MA796, Coxeter groups
Solutions/comments for homework assignment 4

Problem 5

As in the statement of the problem, $x = 316725948$. I got:

J	x^J	${}^Jx$
$\{1,7\}$	136725498	316725948
$\{1\}$	136725948	316725948
$\{7\}$	316725498	316725948
$\emptyset$	316725948	316725948
$\{1,2,3,4,5,6,7,8\}$	123456789	123456789

Confession: I gave all of you a "4" on this problem because it is routine. So I'll count on you to check your answers and approach me if you have concerns. If you had trouble with this problem, it was an issue of getting the definitions right.

Problem 11

Let $s_1 \cdots s_k$ be a reduced word for w in the alphabet J. In particular, $s_1 \cdots s_k$ is a word for w in the alphabet S. Proposition 2.4.1(ii) says that the length function ℓ (length in W) agrees with the length function ℓ_J (length in W_J), so the word $s_1 \cdots s_k$ is a reduced word in (W, S) as well as in (W_J, J). The subword property says that $u \leq w$ in Bruhat order on (W_J, J) if and only if there is a subword of $s_1 \cdots s_k$ that is a reduced word for u. The subword property also says that $u \leq w$ in Bruhat order on (W, S) if and only if there is a subword of $s_1 \cdots s_k$ that is a reduced word for u.

Problem 12

Consider for a moment a more general situation: Let G be a group, let I be an arbitrary subset of G and let H be a subgroup of G. Showing that I is a union of cosets of H is equivalent to showing that

$$\forall x \in I,\ xH \subseteq I. \tag{1}$$

Equivalently:

$$\forall x \in I,\ \forall y \in H,\ xy \in I. \tag{2}$$

Now suppose further that H is generated by the set J. Then a simple inductive argument shows that (2) is equivalent to

$$\forall x \in I,\ \forall s \in J,\ \big(xs \in I \text{ and } xs^{-1} \in I\big). \tag{3}$$

Finally, if all of the elements of J are involutions, (3) can be conveniently rephrased as:

$$\forall x \in G,\ \forall s \in J,\ (x \in I \iff xs \in I). \tag{4}$$

Now, turning to the assigned problem, we see that the task is to verify (4) with $G = W$ and $I = [u, w]$. Let $x \in W$. By the symmetry of the statement "$x \in I \iff xs \in I$", we can assume that $x < xs$. By lifting and the hypotheses on w, we have $x \leq w$ if and only if $xs \leq w$. By lifting and the hypotheses on u, we have $x \geq u$ if and only if $xs \geq u$. Thus $x \in I$ if and only if $xs \in I$.