MA796, Coxeter groups
Solutions/comments for homework assignment 2

Problem 6a

We say that a set A of edges *spans* $[n]$ if and only if, for any $i, j \in [n]$, there is a path from i to j using edges in A.

Claim 1. *$A \subseteq T$ generates S_n if and only if A spans $[n]$.*

Proof of Claim 1. If A does not span then let $[n] = I \cup J$ be a disjoint union such that no path connects an vertex in I to a vertex in J. Then the subgroup $\langle A\rangle$ of S_n generated by A fixes I as a set. In particular $\langle A\rangle \subsetneq S_n$.

On the other hand, if A spans, then for any $i, j \in [n]$, let $i = v_1 — v_2 — \cdots — v_k = j$ be a path from i to j. Then $(i\ j) = (v_1\ v_2)(v_2\ v_3)\cdots(v_{k-1}\ v_k)\cdots(v_2\ v_3)(v_1\ v_1)$ is in $\langle A\rangle$. Since transpositions generate S_n, $\langle A\rangle = S_n$. □

The claim is all you need. A spanning tree is a minimal set of edges that spans $[n]$. Thus the claim says that A is a minimal generating set if and only if it is a spanning tree.

Problem 6b

If (S_n, A) is to be a Coxeter system then the relations will be $m(t, t') = 3$ if t and t' share one vertex and $m(t, t') = 2$ if t and t' are disjoint.

If the tree is linear, then up to relabeling of the elements of $[n]$, we have Example 1.2.3. So (S_n, A) is a Coxeter group in this case.

If the tree is not linear, then it contains three edges of the form $a — b$, $a — c$ and $a — d$. There are two ways to show that (S_n, A) cannot be a Coxeter system in this case.

First way. In this case, the Coxeter graph, restricted to the vertices $(a\ c)$, $(a\ c)$ and $(a\ d)$ would have to be a triangle. But we know from our rank-3 results in class (or from the classification of finite Coxeter groups in the book) that a Coxeter group whose Coxeter graph is a triangle is not finite.

Second way. We can find a violation of the Deletion Property among the generators $(a\ b)$, $(a\ c)$ and $(a\ d)$ in S_n. Consider the word $(a\ b)(a\ c)(a\ b)(a\ d)(a\ b)$ representing an element w. Then $w = (a\ c\ b\ d)$. Notice that another word for w is $(a\ d)(a\ b)(a\ c)$, so the word $(a\ b)(a\ c)(a\ b)(a\ d)(a\ b)$ is not reduced. The Deletion Property requires us to delete two factors from $(a\ b)(a\ c)(a\ b)(a\ d)(a\ b)$ and still get $(a\ c\ b\ d)$. Note that if I delete $(a\ c)$ or $(a\ d)$ then the resulting element will fix c or d and thus cannot be $(a\ c\ b\ d)$. So I have to delete two of the factors among the first, third and fifth factors. Trying all three possibilities $(a\ c)(a\ d)(a\ b)$, $(a\ c)(a\ b)(a\ d)$ and $(a\ b)(a\ c)(a\ d)$, we see that $(a\ c\ b\ d)$ never appears.

Problem 8

We prove the contrapositive: Suppose the word $s_1 \cdots s_k$ is *not* reduced and let w be the element $s_1 \cdots s_k$. Then the Deletion Property says that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ for some $1 \le i < j \le k$. Thus $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k = t_i t_j s_1 \cdots s_k = t_i t_j w$, so $t_i t_j = 1$, and since both are involutions, therefore $t_i = t_j$.

Problem 13

Notice that we can use "right" versions of (Strong or ordinary) Exchange Property and other facts as needed. This is because the symmetry of passing from w to w^{-1} preserves the set of reflections (and the set of simple reflections), preserves length, and simply reverses reduced words. As the class moves forward, we will use either version without comment.

<u>(a) $\Longrightarrow$ (d)</u>: Suppose (a) and write reduced words $u = s_1 \cdots s_k$ and $w = s_{k+1} \cdots s_\ell$. Then (a) implies that $s_1 \cdots s_\ell$ is a reduced word for uw. For each $i \in [\ell]$, write t_i for $s_\ell \cdots s_i \cdots s_\ell$. (Notice, these are "right palindromes" for the word $s_1 \cdots s_\ell$, so this is a different definition of t_i than we used in the book and in class!) By the "right version" of Lemma 1.3.1, each of the t_i is distinct. For each $i \in [k]$, define $t'_i = s_k \cdots s_i \cdots s_k$ (a right palindrome for $s_1 \cdots s_k$). Thus if $i \in [k]$, we have

$t_i = s_k \cdots s_{k+1} t'_i s_{k+1} \cdots s_\ell = w^{-1} t'_i w$. Since these are all reduced words, we can use the right version of Corollary 1.4.4 three times to conclude:

$$T_R(uw) = \{t_i : i \in [\ell]\} = w^{-1}\{t'_i : i \in [k]\}w \ \uplus \ \{t_i : i \in [k+1, \ell]\} = w^{-1}T_R(u)w \ \uplus \ T_R(w).$$

(d) $\Longrightarrow$ (c): This immediate, because (d) is a formally stronger assertion than (c).

(c) $\Longrightarrow$ (a): Again, write reduced words $u = s_1 \cdots s_k$ and $w = s_{k+1} \cdots s_\ell$ and define the t_i and t'_i as above.

We first point out that one containment in (c) always holds. If $t \in T_R(uw)$, then the right version of the Strong Exchange property implies that $t = t_i$ for some $i \in [\ell]$. If $i \in [k]$, then t_i is in $w^{-1}T_R(u)w$ as before, and if $i \in [k+1, \ell]$, then $t_i \in T_R(w)$.

Suppose (a) fails. We will show that the other containment in (c) fails. To do so, it is enough to show that $T_R(w) \subseteq T_R(uw)$ fails.

The failure of (a) means that $s_\ell \cdots s_i \cdots s_\ell$ is *not* reduced. The Deletion Property says there exist i and j with $1 \le i < j \le \ell$ such that $uw = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_\ell$. If $j \le k$, then we can delete the letters $s_{k+1} \cdots s_\ell$ from $s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_\ell$ to get a shorter reduced word for u, contradicting the fact that $s_1 \cdots s_k$ is reduced. We get a similar contradiction if $i > k$, and we conclude that $1 \le i \le k < j \le \ell$. Thus $uw = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots s_\ell$ (allowing $i = k$ and/or $j = k+1$).

If $T_R(w) \subseteq T_R(uw)$, then in particular $t_j \in T_R(uw)$. By Strong Exchange, there exists $m \in [\ell] \setminus \{i, j\}$ such that deleting s_m from $s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots s_\ell$ gives a word for uwt_j. If $m \le k$, then we conclude that $uw = s_1 \cdots \hat{s}_i \cdots \hat{s}_m \cdots s_k s_{k+1} \cdots s_\ell$ (with the $\hat{s}_m$ before or after the $\hat{s}_i$), and we conclude that $u = s_1 \cdots \hat{s}_i \cdots \hat{s}_m \cdots s_k$, contradicting the fact that $s_1 \cdots s_k$ is reduced. If $m > k$, then $uwt_j = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots \hat{s}_m \cdots s_\ell$ (with the $\hat{s}_m$ before or after the $\hat{s}_j$). But also $uwt_j = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots s_\ell$, so we conclude that $s_{k+1} \cdots \hat{s}_j \cdots \hat{s}_m \cdots s_\ell = s_{k+1} \cdots s_\ell$, contradicting the fact that the latter is reduced. By this contradiction, we see that $T_R(w) \subseteq T_R(uw)$ fails.

(a) $\Longleftrightarrow$ (b): Write reduced words $u = s_1 \cdots s_k$ and $w = s_{k+1} \cdots s_\ell$. If (a) fails, then arguing as above, we see that $uw = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots s_\ell$. Let $t = s_k \cdots s_i \cdots s_k$ and let $t' = s_{k+1} \cdots s_j \cdots s_{k+1}$. Then $ut'tw = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots s_\ell$, which equals uw. Thus $t't$ is the identity, and since both are palindromes, $t' = t$. Using the right version and the left version of Corollary 1.4.4, we conclude that $t \in T_R(u) \cap T_L(w)$.

If (b) fails, then again using the right version and the left version of Corollary 1.4.4, we find a reflection $t = s_k \cdots s_i \cdots s_k = s_{k+1} \cdots s_j \cdots s_{k+1}$ for some $i \in [k]$ and $j \in [k+1, \ell]$. Then $uw = uttw = s_1 \cdots \hat{s}_i \cdots s_k s_{k+1} \cdots \hat{s}_j \cdots s_\ell$, contradicting (a).