Lattice congruences and Hopf algebras

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Some maps

$S_n :=$ {permutations of $[n]$}.

$Y_n :=$ {planar binary trees with $n+1$ leaves}.

$2^{[n-1]} :=$ {subsets of $\{1, 2, \ldots, n-1\}$}.

$$\begin{tikzcd}
S_n \arrow[dd, "\delta"'] \arrow[dr, "\eta"] & \\
& Y_n \arrow[dl, "\delta'"] \\
2^{[n-1]} &
\end{tikzcd}$$

We have $\delta = \delta' \circ \eta$. More details soon.

Maps on polytopes

Permutohedron: vertices are permutations.

Associahedron: vertices are planar binary trees.

The maps η and δ on vertices extend to maps on faces.

$$
\begin{tikzcd}
\text{permutohedron} \arrow[dd, "\delta"'] \arrow[dr, "\eta"] & \\
& \text{associahedron} \arrow[dl] \\
\text{cube} &
\end{tikzcd}
$$

(Billera & Sturmfels, Tonks, Loday & Ronco)

Maps on Hopf Algebras

Hopf algebras of:

- permutations: MR (Malvenuto-Reutenauer)
- planar binary trees: PBT
- subsets: NCSym

$$\begin{tikzcd}
\mathrm{MR} & \\
 & \mathrm{PBT} \arrow[ul, "\eta*"'] \\
\mathrm{NCSym} \arrow[uu, "\delta*"] \arrow[ur] &
\end{tikzcd}$$

(Loday & Ronco.)

Maps on Posets

Loday & Ronco described the products in these three Hopf algebras in terms of partial orders (in fact lattices) on basis elements. The maps are order-preserving.

$$
\begin{tikzcd}
\text{Weak order} \arrow[dd, "\delta"'] \arrow[dr, "\eta"] & \\
& \text{Tamari lattice} \arrow[dl] \\
\text{Boolean algebra} &
\end{tikzcd}
$$

(also Björner and Wachs.)

Questions

What is it about these maps that gives them such nice geometric and algebraic properties?

Can we generalize?

That is, can we find other maps from permutations to objects which are just as well-behaved geometrically and (Hopf) algebraically?

Answer

$\boxed{\text{Both } \eta \text{ and } \delta \text{ are lattice homomorphisms.}}$

(For δ, Le Conte de Poly-Barbut, and for η, almost proven by Björner and Wachs.)

Note: "lattice homomorphisms" means more than "order-preserving." A lattice homomorphism must respect meets and joins.

The geometric and algebraic properties generalize to other lattice homomorphisms.

Geometry

Any lattice quotient (i.e. lattice-homomorphic image) of the weak order defines a complete fan of convex cones such that:

• The partial order is induced on maximal cones by a linear functional.

• Homotopy types of intervals are determined by the facial structure of the fan.

• Any linear extension of the partial order is a shelling of the fan.

It is not clear from this construction whether the fans are normal fans of polytopes.

This is all true, even replacing S_n by the weak order on any finite Coxeter groups. One case of the construction gives the normal fan of Fomin and Zelevinsky's generalized associahedron (in types A and B, and conjecturally in all types).

Hopf Algebras

For each n, let Z_n be a lattice quotient of weak order on S_n with some compatibility requirements, and let $\mathbb{K}[Z_\infty]$ be the vector space indexed by the elements of the Z_n's.

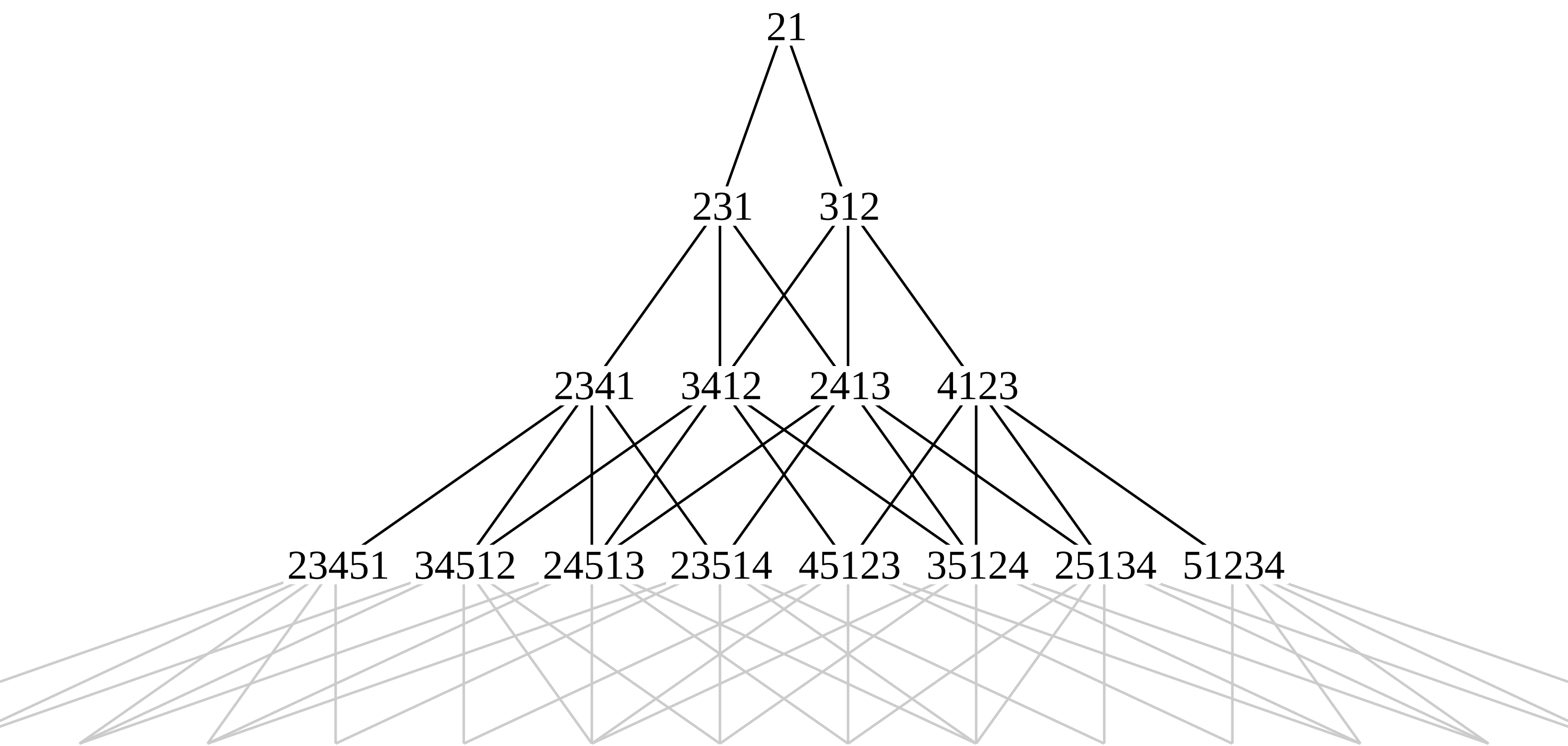
Then $\mathbb{K}[Z_\infty]$ has a graded Hopf-algebra structure and embeds as a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra.

The Hopf algebras constructed in this manner correspond to order-ideals in an infinite partial order $\mathcal{H}_\infty$.

Each of these Hopf algebras has a basis consisting of permutations satisfying a condition similar to pattern-avoidance.

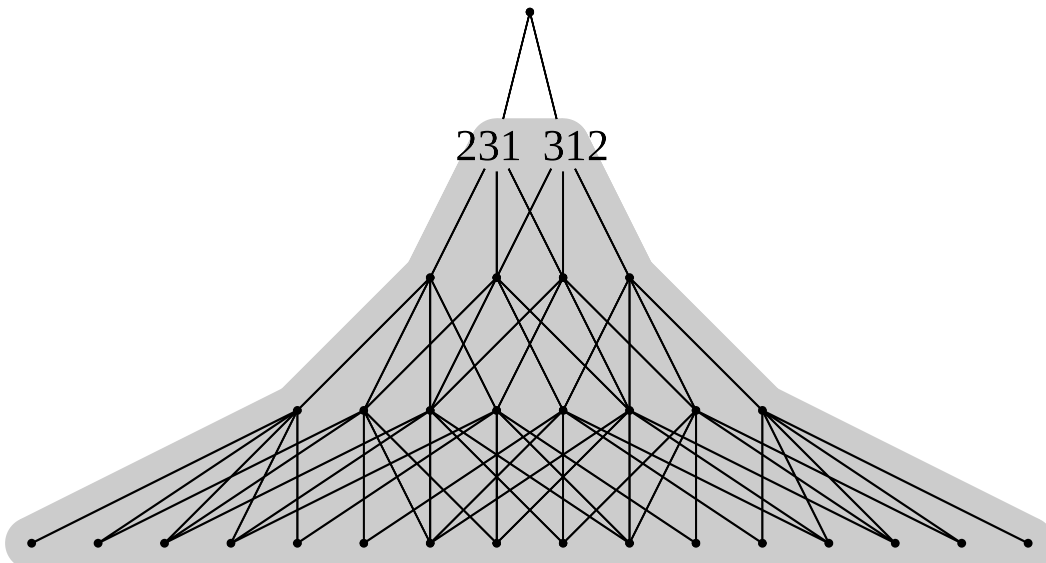
Order Ideals in $\mathcal{H}_\infty$

Here are the top 4 ranks of $\mathcal{H}_\infty$:

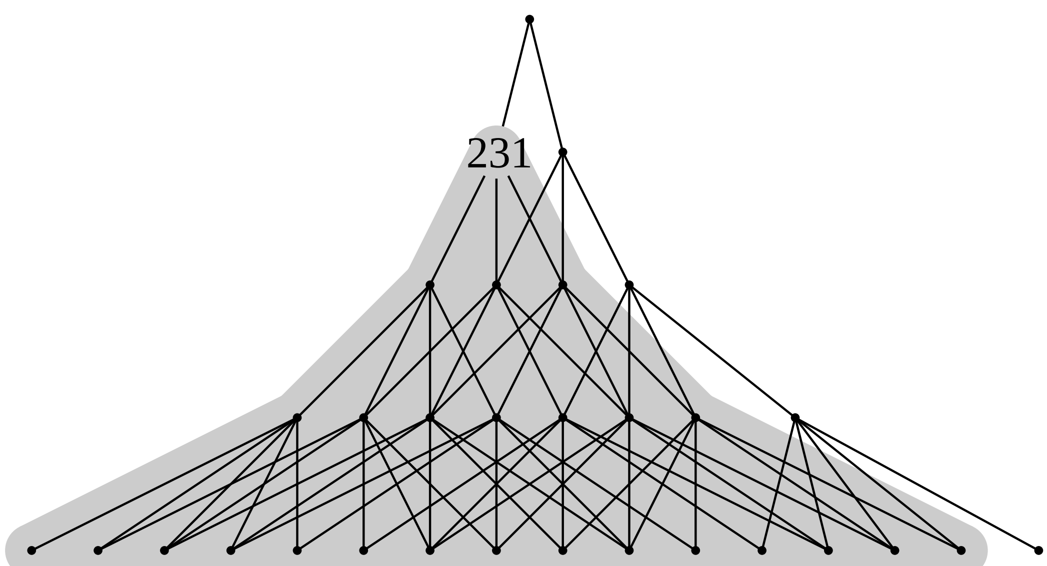


The ideal represents what is "modded out." Thus the empty ideal is the MR algebra. Taking the ideal to be the whole poset gives a Hopf algebra with one-dimensional graded pieces.

The ideal for NCSym:



The ideal for PBT:



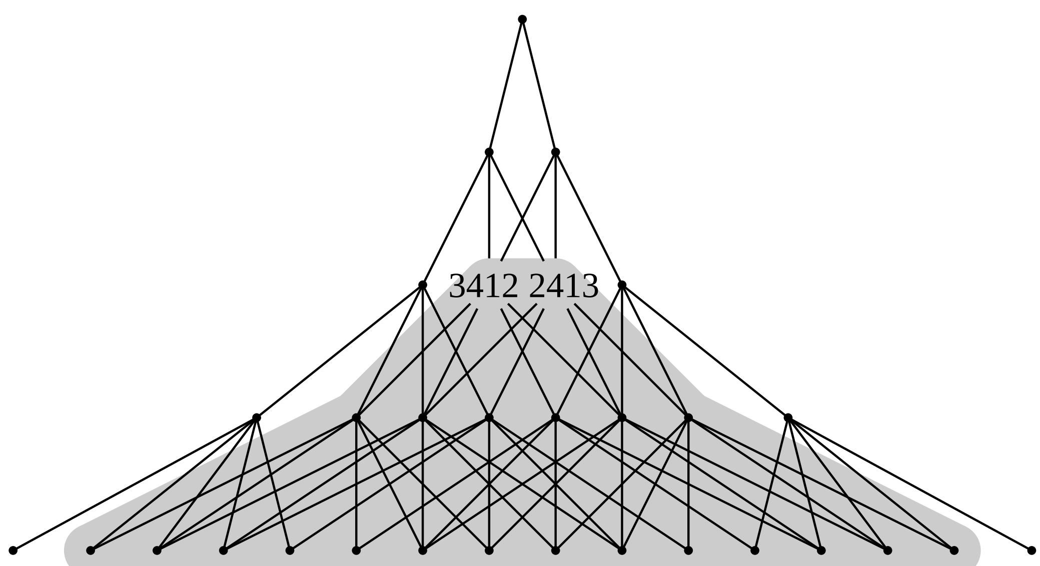
Baxter permutations

The Baxter permutations can be defined by a pattern avoidance criterion and are counted by

$$\frac{\sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

(Chung, Graham, Hoggatt, Kleiman)

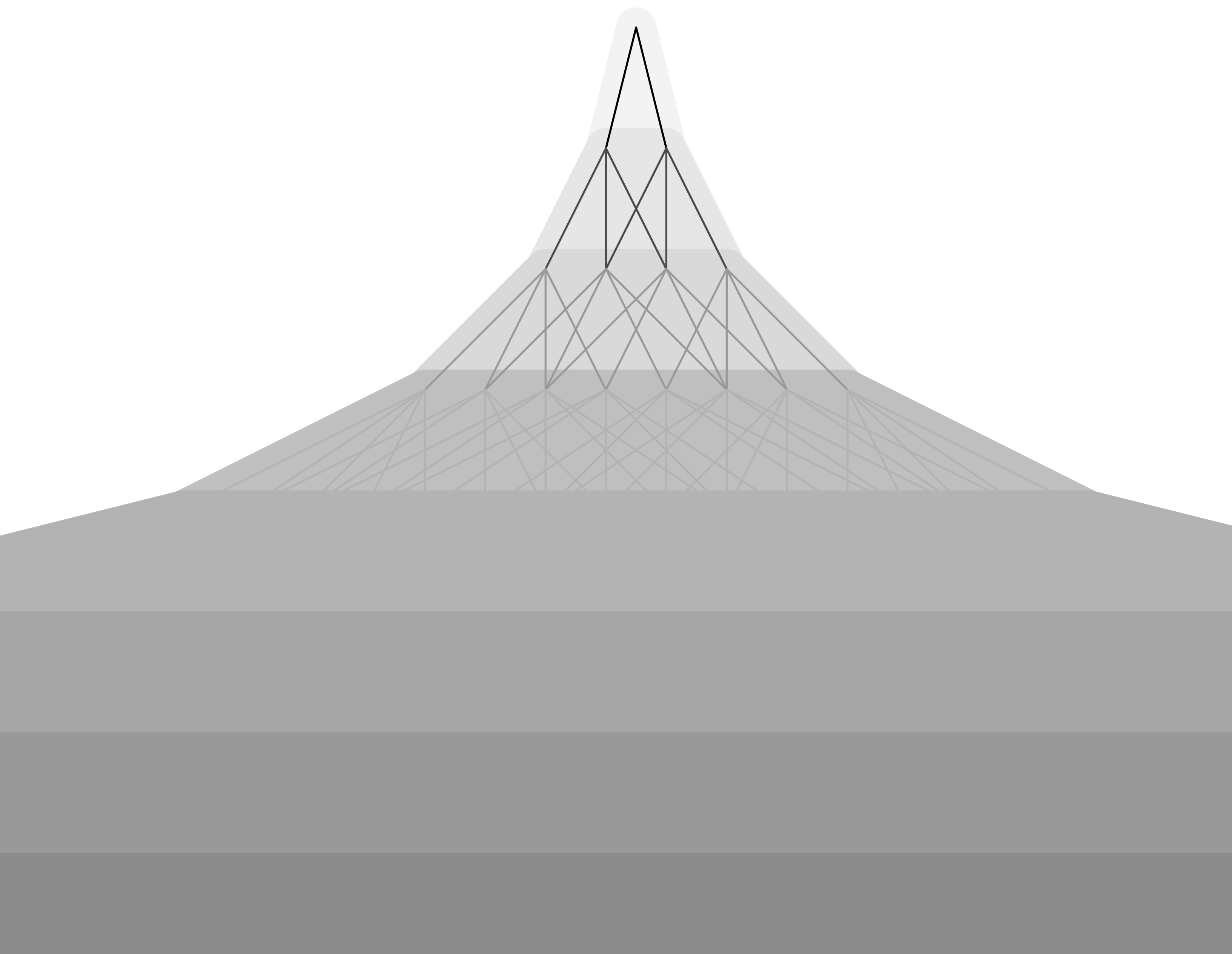
The ideal generated by $\{3412, 2413\}$ gives a Hopf algebra with a basis of *twisted Baxter permutations*, which appear to be equinumerous with Baxter permutations (up to $n = 15$).



A thought

Use ideals to build MR as the limit of an infinite sequence, starting with NCSym.

Can one use this limiting process to "lift" properties or constructions from NCSym to MR?



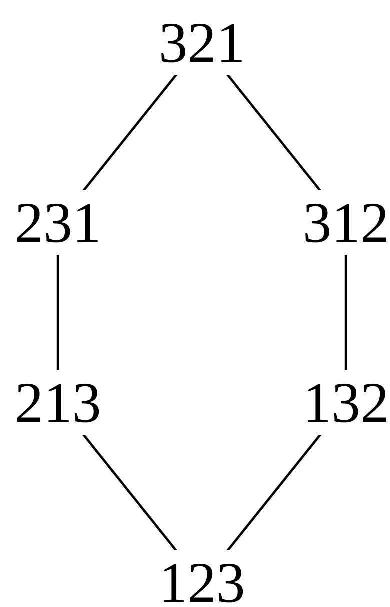
(Right) weak order

A partial order (in fact a lattice) on permutations in S_n.

Covers are transpositions of adjacent entries.

Going "up" means putting the entries out of numerical order.

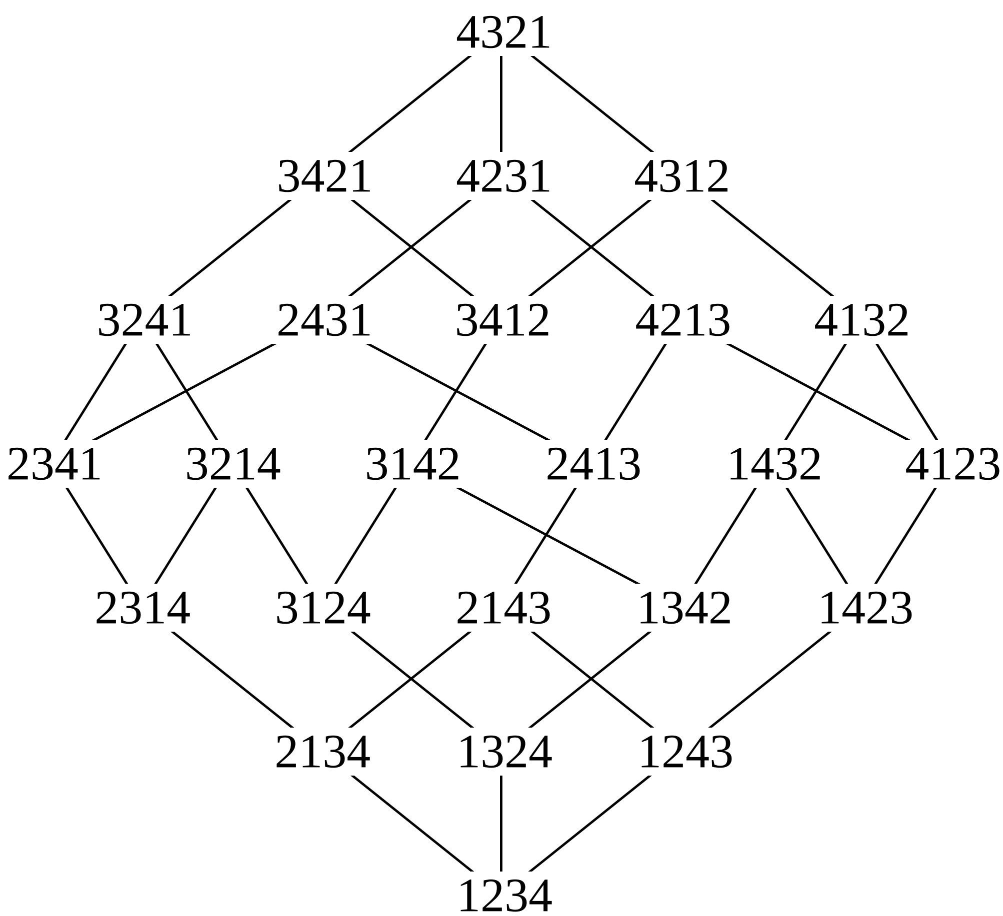
The weak order on S_3:



From now on "S_n" means "the weak order on S_n."

Example

The weak order on S_4:



The cover relations in the weak order are exactly the edges of the permutohedron.

Lattice congruences

- **Definition:** equivalence relations respecting meet and join. (Analogous to congruences on rings).

- They arise as the fibers of lattice homomorphisms.

- Conversely, given a lattice congruence, the quotient map (from elements to congruence classes) is a lattice homomorphism.

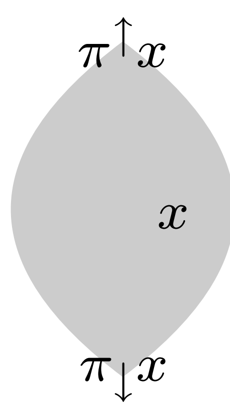
Lattice congruences (cont'd)

A equivalence relation Θ on a finite lattice L is a congruence if and only if:

(i) Every equivalence class is an interval.

(ii) The projection $\pi_\downarrow$ which takes each element to the bottom of its equivalence class, is order-preserving.

(iii) The analogous map $\pi^\uparrow$ is also order-preserving.



The quotient lattice L/Θ is isomorphic to the subposet of L consisting of bottoms of congruence classes of Θ.

Permutations to trees

$\eta(x)$ for $x = x_1 x_2 \cdots x_n \in S_n$:

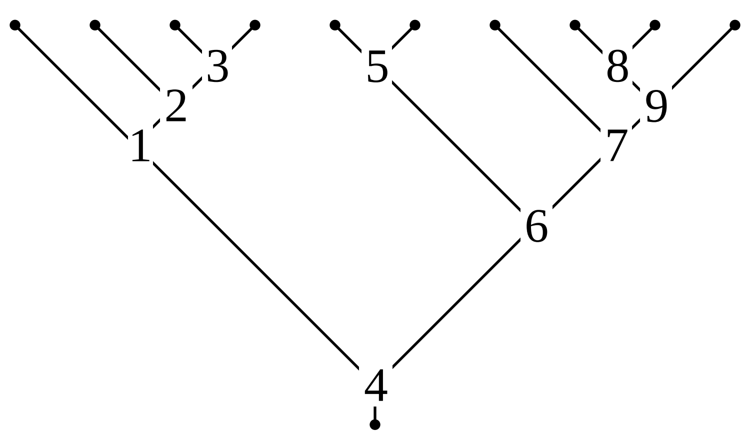
First label the lowest branch point by x_n.

Then construct the subsequence of x consisting of entries less than x_n, and the subsequence of entries greater than x_n.

For $x = 385297614$, these sequences are 321 and 85976.

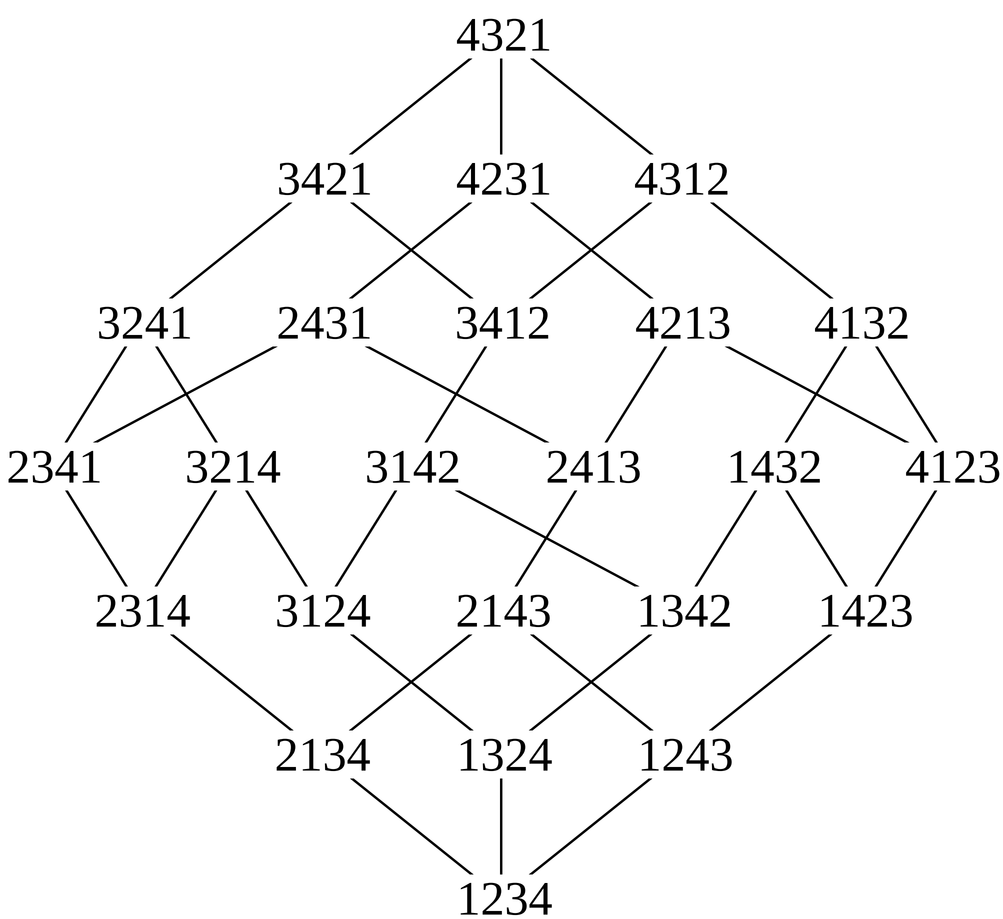
Use these to make the left and right subtrees recursively. An empty sequence makes a leaf.

For $x = 385297614$, $\eta(x)$ is



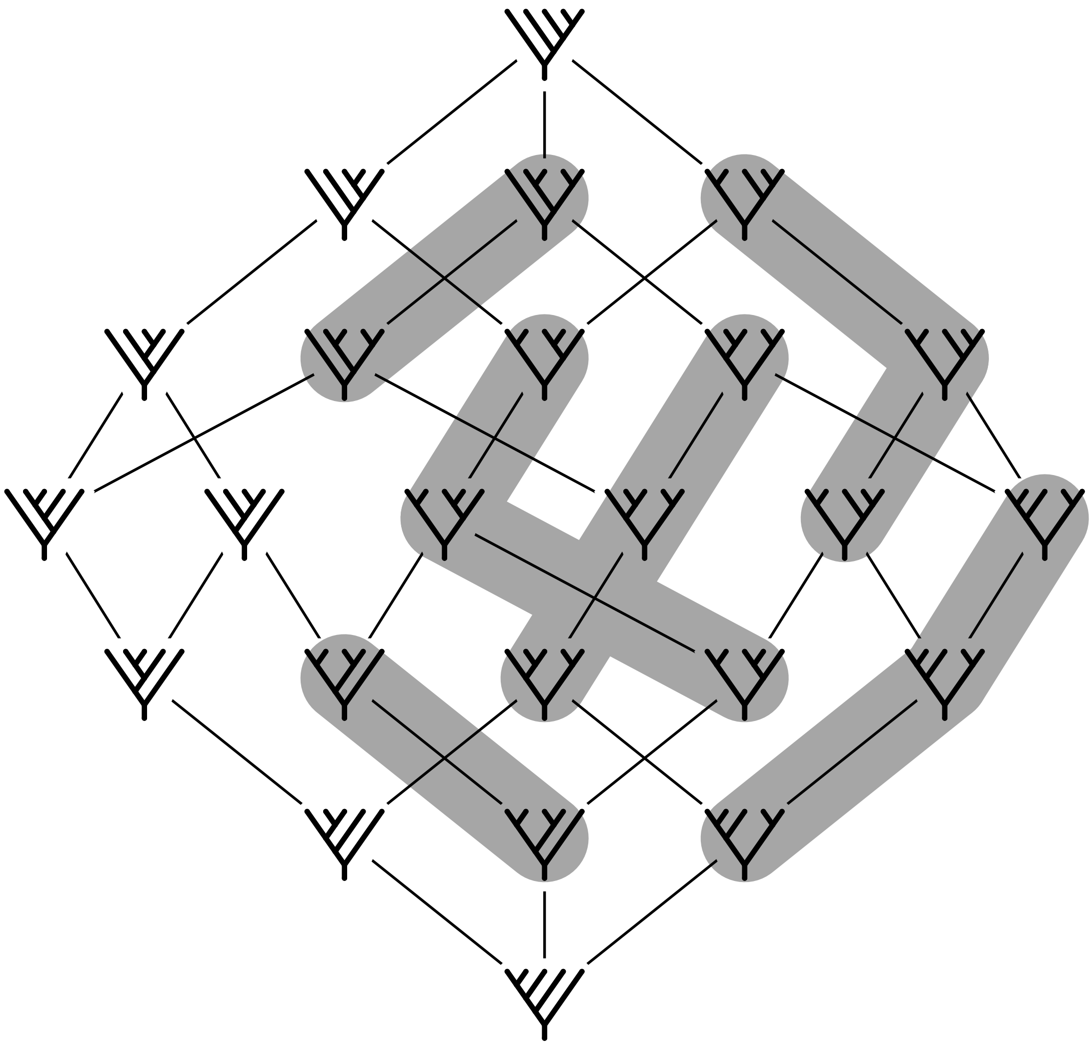
Example

The weak order on S_4:



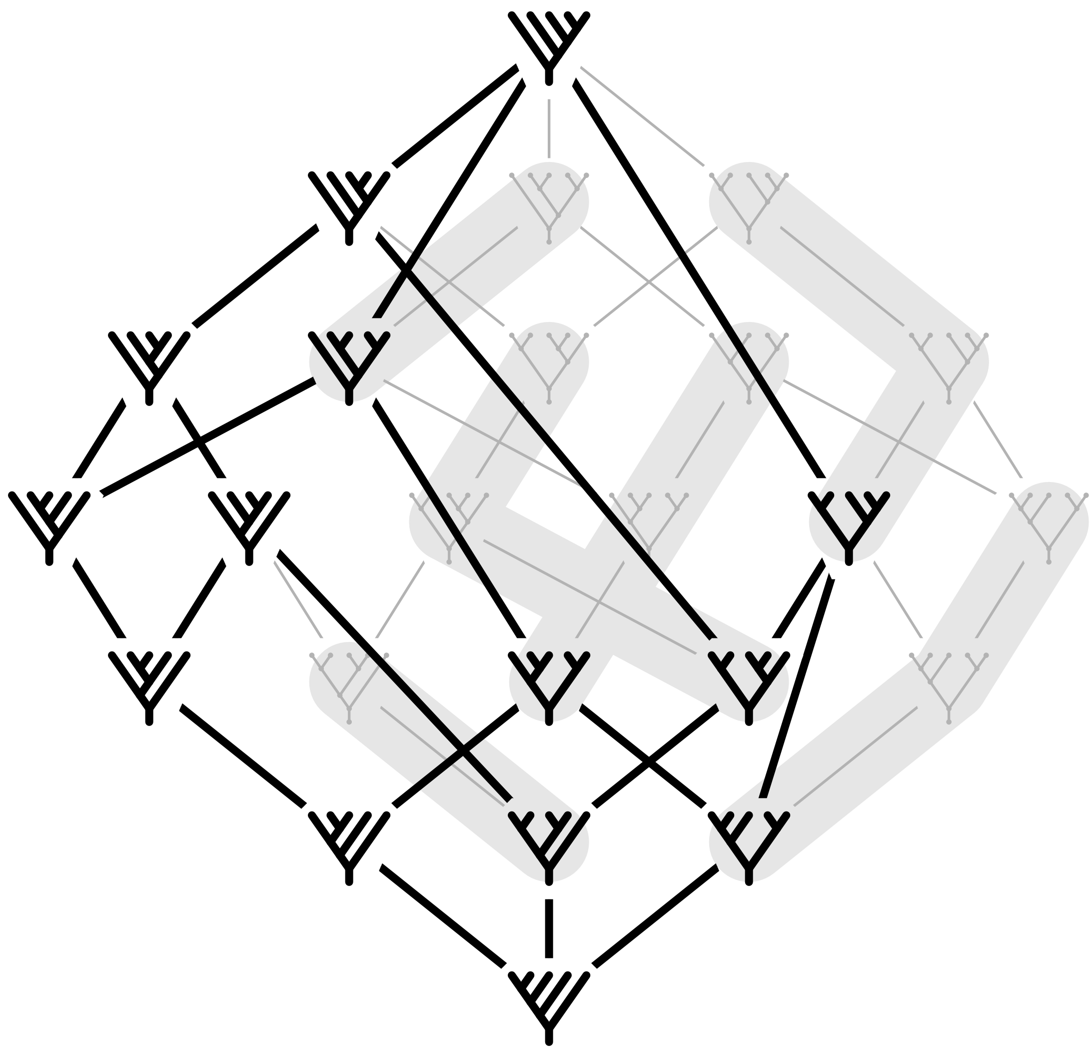
Example

The map η on S_4.



Example (continued)

The quotient of S_4 mod the fibers of η. (The Tamari lattice).



More on η

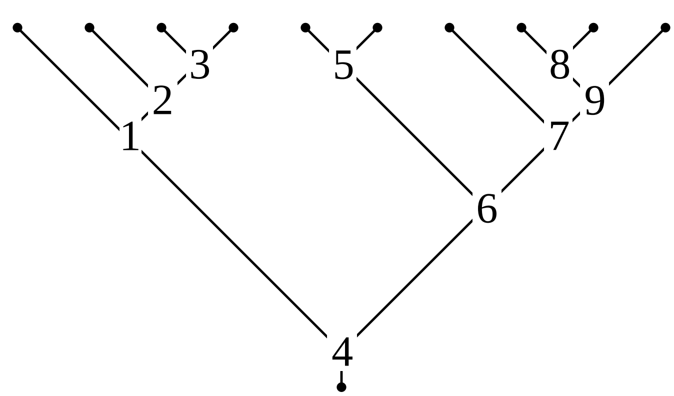
The fibers of η define a lattice congruence on weak order (proof: use inversion sets).

planar binary trees
$\longleftrightarrow$ Bottoms of congruence classes
$\longleftrightarrow$ 312-avoiding permutations.

The map η is a lattice homomorphism to the lattice induced on trees (the Tamari lattice).

Read $\pi_\downarrow(x)$ recursively from the (labeled) tree: Last element is the bottom branch point; the left subtree precedes the right subtree.

Thus for $x = 385297614$, we obtain $\pi_\downarrow(x) = 321589764$.



Permutations to sets

(Solomon) descent map.

A (left) descent of a permutation π is a pair $(i, i{+}1)$ which is inverted in π.

For example, $\delta(385297614)$ is

$$\{(1,2),(2,3),(4,5),(6,7),(7,8)\}.$$

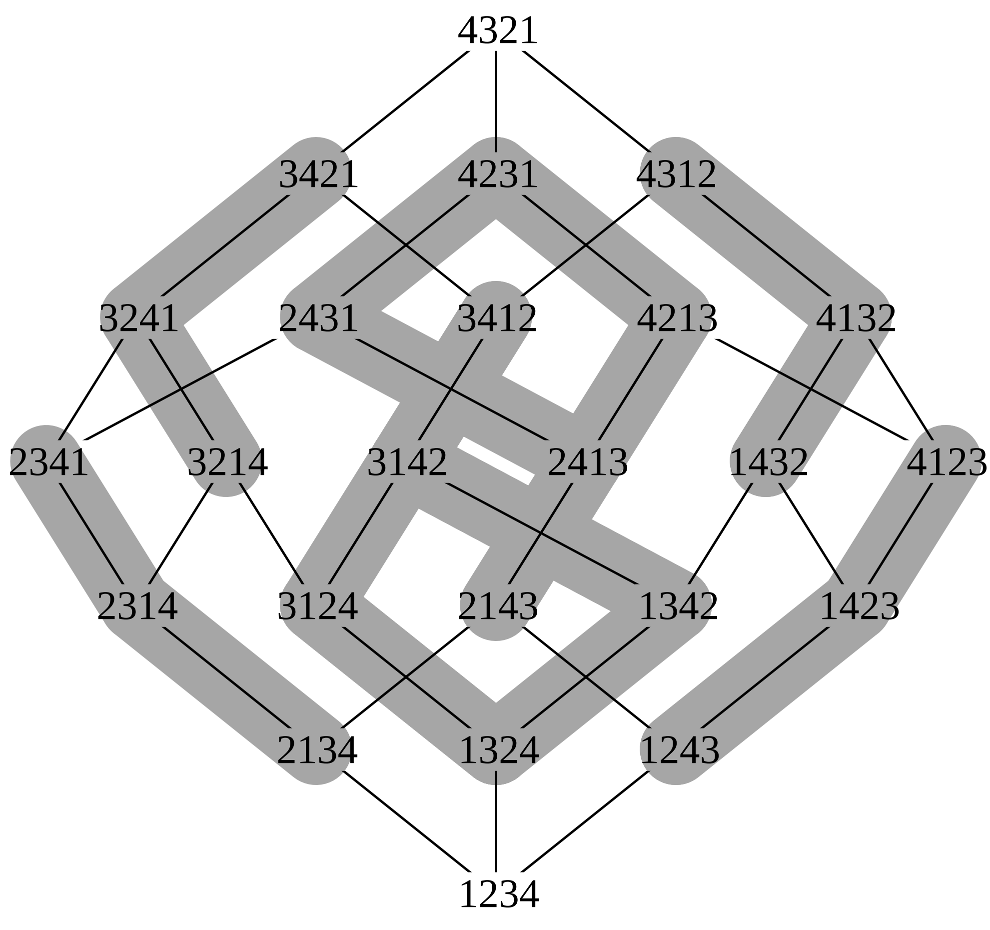
The fibers of this map define a lattice congruence. The quotient is the Boolean algebra.

The descent set of x consists of simple transpositions. Then $\pi_{\downarrow}(x)$ is the maximal element of the (parabolic) subgroup generated by those transpositions.

For example, $\pi_{\downarrow}(385297614) = 321548769$.

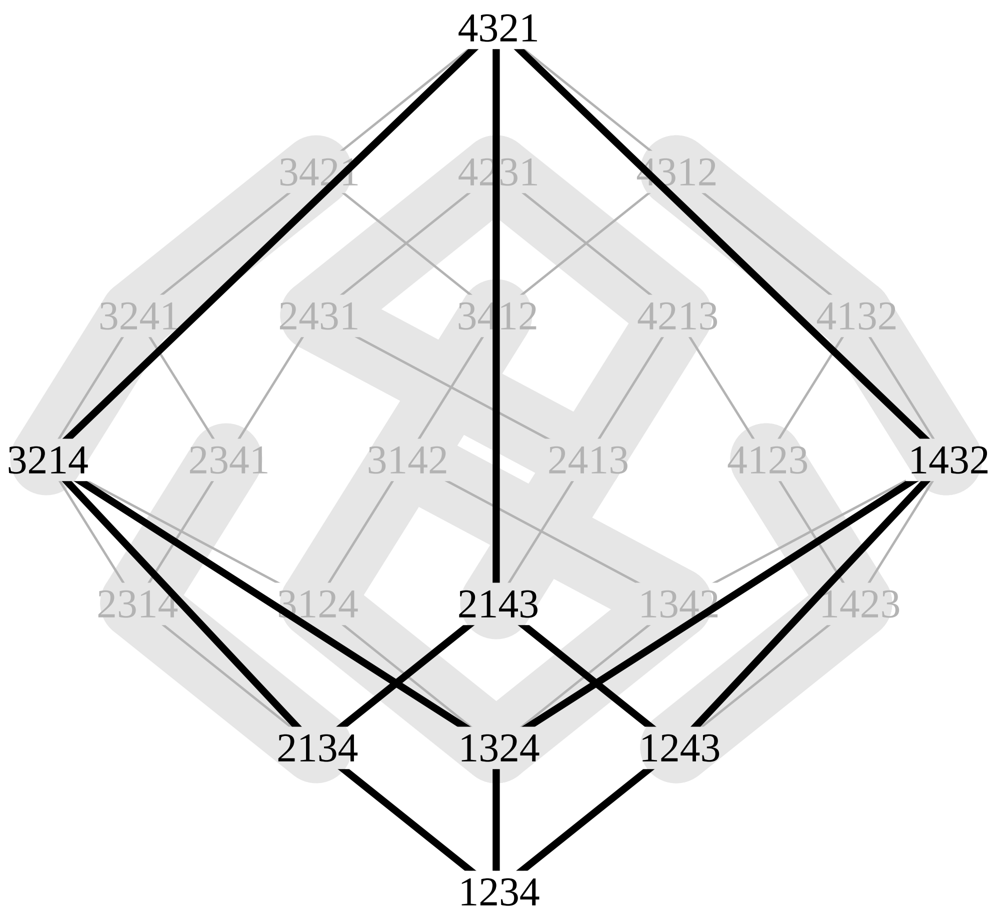
Example

The descent congruence on S_4:



Example

The quotient of S_4 mod the descent congruence:



The MR Hopf algebra

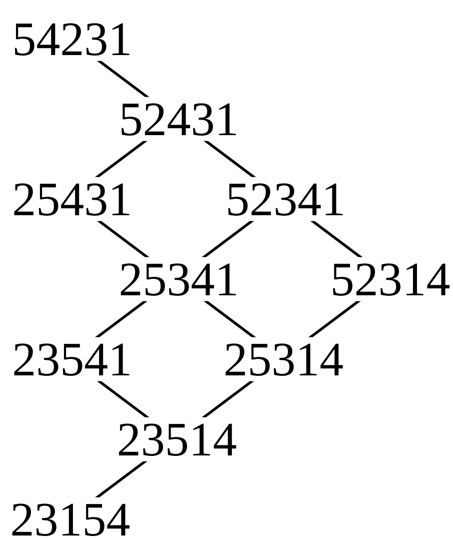
As a graded vector space:

$$\mathbb{K}\,[S_\infty] := \bigoplus_{n\geq 0} \mathbb{K}\,[S_n].$$

Product: "sum over all shifted shuffles."

Order-theoretic description of product (Loday & Ronco):

$312 \bullet_S 21$ is the sum of the elements of the interval $[31254, 54312]$ in weak order.



MR Hopf algebra (continued)

The coproduct:

$\Delta_S(x) = \sum_{p=0}^{n} \mathsf{st}(x_1, \ldots, x_p) \otimes \mathsf{st}(x_{p+1}, \ldots, x_n)$.

"st" is the *standard permutation* of a word.

Example:

$$\begin{aligned}\Delta_S(35124) = \emptyset \otimes 35124 + 1 \otimes 4123 \\ + 12 \otimes 123 + 231 \otimes 12 \\ + 3412 \otimes 1 + 35124 \otimes \emptyset.\end{aligned}$$

A family of quotients

- Fix a congruence Θ_n on each S_n. The notation $x \equiv y$ and $\pi_{\downarrow} x$ refers to the congruence Θ_n on the appropriate S_n.

- Define $Z_n := S_n / \Theta_n$ for each n. We will always think of Z_n as:

$$\left\{ x \in S_n : \pi_{\downarrow} x = x \right\} \subseteq S_n.$$

- Let $\mathbb{K}[Z_\infty]$ denote $\bigoplus_{n \geq 0} \mathbb{K}[Z_n]$.

Maps

- Define $c : \mathbb{K}[Z_\infty] \longrightarrow \mathbb{K}[S_\infty]$ by
$$x \in Z_n \longmapsto \sum_{x' \equiv x} x'.$$
("class" map).

- Define $r : \mathbb{K}[S_\infty] \longrightarrow \mathbb{K}[Z_\infty]$ by
$$x \in S_n \longmapsto \begin{cases} x & \text{if } \pi_{\downarrow} x = x, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$
("representative" map).

- Note that $r \circ c = \mathrm{id} : \mathbb{K}[Z_\infty] \longrightarrow \mathbb{K}[Z_\infty]$.

Product

For $u \in Z_p$ and $v \in Z_q$, define $u \bullet_Z v \in Z_{p+q}$ by

$$u \bullet_Z v = r(u \bullet_S v).$$

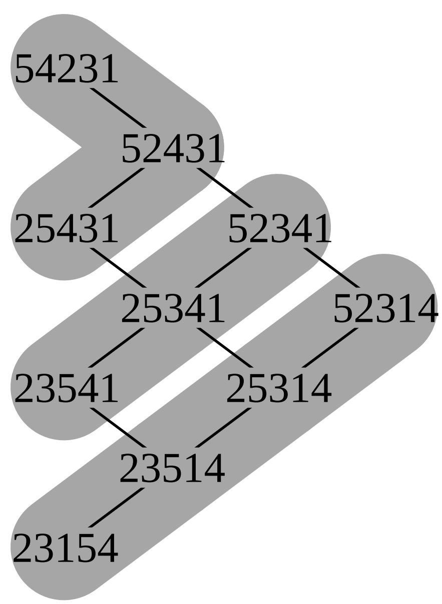
In words: sum over those shifted shuffles of u and v which are in Z_{p+q}.

Example

Let Θ_n be the "312-avoidance" congruence and let $Z_n := S_n/\Theta_n$ for each n. (So elements of Z_n correspond to planar binary trees).

We have $231 \in Z_3$ and $21 \in Z_2$.

$231 \bullet_Z 21 = 23154 + 23541 + 25431$.

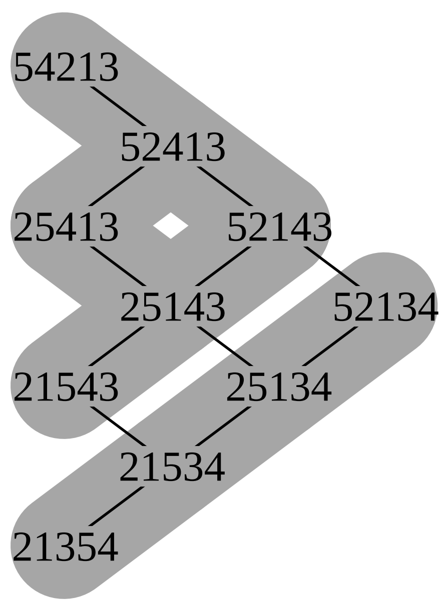


Example

Let Θ_n be the descents congruence and let $Z_n := S_n/\Theta_n$ for each n.

We have $213 \in Z_3$ and $21 \in Z_2$.

We have $213 \bullet_Z 21 = 21354 + 21543$.



In this algebra, the product of two basis elements is always a sum of two basis elements. (Why?)

Coproduct

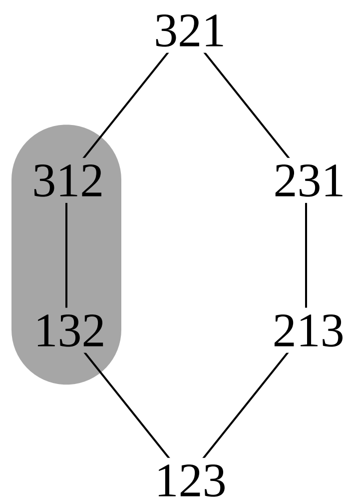
Define $\Delta_Z = (r \otimes r) \circ \Delta_S \circ c$.

In words: . . . well, it doesn't get much simpler in words, but this formula says no more than:

"We want $\mathbb{K}[Z_\infty]$ to embed as a sub-coalgebra of $\mathbb{K}[S_\infty]$."

Example

Again, let Z_n arise from the "312-avoidance" congruence.



We have $132 \in Z_3$ and

$$\begin{aligned}
\Delta_Z(132) &= (r \otimes r) \circ \Delta_S \circ c(132) \\
&= (r \otimes r) \circ \Delta_S(132 + 312) \\
&= (r \otimes r)\,[\emptyset \otimes 132 + 1 \otimes 21 + 12 \otimes 1 + 132 \otimes \emptyset \\
&\qquad\qquad \emptyset \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \emptyset] \\
&= \emptyset \otimes 132 + 1 \otimes (12 + 21) + \\
&\qquad\qquad (12 + 21) \otimes 1 + 132 \otimes \emptyset.
\end{aligned}$$

Sub Hopf algebras

Two compatibility conditions on $\{\Theta_n\}$, guarantee that the map c embeds $\mathbb{K}[Z_\infty]$ as a sub Hopf algebra of $\mathbb{K}[S_\infty]$.

To understand these conditions, we need to consider join-irreducible elements:

An element j of L is *join-irreducible* if it covers exactly one element j^*.

For the weak order on S_n, these are permutations which are increasing everywhere except at one position. For example, 23678145.

Specifying a join-irreducible is equivalent to specifying its *left set* (the elements before the decrease). Equivalently, specify the *right set*.

The left set of 23678145 is $\{2,3,6,7,8\}$ and the right set is $\{1,4,5\}$.

One last slide about
Lattice Congruences

Say a congruence *contracts* a join-irreducible j if $j \equiv j^*$.

A key theorem on congruences: A congruence is determined by the set of join-irreducibles it contracts.

Contracting one join-irreducible typically forces others to be contracted.

One step in finding the compatibility conditions was determining these forcing relationships for join-irreducibles of the weak order on S_n.

Two operations

Insertion: Given a join-irreducible j in S_n and $i \in [n+1]$, insert i into the right or left set. Every entry $\geq i$ is increased by 1.

Example: right-insertion of 3 into 23678145:

$$R_3(23678145) = 247891356.$$

Translation: A special case of insertion. The two *translates* of j are $L_1(j)$ and $R_{n+1}(j)$.

Example: the translates of 23678145 are:

$$L_1(23678145) = 134789256,$$

$$R_9(23678145) = 236781459.$$

1^{st} compatibility criterion

Translation:

$$L_1(j) \text{ is contracted by } \Theta_{n+1}$$

$$\Updownarrow$$

$$j \text{ is contracted by } \Theta_n$$

$$\Updownarrow$$

$$R_{n+1}(j) \text{ is contracted by } \Theta_{n+1}.$$

This is why $\mathcal{H}_\infty$ consists of "untranslated" join-irreducibles.

2^{nd} **compatibility criterion**

<u>Insertion:</u>

$$j \text{ is contracted by } \Theta_n$$

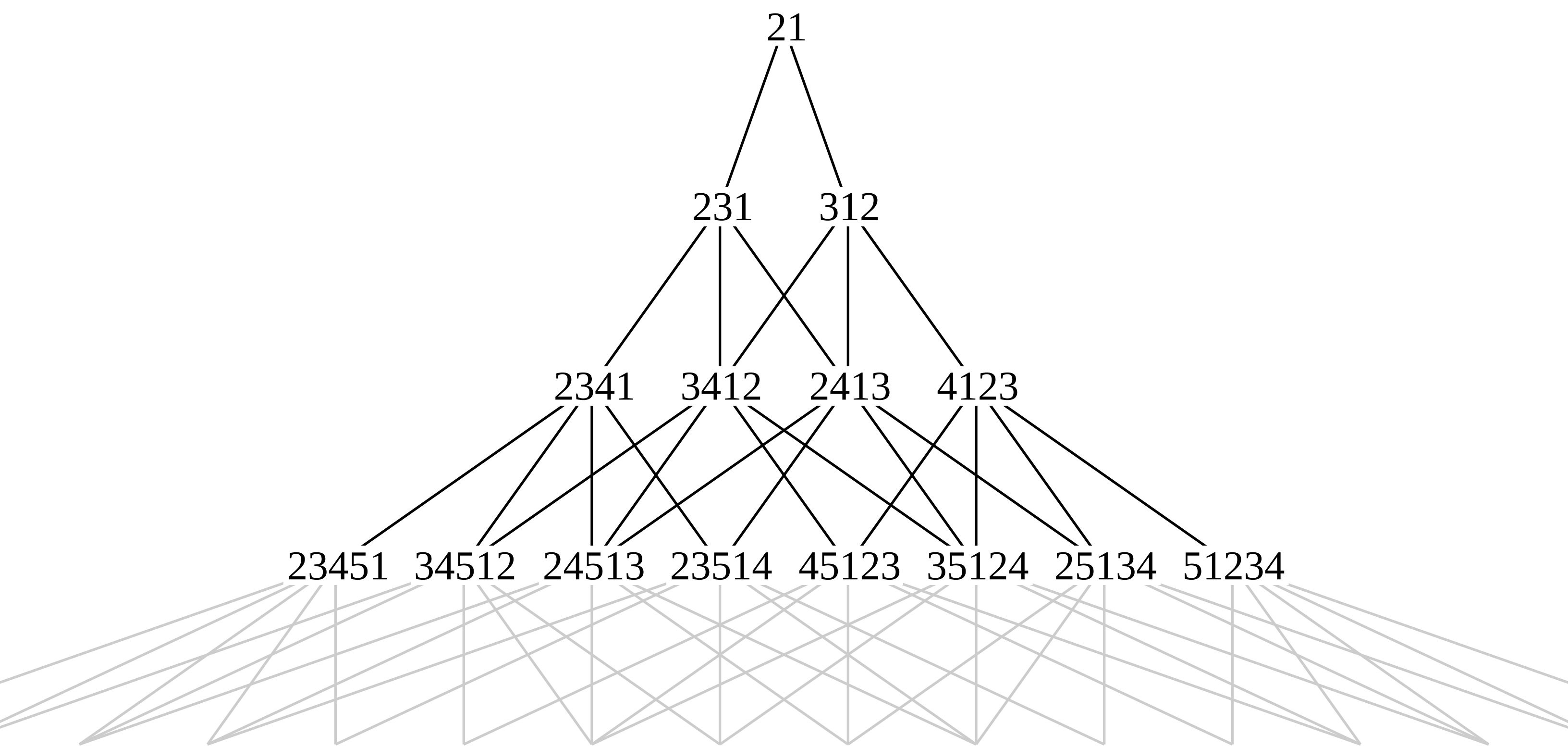
$$\Downarrow$$

$$L_i(j) \text{ and } R_i(j) \text{ are contracted by } \Theta_{n+1}$$
$$\text{for any } i \in [n+1].$$

The cover relations on $\mathcal{H}_\infty$ are these insertion relations, restricted to untranslated join-irreducibles.

$\mathcal{H}_\infty$

Thus choosing an ideal in $\mathcal{H}_\infty$ defines a family of congruences such that $\mathbb{K}[Z_\infty]$ embeds as a sub Hopf algebra.



Given an ideal, there is a pattern-avoidance condition which describes membership in Z_n.

Pattern avoidance

For example, choose the principal ideal generated by 256134.

A permutation x is **not** in Z_n if and only if it has entries $a < b < c < d < e < f$ such that:

(i) f occurs immediately before a,
(ii) b and e occur before f (in any order), and
(iii) c and d occur after a (in any order).

For example, with this choice of ideal, $\underline{7}1\underline{3}5\underline{92}8\underline{46}$ is not in Z_n. (Take $(a, b, c, d, e, f) = (2, 3, 4, 6, 7, 9)$).

For general ideals, impose a condition like this for each maximal element of the ideal.

Pattern avoidance (continued)

This pattern-avoidance condition is why the product formula was so simple.

A priori, the product should have been:

$$u \bullet_Z v = r(c(u) \bullet_S c(v)).$$

But it is easy to check from the pattern-avoidance condition that if $u' \notin Z_p$, then the shifted shuffle of u' with any permutation in S_q is not in Z_{p+q}.

So we may as well write:

$$u \bullet_Z v = r(u \bullet_S v).$$

Examples

This pattern-avoidance condition applies to the examples discussed previously:

PBT: 312-avoiding permutations.

NCSym: permutations x with $x_{i+1} \geq x_i - 1$.

$\mathbb{K}[S_{n,k}]$: permutations x with $x_{i+1} \geq x_i - k + 1$.

BAX: permutations avoiding 3412 with "4" and "1" adjacent and 2413 with "4" and "1" adjacent ("twisted Baxter permutations").

Preprints available on the arXiv.

Thank you

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