MA796, Coxeter groups
Solutions/comments for homework assignment 1

**Grading system.** I am grading each problem on a 0–4 point scale. A "4" means that the answer is correct. A "4∼" means that the answer is basically correct, but there is an important point I want you to consider. When I total the scores, a "4 ∼" counts as a 4. If there is something mathematically wrong or unclear, the score decreases to a "3," then decreases further as problems with the answer increase. If the answer is astoundingly good—it must literally astound me—then I will score it a "5." Sometimes I will score a multipart assignment not according to the number of parts it has, but by an estimate of "how many problems" it is worth.

**Problem 1.** In each line below, the left and right terms are identical except in the places underlined. In the underlined places, a relation has been applied (involving $a$ and $b$ or $a$ and $c$). The right term of each line is identical to the left term of the next line, except that a different location is underlined.

$$\begin{aligned} bcb\underline{caca}babcacbcabacbabacbc &= bcb\underline{\ }babcacbcabacbabacbc \\ = bc\underline{bb}abcacbcabacbabacbc &= bc\underline{\ }abcacbcabacbabacbc \\ = bcab\underline{cac}bcabacbabacbc &= bcab\underline{a}bcabacbabacbc \\ = bc\underline{abab}cabacbabacbc &= bc\underline{ba}cabacbabacbc \\ = bcb\underline{aca}bacbabacbc &= bcb\underline{c}bacbabacbc \\ = bcbcbac\underline{baba}cbc &= bcbcbac\underline{ab}cbc \\ = bcbcb\underline{aca}bcbc &= bcbcb\underline{c}bcbc \\ = bcbcbcbcbc. & \end{aligned}$$

Setting this equal to the identity is equivalent to saying that $x = 5$. (But be careful: If the equation simplified to $bcbcbcbc = e$, we would have two choices, $x = 2$ or $x = 4$.)

**Problem 2.** Following the hint, in light of Example 1.2.7, the dihedral group of order 12 is isomorphic to the Coxeter group $I_2(6)$. The key is to convince yourself that this dihedral group is also isomorphic to $I_2(3) \times \mathbb{Z}_2$. The latter is a Coxeter group whose Coxeter graph has three vertices and exactly one edge, and this edge is unmarked.

There are several ways to convince yourself of this isomorphism. You could use the universal property to show the existence of a surjective homomorphism, then use the fact that both groups have the same size. You could argue directly, naming all the elements of the dihedral group and constructing an explicit isomorphism. Or if you like to think pictorially, here is a fun way: Consider two equilateral triangles superimposed to form a "Star of David." The symmetry group of the union of the two is the same as the symmetry group of a regular hexagon, namely the dihedral group $I_2(6)$. You can check that the symmetry group of the union of the two is also the direct product of the symmetry group $I_2(3)$ of one of the triangles with a two-element group whose nontrivial element is a rotation taking one triangle to the other. (There is a basic theorem that says when a group is a direct product of two of its subgroups.)

**Problem 3.** We have $a_1a_2 = (3\ 4\ 5)$, an element of order 3 in $S_5$. Similarly, $a_1a_3 = (1\ 3)(2\ 4)$, an element of order 2. The product $a_2a_3 = (1\ 5\ 4\ 2\ 3)$ is of order 5. Let $H_3$ be realized explicitly as the Coxeter system $(W, S)$ with $S = \{s_1, s_2, s_3\}$ and $m(s_1, s_2) = 5$, $m(s_1, s_3) = 2$ and $m(s_2, s_3) = 3$.

The elements $a_1$, $a_2$ and $a_3$ are all in the alternating group. By the universal property, we conclude that the map sending $s_i \mapsto a_{4-i}$ for $i = 1, 2, 3$ extends uniquely to a surjective group homomorphism $\eta$ from $H_3$ to the subgroup generated by $\{a_1, a_2, a_3\}$.

Let $G$ be the subgroup generated by $\{a_1, a_2, a_3\}$. Since $G$ contains an element $a_1a_2$ of order 3, the order of $G$ is divisible by 3. Similarly, since $G$ contains $a_2a_3$, the order of $G$ is divisible by 5. Since the subgroup of $G$ generated by $\{a_1, a_3\}$ is of order 4, the order of $G$ is divisible by 4. In particular, the order of $G$ is divisible by $\mathrm{lcm}(3, 5, 4) = 60$. But the alternating subgroup of $S_5$ has order $(5!)/2 = 60$, so $G$ is the whole alternating subgroup of $S_5$. Thus $\eta$ maps $H_3$ surjectively onto the alternating subgroup of $S_5$.

The kernel of $\eta$ has $120/60 = 2$ elements, so to understand it, we only need to find one nontrivial element that maps to the identity. Let $w_0$ be the element of $H_3$ given by the word $s_1s_2s_1s_2s_1s_3s_2s_1s_2s_1s_3s_2s_1s_2s_3$. (There are motivations for thinking about this element: as we will discuss later in the course, it is the longest element of $H_3$.) It is routine to check that $\eta(w_0)$ is the identity. Note also that $w_0$ has odd length.

By one of the standard isomorphism theorems for groups, the alternating subgroup of $S_5$ is isomorphic to the quotient $H_3/\{e, w_0\}$. But each coset of $\{e, w_0\}$ contains exactly one even-length element, and so the map $\varphi$ sending each coset to its even element is a surjective homomorphism from $H_3/\{e, w_0\}$ onto the alternating subgroup of $H_3$. (To see that $\varphi$ is a homomorphism, let $\{x, xw_0\}$ and $\{y, yw_0\}$ be any two cosets, and we may as well have written them so that $x$ and $y$ are even. Then $\varphi(\{x, xw_0\}) \cdot \varphi(\{y, yw_0\}) = xy$. By the definition of the quotient modulo a normal subgroup, the product $\{x, xw_0\} \cdot \{y, yw_0\}$ equals $\{xy, xyw_0\}$. Since $xy$ is also even, $\varphi(\{x, xw_0\} \cdot \{y, yw_0\}) = xy$ as desired.) Since these two groups have the same number of elements, they must in fact be isomorphic.

*Alternate solution method:* Start from where we had established (above) a surjective group homomorphism $\eta$ from $H_3$ to the subgroup generated by $\{a_1, a_2, a_3\}$, and don't bother checking that $\eta$ is surjective onto the alternating subgroup of $S_5$. Instead, because the restriction of $\eta$ to the alternating subgroup of $H_3$ is still a homomorphism, you can just check that this restriction is surjective onto the alternating subgroup. In fact, that can be done with the same calculation we did above. And then, you have a surjective homomorphism between finite groups of the same size, so you're done. (This is easier—much less fuss about normal subgroups—but somehow I don't like it as well. Maybe I *like* all the fuss about normal subgroups?)

*Non-mathematical question*: Why don't I use the name "$A_5$" for the alternating subgroup of $S_5$?

**Additional Problem.** Suppose $r$ and $s$ are distinct Euclidean reflections in an $n$-dimensional real vector space $V$ and let $v$ and $w$ be unit $(-1)$-eigenvectors of $r$ and $s$ respectively. Then both of them fix the subspace $v^\perp \cap w^\perp$. Choose an orthonormal basis for $V$ containing $w$ and having $n-2$ of its vectors in $v^\perp \cap w^\perp$. In this basis, every matrix for an element of $\{r, s\}$ is in block diagonal form, with one block being an $(n-2) \times (n-2)$ identity matrix. We might as well ignore that block throughout (or equivalently, mod out by the subspace $v^\perp \cap w^\perp$). We write $s$ (listing the basis with $v$ first) as $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If $\theta$ is the angle between $v^\perp$ and $w^\perp$, then (up to replacing the first basis vector by its negative), we can write $r$ as $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. Then

$$rs = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix},$$

which is the matrix for rotation through an angle $2\theta$.