

PROBLEM 1

For the purposes of this problem, we'll call the Bruhat graph and Bruhat order defined in class and in the book the "right" Bruhat graph and the "right" Bruhat order. That is, the right Bruhat graph is the digraph with arrows  $u \rightarrow_R w$  whenever  $\ell(u) < \ell(w)$  and there exists  $t \in T$  such that  $w = ut$ . The right Bruhat order is the transitive closure of the right Bruhat graph. Similarly, define a "left" Bruhat graph and the "left" Bruhat order. The left Bruhat graph is the digraph with arrows  $u \rightarrow_L w$  whenever  $\ell(u) < \ell(w)$  and there exists  $t' \in T$  such that  $w = t'u$ . The left Bruhat order is the transitive closure of the left Bruhat graph.

**Claim 1.** For all  $u, w \in W$ , we have  $u \rightarrow_R w$  if and only if  $u \rightarrow_L w$ .

*Proof.* This amounts to showing that there exists  $t \in T$  such that  $w = ut$  if and only if there exists  $t' \in T$  such that  $w = t'u$ . If  $w = ut$  then set  $t' = utu^{-1}$ , so that  $t'u = utu^{-1}u = ut = w$ . If  $t \in T$  (i.e.  $t$  is conjugate to an element of  $S$ ) then  $t' = utu^{-1}$  is conjugate to the same element of  $S$ , so  $t' \in T$ . Conversely, if  $w = t'u$  then set  $t = u^{-1}t'u$ , so that  $ut = uu^{-1}t'u = t'u = w$ . If  $t' \in T$  then  $t = u^{-1}t'u$  is also in  $T$ .  $\square$

By the claim, the right and left Bruhat graphs are identical as directed graphs. Thus the transitive closures (left and right Bruhat order) of the two digraphs are identical.

PROBLEM 3A

(We did this in class, so it was not assigned.)

In general, let  $G$  be a group and let  $G_1$  and  $G_2$  be subgroups such that  $G_1 \cap G_2 = \{e\}$ ,  $G = \langle G_1, G_2 \rangle$  and every element of  $G_1$  commutes with every element of  $G_2$ . First, there is a **unique** way to write  $g \in G$  as  $g_1g_2$  for  $g_1 \in G_1$  and  $g_2 \in G_2$ . Indeed, since elements of  $G_1$  commute with elements of  $G_2$ , any element of  $\langle G_1, G_2 \rangle$  can be written as  $g_1g_2$  for  $g_1 \in G_1$  and  $g_2 \in G_2$ . If  $g_1g_2 = g'_1g'_2$  for  $g_1, g'_1 \in G_1$  and  $g_2, g'_2 \in G_2$  then  $e = g'_1g'_2(g_2)^{-1}(g_1)^{-1} = g'_1(g_1)^{-1}g'_2(g_2)^{-1}$ , so  $[g'_1(g_1)^{-1}]^{-1} = g'_2(g_2)^{-1}$ . Since  $G_1 \cap G_2 = \{e\}$ , each of these elements is  $e$ , so  $g_1 = g'_1$  and  $g_2 = g'_2$ .

Let  $\eta : G \rightarrow G_1 \times G_2$  given by  $\eta(g) = (g_1, g_2)$  as above. This map is well-defined as argued above and has inverse  $(g_1, g_2) \mapsto g_1g_2$ . Furthermore  $\eta(g) \cdot \eta(h) = (g_1, g_2) \cdot (h_1, h_2) = (g_1h_1, g_2h_2)$ . Since  $g_1h_1g_2h_2 = g_1g_2h_1h_2 = gh$ , uniqueness of the factorization implies that  $\eta(gh) = (g_1h_1, g_2h_2)$ . Thus  $\eta$  is a group isomorphism from  $G$  to  $G_1 \times G_2$ .

Now, for the actual problem: It is enough to prove that if  $S$  is the disjoint union  $I \cup J$  and the Coxeter graph has no edges linking vertices of  $I$  to vertices of  $J$ , then  $W$  is the direct product of  $W_I$  and  $W_J$ . We have  $W_I \cap W_J = \{e\}$  by Proposition 2.4.1(iii),  $W = \langle W_I, W_J \rangle$  by Proposition 2.4.1(iv), and  $W_I$  commutes with  $W_J$  because there are no edges linking vertices of  $I$  to vertices of  $J$  (and we showed that the lack of an edge implies commuting generators).

PROBLEM 3B

Again, we need only consider the case where  $S$  is the disjoint union  $I \cup J$  and the Coxeter graph has no edges linking vertices of  $I$  to vertices of  $J$ . We will show that the map  $\eta$  from Problem 3b has the property that  $u \leq w$  in Bruhat order on  $W$  if and only if  $\eta(u) \leq \eta(v)$  in the product of the Bruhat order on  $W_I$  and the Bruhat order on  $W_J$ . Write a reduced word  $s_1 \cdots s_q$  for  $w$  such that  $s_1 \cdots s_k$  is a reduced word for  $w_1$  and  $s_{k+1} \cdots s_q$  is a reduced word for  $w_2$ . If  $u \leq w$  then there exists a reduced word for  $u$  that is a subword of  $s_1 \cdots s_q$ . The restriction of the subword to positions  $[k]$  is a reduced word for  $u_1$  which is a subword of  $s_1 \cdots s_k$ , so  $u_1 \leq w_1$ . Furthermore, the restriction of the subword to positions  $\{k+1, \dots, q\}$  is a reduced word for  $u_2$  which is a subword of  $s_{k+1} \cdots s_q$ , so  $u_2 \leq w_2$ . Thus  $(u_1, u_2) \leq (w_1, w_2)$ . Conversely, if  $(u_1, u_2) \leq (w_1, w_2)$  then there is a reduced word for  $u_1$  which is a subword of  $s_1 \cdots s_k$  and a reduced word for  $u_2$  which is a subword of  $s_{k+1} \cdots s_q$ , so  $u_2 \leq w_2$ . Concatenating these two words, we get a reduced word for  $u$  which is a subword of  $s_1 \cdots s_q$ .

Some of you did this with arrows in the Bruhat graph, and that can work. The key observation for that approach is that if you have an arrow in the Bruhat graph for  $W$ , then the label  $t$  on the arrow actually is an element of  $W_I$  or of  $W_J$ .

## PROBLEM 10

**Part a.** Let  $s \in S$ . Then

$$s \in D_L(ww_0) \iff \ell(sw_0) < \ell(w) \iff \ell(sw) > \ell(w) \iff s \in S \setminus D_L(w).$$

The first and last equivalences are by definition and the middle equivalence follows immediately from Proposition 2.3.2(ii).

Alternate proof: Proposition 2.3.2(ii) says that  $T_L(ww_0) = T \setminus T_L(w)$ , so

$$D_L(ww_0) = S \cap (T_L(ww_0)) = S \cap (T \setminus T_L(w)) = S \setminus (S \cap T_L(w)) = S \setminus D_L(w).$$

**Part b.** Let  $s \in S$ . Then

$$s \in D_L(w_0w) \iff \ell(sw_0w) < \ell(w_0w) \iff \ell(w_0sw_0w) > \ell(w) \iff w_0sw_0 \in S \setminus D_L(w).$$

The first equivalence is by definition. The second equivalence follows immediately from Proposition 2.3.2(i) and Proposition 2.3.3(i). The third equivalence is by definition and by the fact that conjugation by  $w_0$  is a permutation of  $S$  (so that  $w_0sw_0 \in S$ ). Now

$$w_0sw_0 \in S \setminus D_L(w) \iff s \in w_0(S \setminus D_L(w))w_0 = S \setminus w_0D_L(w)w_0.$$

The implication again uses the Proposition 2.3.2(i) and the equality again uses the fact that conjugation by  $w_0$  is a permutation of  $S$  (so that  $w_0Sw_0 = S$ ).

**Part c.** By parts a and b,

$$D_L(w_0ww_0) = S \setminus D_L(w_0w) = S \setminus (S \setminus w_0D_L(w)w_0) = w_0D_L(w)w_0.$$

**Part d.** Let  $t \in T$ . Then

$$t \in T_L(w_0ww_0) \iff \ell(tw_0ww_0) < \ell(w_0ww_0) \iff \ell(w_0tw_0ww_0) > \ell(w) \iff w_0tw_0 \in T \setminus T_L(w).$$

The first equivalence is by definition. The second equivalence follows immediately from Proposition 2.3.2(i) and Proposition 2.3.3(i). The third equivalence is by definition and by the fact that conjugation by  $w_0$  is a permutation of  $T$  (so that  $w_0tw_0 \in T$ ). Now Proposition 2.3.2(iii) says that  $T \setminus T_L(w) = T_L(w)$ , and  $w_0tw_0 \in T_L(w)$  if and only if  $t \in w_0T_L(w)w_0$  by Proposition 2.3.2(i).

## PROBLEM 14

**Proposition 1.** Suppose  $\leq_?$  is a partial order on  $W$  with the following lifting property:

For all  $u, w \in W$  and  $s \in S$  with  $\ell(su) > \ell(u)$  and  $\ell(sw) < \ell(w)$ , the following are equivalent:

- (i)  $su \leq_? w$
- (ii)  $u \leq_? w$
- (iii)  $u \leq_? sw$ .

Then for all  $u, w \in W$  we have  $u \leq_? w$  if and only if  $u \leq_B w$ , where  $\leq_B$  means Bruhat order.

*Proof.* We first note that whenever  $\ell(sx) < \ell(x)$ , also  $sx \leq_? x$ . (This is lifting with  $u = sx$  and  $w = x$ . Since  $x \leq_? x$ , the  $?$ -lifting property says that also  $sx \leq_? x$ .)

We argue by induction on  $\ell(u) + \ell(w)$ . If  $\ell(u) + \ell(w) = 0$ , then  $u = w = 1$ , so  $u \leq_? w$  and  $u \leq_B w$ .

If  $\ell(w) > 0$ , then  $\exists s \in S$  such that  $\ell(sw) < \ell(w)$ . If also  $\ell(su) < \ell(u)$  then by Bruhat lifting, then induction, then  $?$ -lifting, we have  $u \leq_B w \iff su \leq_B w \iff su \leq_? w \iff u \leq_? w$ . If instead  $\ell(su) > \ell(u)$ , then again by Bruhat lifting, then induction, then  $?$ -lifting, we have  $u \leq_B w \iff u \leq_B sw \iff u \leq_? sw \iff u \leq_? w$ .

If  $\ell(w) = 0$  and  $\ell(u) > 0$ , then  $w = 1$  and  $u \neq 1$ , and  $u \not\leq_B 1$ . Suppose for the sake of contradiction that  $u \leq_? 1$ . There exists  $s \in S$  such that  $\ell(su) < \ell(u)$ . As noted in the first paragraph,  $su \leq_? u$  in this case. By transitivity,  $su \leq_? 1$ . Thus by induction,  $su \leq_B 1$ , which implies  $su = 1$ . Then  $u = s$ , so  $u \leq_? 1$  says  $s \leq_? 1$  and  $su \leq_? u$  says  $1 \leq_? s$ . Since  $\leq_?$  is a partial order,  $1 = s$ , contradicting Proposition 1.1.1. This contradiction lets us conclude that  $u \not\leq_? 1$ .  $\square$