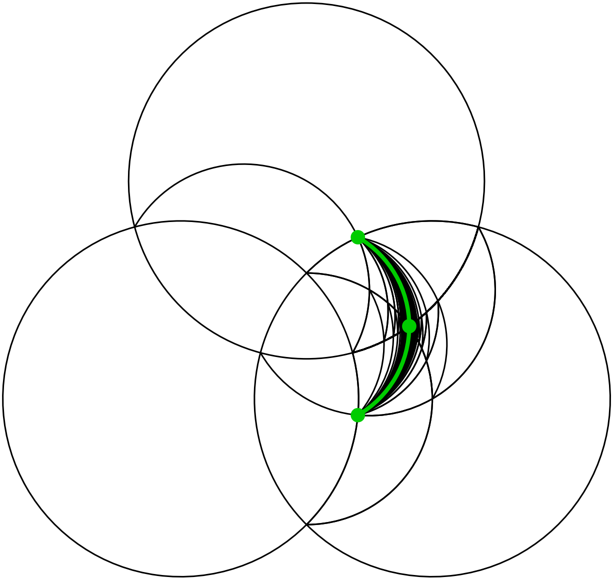
Theorem (R., Stella 2026). For $\mathbf{k}$ a mutation symmetry ($+$ *hypotheses*), a product of theta functions in finite $\eta_{\mathbf{k}}^{B^T}$-orbits is a linear combination of theta functions in finite $\eta_{\mathbf{k}}^{B^T}$-orbits.

The *hypotheses* have to do with making sure each theta function comes from finitely many broken lines. (For experts: $\lambda, \nu \in \Theta$.)

Acyclic affine case: If $c = s_1 \cdots s_n$, then $\mathbf{k} = 1 \cdots n$ is a symmetry.

Corollary. A product of theta functions in the imaginary wall expands as a combination of theta functions in the imaginary wall.

Mutation symmetry example



Tool: Mutating to one side of the imaginary wall

Very briefly:

Broken lines also mutate. For broken lines coming from a direction in the imaginary wall, you can use the mutation symmetry to move all the “interesting” parts of the broken line near on side of the imaginary wall, where you can show that nothing interesting can happen.

Upshot: You can place severe restrictions on where these broken lines can bend.

Dominance regions

Definition. Given a vector $\lambda \in \mathbb{R}^n$,

$$\mathbb{P}^B_{\lambda,\mathbf{k}} = \left(\eta_{\mathbf{k}}^{B^T}\right)^{-1}\left\{\eta_{\mathbf{k}}^{B^T}(\lambda) + \mu_{\mathbf{k}}(B)\alpha : \alpha \in \mathbb{R}^n, \alpha \geq 0\right\}.$$

The dominance region of λ is $\mathbb{P}^B_\lambda = \bigcap_{\mathbf{k}} \mathbb{P}^B_{\lambda,\mathbf{k}}$.

Initial motivation: Fan Qin (2021) constructed all bases that are "nice" in similar ways to the theta basis. (Difference: B or $\tilde{B}$.)

Work with Rupel and Stella (2025) determines these dominance regions (both kinds) in affine type. Rupel and Stella will apply these results to the question of bases. Our interest is:

Theorem (R., Stella 2026). If λ and ν are in the same cone of the scattering fan (mutation fan), then their product is a linear combination of theta functions in the dominance region of $\lambda + \nu$.

Combining these: We drastically cut down the possible nonzero structure constants for certain products.

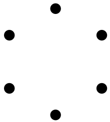
Section 5: Combinatorial details I won't have time for

Combinatorial details: Almost positive Schur roots

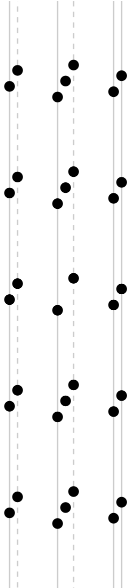
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$c = s_1 s_2 s_3$

- Start with a finite root system (A_2 in this case)



Combinatorial details: Almost positive Schur roots

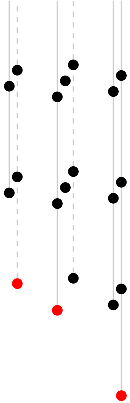


Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$c = s_1 s_2 s_3$

- Start with a finite root system (A_2 in this case)
- Construct an affine root system ($\tilde{A}_2$) (including imaginary roots)

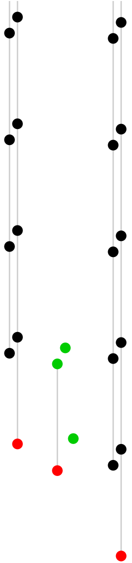
Combinatorial details: Almost positive Schur roots



Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$c = s_1 s_2 s_3$

- Start with a finite root system (A_2 in this case)
- Construct an affine root system ($\tilde{A}_2$) (including imaginary roots)
- Divide them into positive and negative roots.
- Restrict to the almost positive roots.



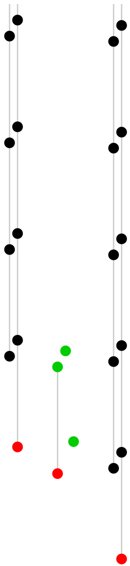
Example: $B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$c = s_1 s_2 s_3$

- Start with a finite root system (A_2 in this case)
- Construct an affine root system ($\tilde{A}_2$) (including imaginary roots)
- Divide them into positive and negative roots.
- Restrict to the almost positive roots.
- Restrict further to the almost positive Schur roots.

Observation: The real Schur roots limit, from two sides, to the direction of the imaginary root.
(This is true in general affine type.)

A.p. Schur roots and clus. scattering diagrams (R., Stella)

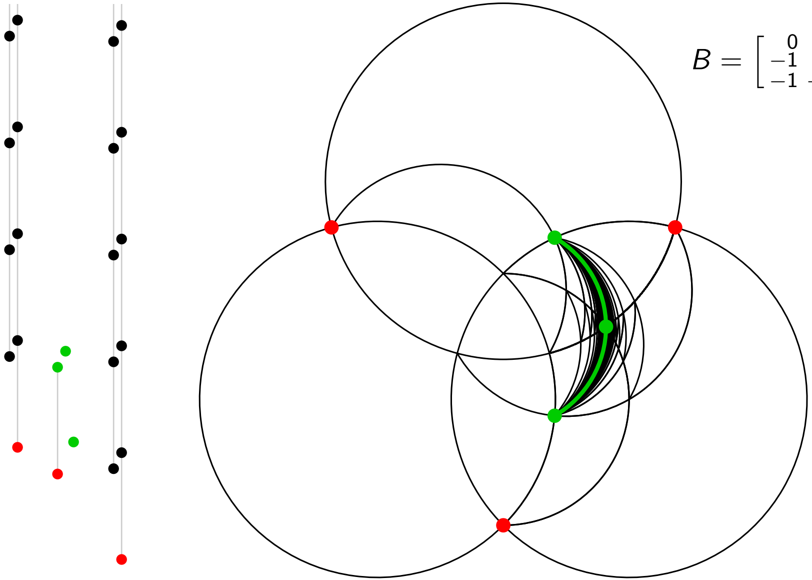


Almost positive Schur roots appear in two ways in affine cluster scattering diagrams:

1. They index the rays of the fan.
   - There is a piecewise-linear bijection from real a.p. Schur roots to $\mathbf{g}$-vectors of cluster variables.
   - The one imaginary a.p. Schur root corresponds to a "limiting ray" outside the cluster fan.
   - There is a combinatorial description of "compatibility" similar to finite type.
2. {Positive Schur roots}

   $= \{$normal vectors to walls$\}$.
   - Exactly one wall normal to each positive Schur root, including one limiting "imaginary wall" normal to the imaginary root δ. (Non-imaginary walls are essentially walls of the cluster fan.)
   - Each wall is the union of the shards in $\beta^\perp$ that contain $-B\beta$. (Only one shard unless $\beta = \delta$.)

Affine scattering fan example



$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Combinatorial details: Doubled Cambrian fans (R., Speyer)

Cambrian fans are build using lattice congruences and/or sortable elements.

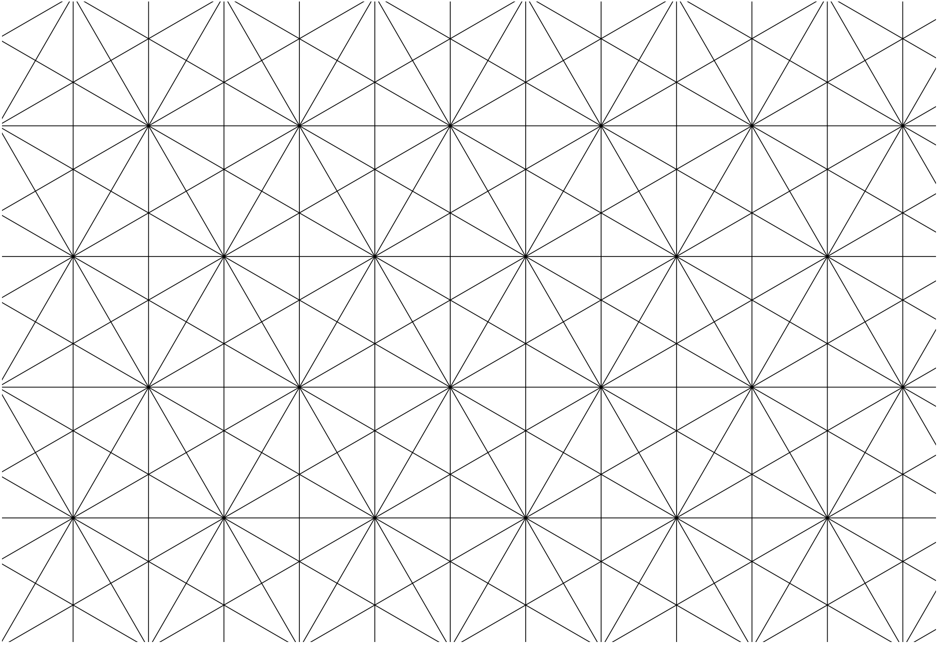
In finite type, Cambrian fans are $\mathbf{g}$-vector fans.

Cambrian fans can be defined even for infinite Coxeter groups. In every case, the Cambrian fan is a subfan of the cluster fan.

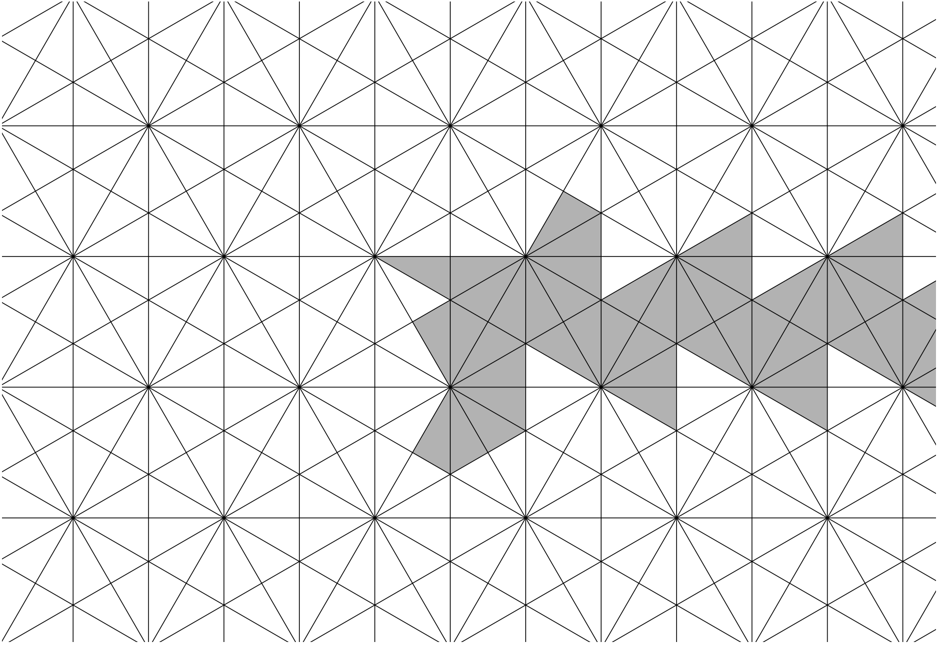
In the infinite case, they are not the whole cluster fan.

In the affine case, the entire cluster fan can be constructed from two overlapping pieces: The c-Cambrian fan and the antipodal opposite of the c^{-1}-Cambrian fan.

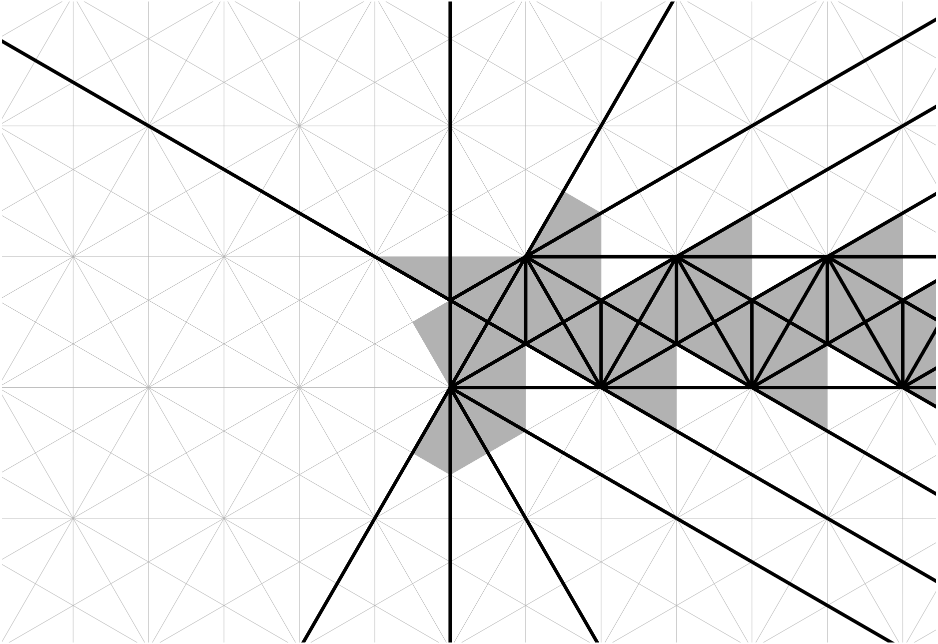
An affine Cambrian fan



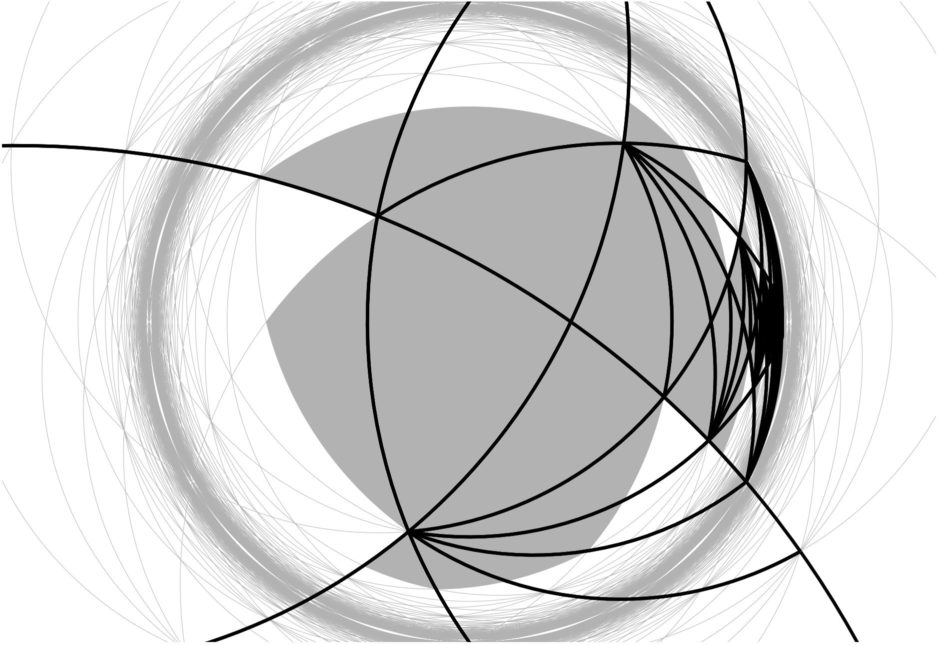
An affine Cambrian fan



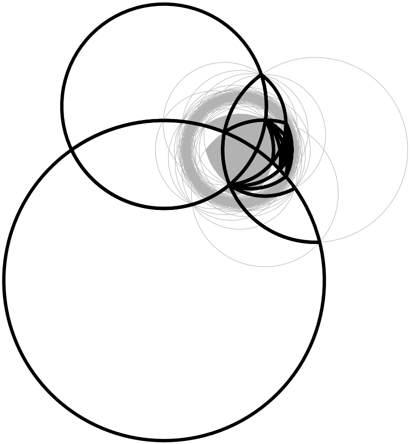
An affine Cambrian fan



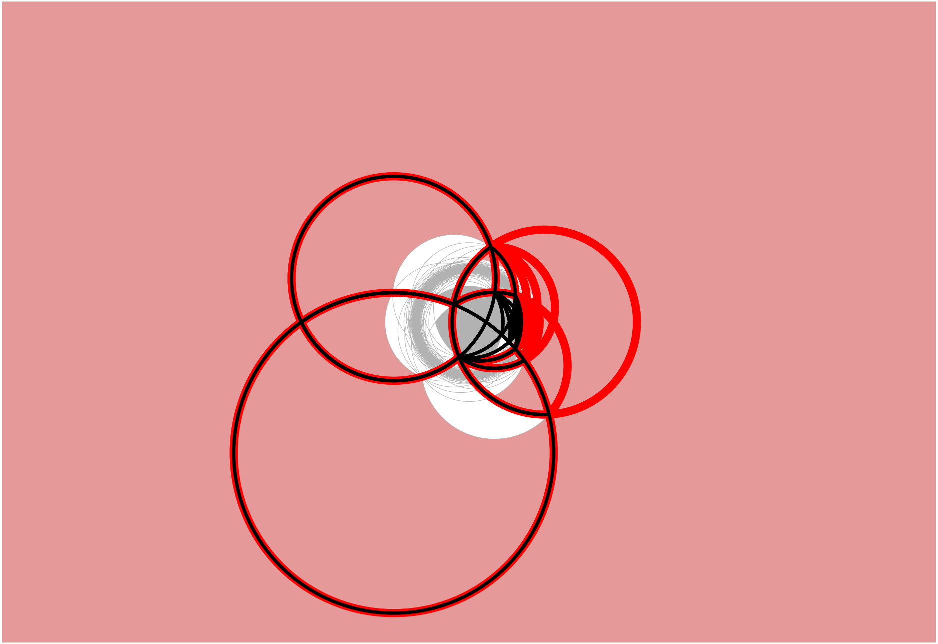
An affine Cambrian fan



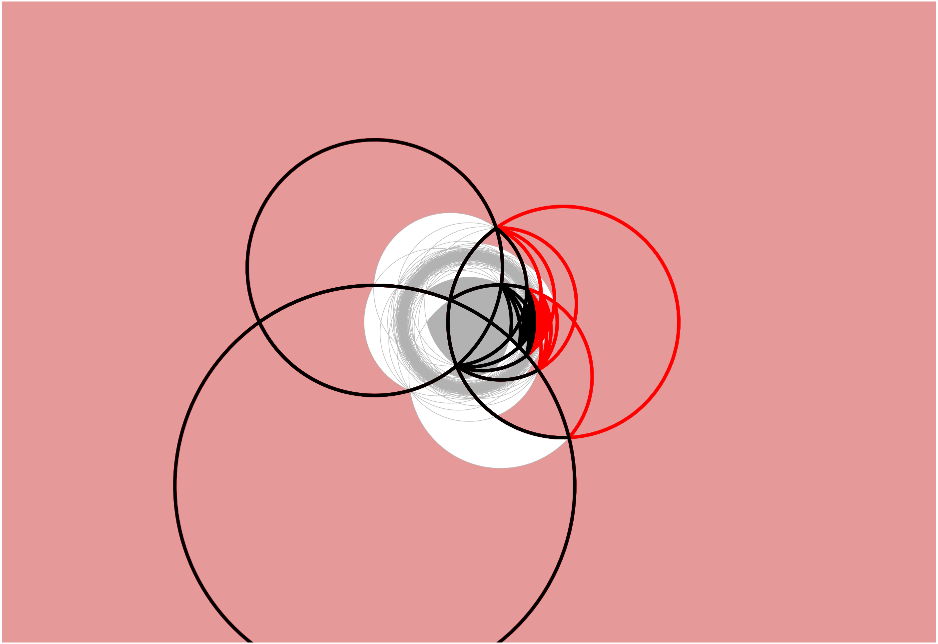
An affine Cambrian fan



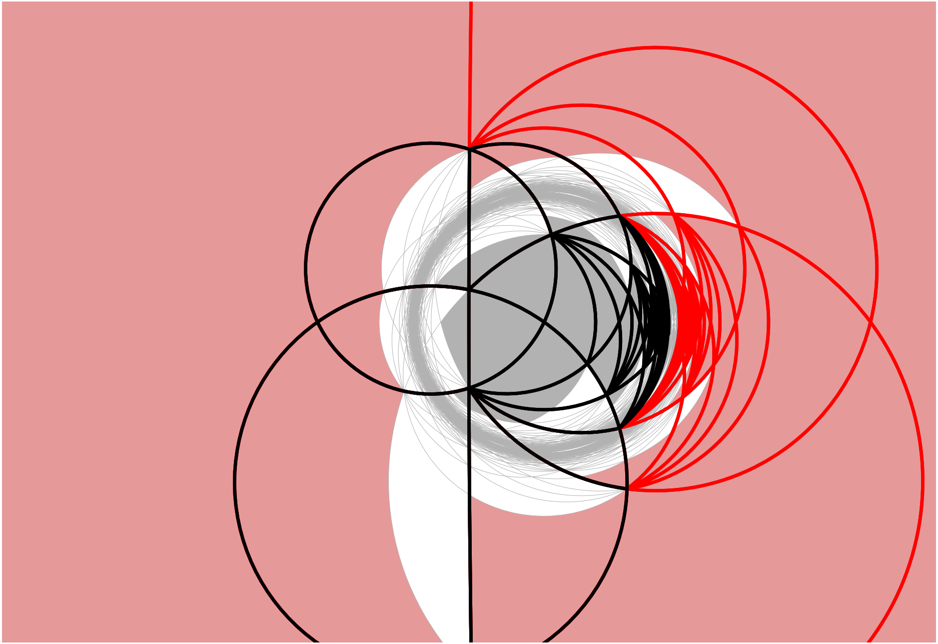
An affine doubled Cambrian fan



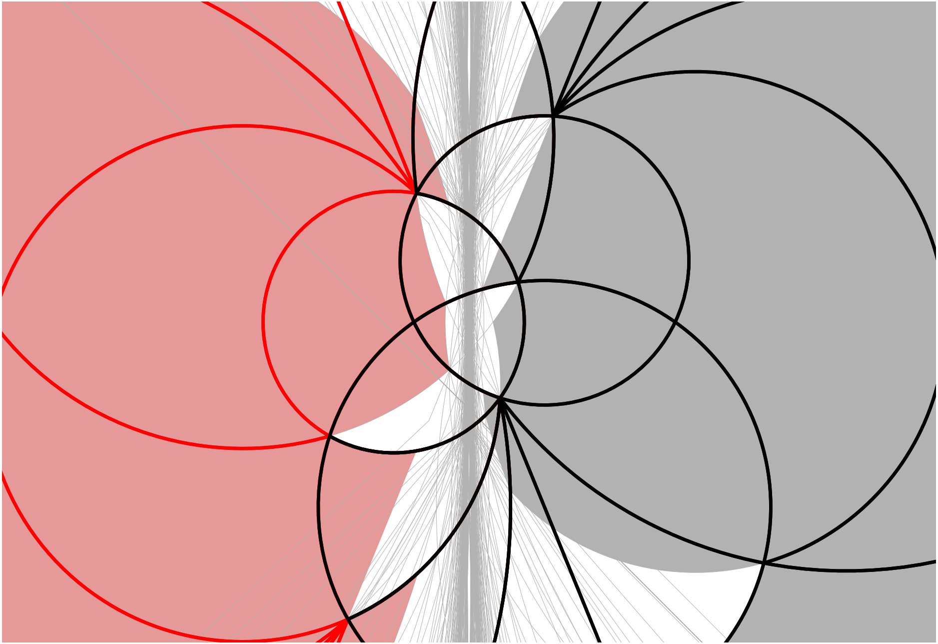
An affine doubled Cambrian fan



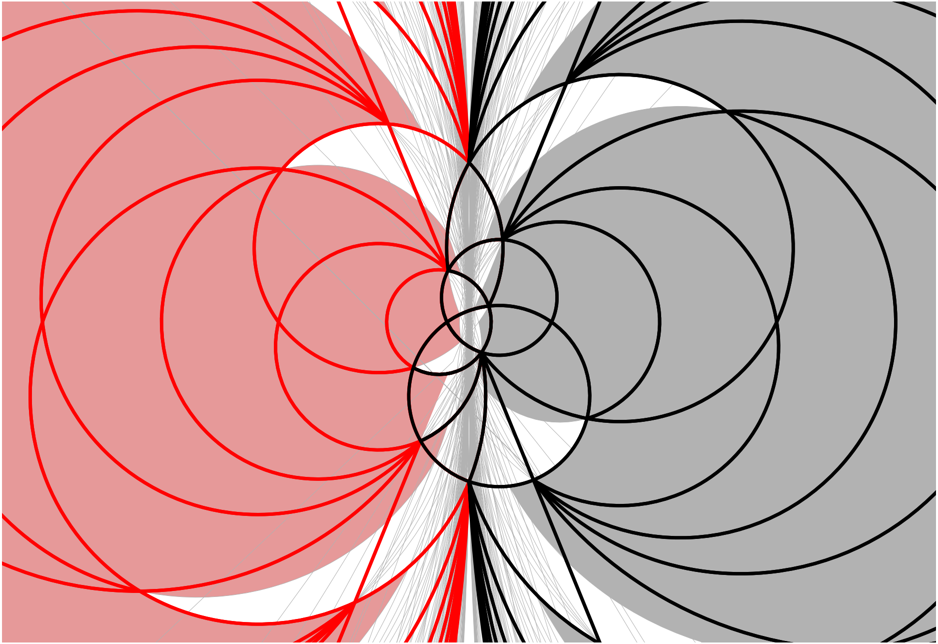
An affine doubled Cambrian fan



An affine doubled Cambrian fan



An affine doubled Cambrian fan



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