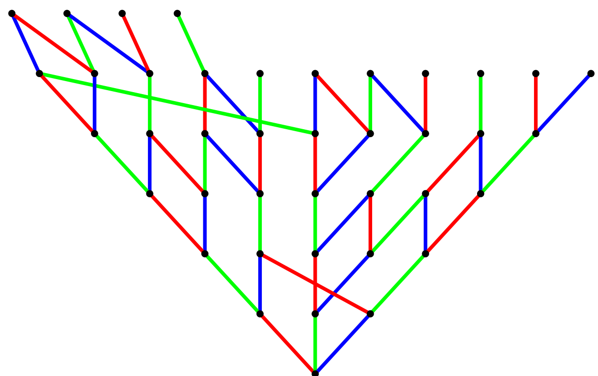
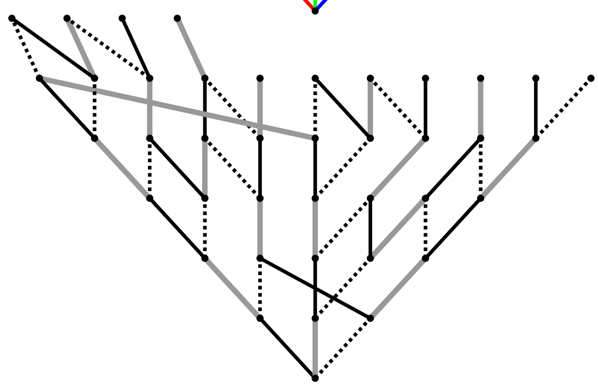
MA796
Solutions/comments for homework assignment 5

Problem 1

The best approach was to label the Hasse diagram of the weak order on H_3. (Hence the photocopier and colored pencils.) Each edge involving x and y is labeled by the simple generator $s \in S$ such that $xs = y$. Here are two versions of (part of) the labeling. One is in color: red edges are a, green edges are b and blue edges are c. The other is black and white: black edges are a, gray edges are b and dotted edges are c. In either case, $m(a, b) = 5$, $m(b, c) = 3$ and $m(a, c) = 2$.





Once you have that labeling, (or just enough of the labels) you can read off a reduced word for x, y and z and try all the possibilities. The answer was $y = zx$.

Problem 5

If $I \subseteq J$, then since $w_0(J)$ is the maximal element of W_J in weak order and since $W_I \subseteq W_J$, we have $w_0(I) \leq_R w_0(J)$.

If $w_0(I) \leq_R w_0(J)$, then since we proved that weak order is a weaker partial order than Bruhat order, we have $w_0(I) \leq w_0(J)$.

Recall that we proved that every element of w has a well-defined support (the set of letters occurring in a reduced word for it). I claim that the support of $w_0(K)$ is K. I can write a reduced word for $w_0(K)$ using only letters from K (as we showed in class), so the support is contained in K. On the other hand, every $s \in K$ has $\ell(w_0(K)s) < \ell(w_0(K))$, so there is a reduced word for $w_0(K)$ ending in s. In particular, s is in the support, and that proves the claim.

Now if $w_0(I) \leq w_0(J)$, there is a reduced subword for $w_0(I)$ occurring as a subword of some reduced word for $w_0(J)$. By the claim, every $s \in I$ is a letter of the reduced subword, and therefore is in the support of $w_0(J)$, which is J.

Problem A

The weak order and the Bruhat order coincide if and only if the Coxeter graph has no edges. Recall that this means that every pair of generators commutes, so the Coxeter graph has no edges if and only if the Coxeter group is abelian.

Proof. If the Coxeter group has some edge, say $m(a, b) \geq 3$, then ab is greater than b in the Bruhat order but not in the weak order. In particular, the two orders don't coincide. On the other hand, if the Coxeter graph has no edges, then the subword property and the prefix property coincide, so the weak order and the Bruhat order coincide. □

Problem B

The general philosophy behind this problem was explained in class. A good way to think about the problem was: Can I "walk" through the Coxeter graph forever, recording the nodes I visit, without causing any braid moves? Or, failing that, can I walk through while only creating some local braid moves that are easily seen not to lead to nil moves?

In Part 0, we noticed that we can walk around a cycle forever, and never make any braid moves. If we don't have any cycles, we have to find other ways to "turn around" in our walk, without creating braid moves. For example, if we have a label on some edge a—b, then when we arrive at a, we can go to b and then back to a and since $m(a, b) > 3$, we know that aba is not a braid move. Thus if we have two labels, we can move between one and the other forever, turning around each time. (This accomplishes Part 3.)

If we have a vertex of degree 3, say x——a, x——b, x——c are all edges (perhaps with labels) then if we arrive at x from a, we can go to b then c and back to a. This allows a braid move $bc = cb$ but no more. Thus we have succeeded in turning around. (Now we can do Part 2 and Part 4. Also, looked at in the right way, Part 1 can be argued just like Part 2.)

The really correct way to do this problem was to draw pictures. But I hope this verbal description explains what pictures you should draw. If not, feel free to talk to me about it.

Problem C

By Parts 1 and 2, either the graph is a path or it has a single vertex of degree 3.

We need to rule out all diagrams with *subgraphs* ruled out below.

If the graph is a path. Then it can have at most one label. If it has no labels then the group is finite, but we need to rule out a lot of possibilities for where and what the label can be. Suppose the graph consists of p unlabeled edges followed by an edge labeled $m > 3$ followed by q more unlabeled edges. We can take $p \leq q$.

> If $p = q = 0$ then any non-infinite m is ok, but you have to rule out infinity labels.
> If $p = 0$ and $m = 4$ then then any q is ok.
> If $p = 0$ and $m = 5$ then then we must have $q \leq 2$. So we have to rule out the graph with $b = 0$, $m = 5$ and $q = 3$. (If $q > 3$ then the graph has a subgraph with $q = 3$, and we already are ruling out that subgraph, i.e. that subgraph already defines an infinite group.)
> If $p = 0$ and $m > 5$ then q must be zero. So we have to rule out the graph with $b = 0$, $m > 5$ and $q = 1$.
> If $p = 1$ (so $q \geq 1$) and $m = 4$ then q must be 1. So we have to rule out the graph with $b = 1$, $m = 4$ and $q = 2$.
> If $p = 1$ (so $q \geq 1$) and $m > 4$ then the graph needs to be ruled out.
> If $p > 1$ then the graph has a subgraph that we have already ruled out.

Thus there are five paths that we need to rule out. (You should preferably draw these.)

If the graph has a degree-3 vertex v. Then it can have no label. Suppose there are a edges emanating from v in one direction, b in another, and c in the third. We can take $1 \leq a \leq b \leq c$. We need to rule out a lot of possibilities for what a, b and c can be.

> If $a \geq 2$ (so $c \geq b \geq 2$) the group is not in the classification. So we need to rule out $a = b = c = 2$.
> If $b \geq 3$ (so $c \geq 3$), the group is not in the classification. So we need to rule out $a = 1$, $b = c = 3$.
> If $a = 1$ and $b = 1$ then c can be anything. The only cases not yet dealt with have $a = 1$ and $b = 2$.
> We are left with the case where $a = 1$ and $b = 2$. In this case we need $c \leq 4$, so we need to rule out $a = 1$, $b = 2$, $c = 5$.

Thus there are three branched graphs that we need to rule out. (Again, you should draw these.)