Para-exceptional sequences and the McCammond-Sulway lattice

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Maurice Auslander Centenary
23 April 2026

Joint with Eric Hanson (and earlier work joint with Laura Brestensky).

Section 1: Introduction

The noncrossing partition poset

(W, S): a Coxeter system with reflections T.

Coxeter element: $c =$ product of S in any order.

Absolute order $u \leq_T w$ is prefix order for T.

The noncrossing partition poset is $[1, c]_T$.

One motivation: combinatorial Garside structures for Artin groups.

The noncrossing partition poset in representation theory

Let Λ be a finite-dimensional hereditary algebra and let $X \in \operatorname{mod} \Lambda$. X is a brick if $\operatorname{End}(X)$ is a division ring and rigid if $\operatorname{Ext}^1(X, X) = 0$. Exceptional module: a rigid brick. Let $\mathcal{E} = \{\text{exceptional modules}\}$.

Let $\mathcal{B}$ be any set of bricks. A $\mathcal{B}$-brick sequence is $(X_1, \ldots, X_k)$ with $X_i \in \mathcal{B}$ for all $1 \leq i \leq k$ and $\operatorname{Hom}(X_i, X_j) = 0 = \operatorname{Ext}^1(X_i, X_j)$ for all $1 \leq j < i \leq k$. Maximal means: can't insert additional terms.

An $\mathcal{E}$-brick sequence is called an exceptional sequence.

Given a finite-dim'l hereditary algebra Λ, read off a Cartan matrix and a Coxeter element c of the associated Coxeter group W.

Key fact: Maximal chains in $[1, c]_T \longleftrightarrow$ maximal exceptional sequences in the category $\operatorname{mod} \Lambda$ (Igusa-Schiffler, Hubery-Krause).

The noncrossing partition poset in rep. thy. (continued)

A subcategory $\mathcal{W} \subseteq \operatorname{mod} \Lambda$ is called wide (or sometimes thick) if it is closed under extensions, kernels, and cokernels.

Gven modules $X_1, \ldots, X_k$, let $\mathsf{W}(X_1, \ldots, X_k)$ be the smallest wide subcategory containing $X_1, \ldots, X_k$.

A wide subcategory $\mathcal{W}$ is called exceptional if there exists an exceptional sequence $(X_1, \ldots, X_k)$ such that $\mathcal{W} = \mathsf{W}(X_1, \ldots, X_k)$.

Key fact: Elements of $[1, c]_T \longleftrightarrow$ exceptional subcategories of $\operatorname{mod} \Lambda$ (Hubery-Krause, Igusa-Schiffler, Ingalls-Thomas).

Finite/affine case (spherical/Euclidean Artin groups)

When W is finite:

• $[1, c]_T$ is a lattice (Bessis, Brady-Watt). This is the critical piece in making a Garside structure that leads to proofs of important properties of the associated Artin group.

• Classical finite types (A, B, D): $\exists$ planar models.

When W not finite:

• $[1, c]_T$ need not be a lattice.

When W is affine: (generated by affine reflections):

• McCammond and Sulway extend the affine Coxeter group W to a larger group, by "factoring translations", thus extending $[1, c]_T$ to a lattice (Garside structure for a supergroup of the Artin group).

Goals for this talk

Realize the McCammond-Sulway construction in terms of representation theory, specifically in terms of para-exceptional sequences and para-exceptional subcategories.

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Describe planar combinatorial models in classical affine types that model $[1, c]_T$ and the McCammond-Sulway lattice. (Thus in particular, these models describe exceptional sequences and exceptional subcategories, and para-exceptional sequences and para-exceptional subcategories, in classical affine type.)

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. . . but not in that order. We'll start with combinatorial models and use them to explain what we need from the representation theory.

Section 2: Planar models for noncrossing partitions

Models for NC partitions in finite type

Heuristic: Project a small orbit to Coxeter plane $\rightarrow$ planar model.

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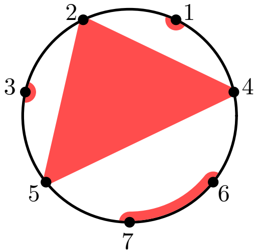
$[1, c]_T \leftrightarrow$ nc partitions of the $(n+1)$-cycle c.

Example: $W = S_7$, $c = s_3 s_5 s_2 s_1 s_6 s_4 = (1\ 4\ 6\ 7\ 5\ 3\ 2)$



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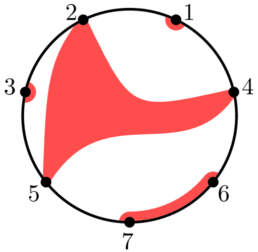

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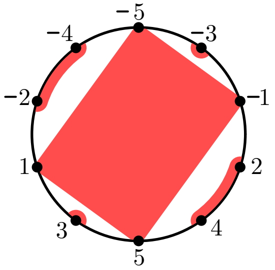
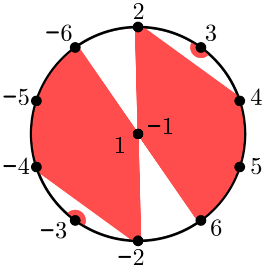
Type B (Reiner/Athanasiadis-Reiner):

centrally symmetric NC partitions of a disk



Type D (Athanasiadis-Reiner):

centrally symmetric NC partitions of a disk with a double point



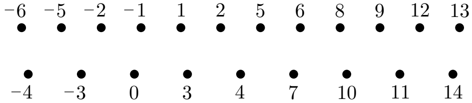
Models for NC partitions in affine type

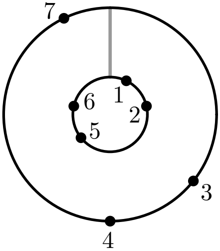
Run the heuristic from finite type: Project a "small" orbit to the "Coxeter plane", then mod out by some symmetries.

Classical cases: The orbit is indexed by $\mathbb{Z}$. The projected orbit is an infinite strip with translational symmetry and becomes an annulus.

Example: Affine type $\widetilde{A}_6$

$c = s_6 s_5 s_2 s_1 s_3 s_4 s_7$





The type $\widetilde{A}_{n-1}$ affine Coxeter group $\widetilde{S}_n$ is affine permutations of $\mathbb{Z}$:

- $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$
- $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$.

Larger group: Remove the condition $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$.

Noncrossing partitions of an annulus

Type $\widetilde{A}$ is as nice as one could hope:

- A natural combinatorial construction of the larger lattice.
- An easy combinatorial restriction obtains $[1, c]_T$.

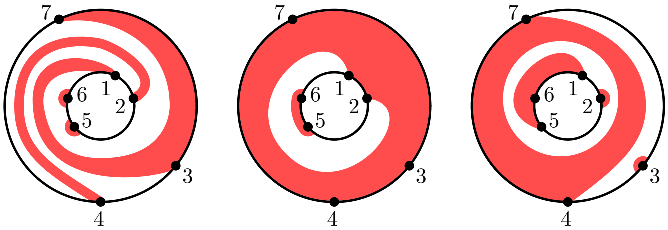
NC partitions: Set partitions of $\{1, \ldots, n\}$ + additional topology, specifically an embedding of each block in an annulus.

Each block of the set partition is embedded as

- a disk block,
- a dangling annular block, or
- a nondangling annular block.

A noncrossing partition is a collection of disjoint embedded blocks, satisfying some easy rules and considered up to isotopy.

Noncrossing partitions of an annulus (continued)



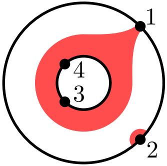
Noncrossing partition lattice $\widetilde{NC}_c^A$:

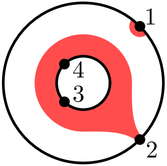
$\mathcal{P} \leq \mathcal{Q}$ iff every block of $\mathcal{P}$ is contained in a block of $\mathcal{Q}$.

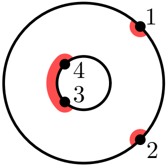
Isomorphism from $\widetilde{NC}_c^A$ to the McCammond-Sulway lattice:
Read boundaries of blocks as cycles, and there is a "date line".

Restrict to noncrossing partitions with no dangling annular blocks:
Get an isomorphism to $[1, c]_T$.

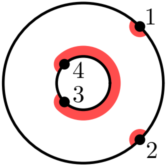
The lattice property needs dangling annular blocks





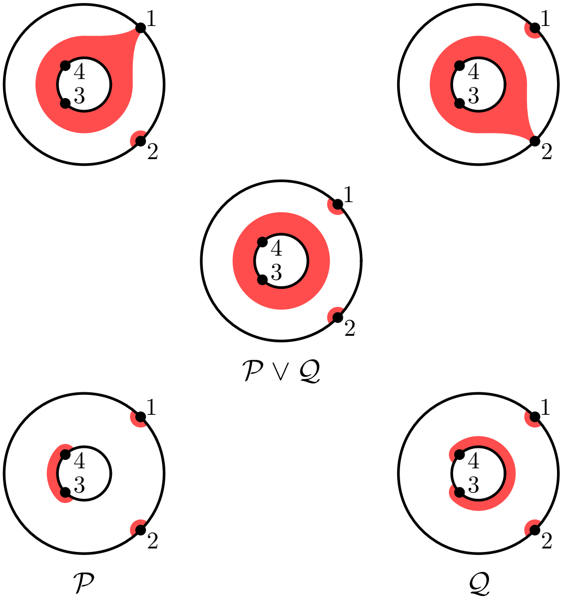


$\mathcal{P}$



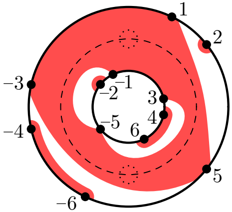
$\mathcal{Q}$

The lattice property needs dangling annular blocks

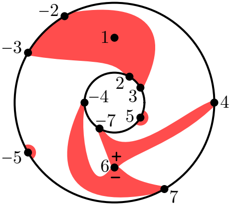


Affine types $\widetilde{C}$, $\widetilde{D}$, and $\widetilde{B}$

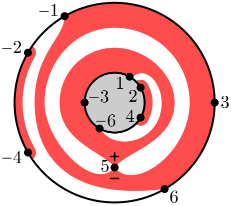
$\widetilde{C}$	$\widetilde{D}$	$\widetilde{B}$



symmetric NC partitions of an annulus



symmetric NC partitions of an annulus with two double points



symmetric NC partitions of an annulus with one double point

Summing up the affine models

Classical affine cases:
The planar model suggested by projecting an orbit to the Coxeter plane captures the noncrossing partition poset $[1, c]_T$.
(Exclude dangling annular blocks.)

Types $\widetilde{A}$, $\widetilde{B}$:
The planar model captures the larger lattice that McCammond and Sulway defined. (Allow dangling annular blocks.)

Type $\widetilde{C}$:
The planar model captures the noncrossing partition poset $[1, c]_T$, which is already a lattice. (Dangling annular blocks don't exist.)

Type $\widetilde{D}$:
To understand the McCammond-Sulway lattice, allow dangling annular blocks, but you also need a small amount of algebraic information.

Aside: Noncrossing partitions of a marked surface

By the way:

Given a general marked (unpunctured) surface, there is a lattice of noncrossing partitions. It is graded, with a nice rank function.

Given a symmetric marked surface with double points, there is a poset of symmetric noncrossing partitions. Again graded, with a nice rank function.

Section 3: Para-exceptional sequences and subcategories

Factored translations and the McCammond-Sulway lattice

McCammond and Sulway build the larger interval (in their larger group) by factoring all translations that appear in $[1, c]_T$.

Let F be the set of all factors that arise.

The set T of reflections and the set F together generate a group larger than the Coxeter group W.

The McCammond-Sulway lattice $[1, c]_{T \cup F}$ is the interval (analogous to the noncrossing partition poset) in this larger group.

Theorem (McCammond-Sulway). $[1, c]_{T \cup F}$ is a lattice.

Corollary (McCammond-Sulway). ...long-conjectured facts about the corresponding Euclidean Artin groups...

Factored translations and dangling annular blocks

Recall in type $\widetilde{A}$:

$[1, c]_T \cong$ poset of n.c. partitions, with no dangling annular blocks.

Translations in $[1, c]_T$ are

$$(\cdots\ i \quad i+n\ \cdots)(\cdots\ j \quad j-n\ \cdots) \text{ for } i \text{ outer and } j \text{ inner}$$

$$\longleftrightarrow$$

Noncrossing partitions with only one nontrivial block—
an annulus with one numbered point on each boundary component.

There is an obvious factorization into two infinite cycles.

one cycle $\longleftrightarrow$ the dangling annular block containing only i.

the other $\longleftrightarrow$ the dangling annular block containing only j.

(There are similar ideas in other classical types.)

A combinatorial abstraction for factored translations

As before, affine type, and c is a Coxeter element.

U^c: A hyperplane in the root space such that a root γ has finite c-orbit iff $\gamma \in U^c$. Roots in U^c form a "sub root system" Υ^c.

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Spoiler: Υ^c corresponds to roots in the non-homogeneous tubes.

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Reflections in $[1, c]_T$ are precisely the reflections t_γ for:
• every positive root $\gamma \notin \Upsilon^c$, and
• every positive root $\gamma \in \Upsilon^c$ with simple-root coordinates componentwise strictly less than δ.

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We characterize $[1, c]_{T \cup F}$ as a binary chain system, a collection of words in the formal symbols t_γ and f_β defined by certain binary compatibility rules, which dictate when one symbol may occur before another in the word. (Maximal chains $\longleftrightarrow$ words.)

The noncrossing partition poset in representation theory

We want to "find" the same binary chain system in rep. theory of a tame connected hereditary algebra Λ.

Leaving out the symbols f_β, we have $[1, c]_T$. Results of Igusa-Schiffler and Hubery-Krause amount to a rep.-theoretic description of the "letters" t_γ and the binary compatibility rules.

The map $X \mapsto \underline{\dim} X$ is a bijection from the set $\mathcal{E}$ of exceptional module to the set of roots γ such that $t_\gamma \in [1, c]_T$.

Furthermore, if $\underline{\dim} X = \gamma$ and $\underline{\dim} Y = \phi$, then t_γ and t_ϕ are compatible (in that order) if and only if (X, Y) is an exceptional sequence $(\mathrm{Hom}(Y, X) = 0 = \mathrm{Ext}^1(Y, X).)$

Therefore, $(X_1, \ldots, X_n) \mapsto t_{\underline{\dim} X_1} \cdots t_{\underline{\dim} X_n}$ is a bijection $\{$maximal exceptional sequences$\} \to \{$maximal chains in $[1, c]_T\}$.

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The set $\mathcal{E}$ of exceptional modules can be extended to $\mathrm{p}\mathcal{E}$ so that $\{$maximal $\mathrm{p}\mathcal{E}$-brick sequences$\} \to \{$maximal chains in $[1,c]_{T\cup F}\}$.

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The set $\mathcal{E}$ of exceptional modules can be extended to $\mathrm{p}\mathcal{E}$ so that $\{$para-exceptional sequences$\} \to \{$max. chains in McCam-Sulway$\}$.

The McC-Sulway lattice in representation theory (continued)

There is an exceptional indecomposable Λ-module (preprojective or preinjective) for each root $\gamma \notin \Upsilon^c$.

The remaining indecomposable modules (the regular modules) have dimension vectors in U^c. The quasi-simple modules are those that are simple in the subcategory of regular modules.

Each τ-orbit of quasi-simples generates a tube of regular modules (closing by extensions). There are infinitely many homogeneous tubes and there are 0, 1, 2, or 3 non-homogenous tubes.

Modules in the non-homogeneous tubes are exceptional if and only if their dimension vectors are componentwise strictly less than δ. These dimension vectors are positive roots $\gamma \in \Upsilon^c$.

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The non-homogenous, non-exceptional bricks are in bijection with the non-homogeneous quasi-simples. Each has dimension vector δ.

Para-exceptional modules and sequences

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Call a module para-exceptional if it is either exceptional or is a non-homogenous non-exceptional brick (equivalently, if it is a non-homogeneous brick).

Let $\mathrm{p}\mathcal{E}$ be the set of para-exceptional bricks. A para-exceptional sequence is a $\mathrm{p}\mathcal{E}$-brick sequence.

Define a map ω on $\mathrm{p}\mathcal{E}$:

$$\omega(X) = \begin{cases} t_{\underline{\dim} X} & \text{if } X \in \mathcal{E} \\ f_\gamma & \text{if } X = F_\gamma \in (\mathrm{p}\mathcal{E} \setminus \mathcal{E}). \end{cases}$$

Theorem (Hanson-R. 2025). Suppose Λ is a tame connected hereditary algebra with more than one non-homogeneous tube. The map $(X_1, \dots, X_k) \mapsto (\omega(X_1), \dots, \omega(X_k))$ is a bijection $\{$para-exceptional sequences$\} \to \{$maximal chains in $[1, c]_{T \cup F}\}$.

Para-exceptional subcategories

For Λ connected tame hereditary and $\mathcal{W}$ a wide subcategory:

$$\operatorname{aug}(\mathcal{W}) = \begin{cases} \mathcal{W} \vee_{\mathsf{wide}\,\Lambda} \mathcal{H} & \text{if } \mathcal{W} \text{ is representation-infinite} \\ \mathcal{W} & \text{if } \mathcal{W} \text{ is representation-finite.} \end{cases}$$

$$\overline{\mathcal{W}} = {}^{\perp}(\operatorname{aug}(\mathcal{W})^{\perp}).$$

A wide subcategory is para-exceptional if it is of the form $\overline{\mathsf{W}(X_1, \ldots, X_k)}$ for some para-exceptional sequence $(X_1, \ldots, X_k)$.

Write pe-wideΛ for the set of para-exceptional subcategories of Λ, partially ordered by containment.

Given $x \in [1, c]_{T \cup F}$ with reduced word $a_1 \cdots a_k$ for x in the alphabet $T \cup F$, let $\pi(x) = \overline{\mathsf{W}(\omega^{-1}(a_1), \ldots, \omega^{-1}(a_k))}$.

Theorem (Hanson-R. 2025). Suppose Λ is a tame connected hereditary algebra with more than one non-homogeneous tube. The map π is an isomorphism from the McCammond-Sulway lattice $[1, c]_{T \cup F}$ to the poset pe-wideΛ of para-exceptional subcategories.

Proof ideas

Proof of the bijection between maximal para-exceptional sequences and maximal chains in McCammond-Sulway: Explicitly determine binary compatibility for reflections and factored transformations, and use known facts (e.g. Ingalls-Paquette-Thomas) to show that length-2 para-exceptional sequences satisfy the same rules.

Proof that the poset of para-exceptional subcategories is isomorphic to the McCammond-Sulway lattice: Show that para-exceptional sequences $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_\ell)$ can both be completed to maximal para-exceptional sequences by adjoining the same postfix $(Z_1, \ldots, Z_m)$ iff $\overline{\mathrm{W}(X_1, \ldots, X_k)} = \overline{(Y_1, \ldots, Y_k)}$. (An ingredient: study regular para-exceptional sequences, in part using work of Igusa-Sen on such sequences in a tube category.)

Remarks

Remark: Digne and McCammond proved (in the language of root systems) that the noncrossing partition poset $[1, c]_T$ is a lattice if and only if there are < 2 non-homogeneous tubes. In that case, we don't factor translations. Also, in that case, para-exceptional subcategories are the same as exceptional subcategories, but the para-exceptional sequences are a proper superset of the exceptional sequences.

Remark: We give an entirely representation-theoretic proof that pe-wideΛ is a lattice. This constitutes a representation-theoretic proof that the McCammond-Sulway lattice is a lattice. Furthermore, we give a representation-theoretic proof that pe-wideΛ is a combinatorial Garside structure when Λ has at least two non-homogeneous tubes.

Remarks (continued)

Remark: A general phenomenon in "Coxeter-Catalan combinatorics/algebra" of affine type: It consists of a piece that with (combinatorial) translational symmetry and a piece that is isomorphic to some product of finite type-C objects. Here, that manifests itself as the fact that the para-exceptional subcategories generated by *regular* para-exceptional sequences are partially ordered as a product of type-C noncrossing partition posets. (Compare Igusa-Sen.)

Remark: In the classical affine types, there are detailed combinatorial models for para-exceptional sequences and para-exceptional subcategories (as we discussed)

Thanks for listening!

L. Brestensky and N. Reading, **Noncrossing partitions of an annulus.**

E. Hanson and N. Reading, **Para-exceptional sequences for tame hereditary algebras and McCammond-Sulway lattices.**

J. McCammond and R. Sulway, **Artin groups of Euclidean type.**

N. Reading. **Noncrossing partitions of a marked surface.**

N. Reading, **Symmetric noncrossing partitions of an annulus with double points.**