1a. $S = \{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$

$$S^{\triangle} = \{[c_1\, c_2] \in (\mathbb{R}^2)^* : c_1 \leq 1, c_1 + c_2 \leq 1\}.$$

1b. $S = P\left(\left[\begin{smallmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]\right)$

$$S^{\triangle} = \{\vec{c}\left[\begin{smallmatrix}1 & 0\\ 0 & 1\\ -1 & -1\end{smallmatrix}\right] : \vec{c} \geq \mathbb{0},\, \vec{c}\vec{1} = 1\}$$

This is the convex hull of the rows of the given matrix.

1c. $S = \operatorname{conv}\left(\left[\begin{smallmatrix}1 & 1 & 2\\ 1 & -2 & 1\end{smallmatrix}\right]\right)$

$$S^{\triangle} = \{\vec{c} \in (\mathbb{R}^2)^* : \vec{c}\left[\begin{smallmatrix}1 & 1 & -2\\ 1 & -2 & 1\end{smallmatrix}\right] \leq \mathbb{1}\}$$

You didn't need to notice that the polytopes in parts b and c are the same. I gave $P^{\triangle}$ in the form ($\mathcal{V}$ or $\mathcal{H}$) that you could get easily from the given form of P. You should be comfortable doing polar duality on a polytope given in $\mathcal{V}$-form or $\mathcal{H}$-form, as long as $\vec{0}$ is in its interior.

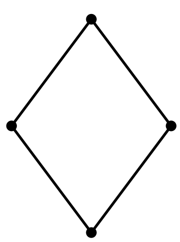
2. The polar dual is also an Egyptian pyramid. One way to see this is to actually draw the Hasse diagram of the face lattice and then turn it upside down. Another way is to think through: There are 5 vertices of $P^{\triangle}$, corresponding to the 5 facets of P. Facets of $P^{\triangle}$ correspond to vertices of P. Since there is one vertex of P contained in 4 facets, there is one facet of P (necessarily 2-dimensional) with 4 vertices, or in other words, a sqaure. Similarly, there are 4 other vertices of P, each contained in 3 facets, and these correspond to triangular facets of $P^{\triangle}$. A bit of thought (possibly thinking of how edges of P become edges of $P^{\triangle}$), shows that $P^{\triangle}$ is an Egyptian pyramid.

3. *A priori*, P is $P(A, \vec{z})$. But if any entry of $\vec{z}$ is negative (say $z_i < 0$), then $\vec{0} \notin P$, because $\vec{0}$ does not satisfy $\vec{a}_i\vec{x} \leq z_i$. Also if any entry of $\vec{z}$ is zero, then $\vec{0}$ is not in the interior of P. (For this, you could remember Lemma 2.8 or just think it through: If $\vec{0} \in \operatorname{int}(P)$ then P must be full-dimensional. But if z_i is zero, then $\vec{0}$ satisfies $\vec{a}_i\vec{x} \leq z_i$ with equality, so $\vec{0}$ is in a proper face $P \cap \{\vec{x} \in \mathbb{R}^d : \vec{a}_i\vec{x} = z_i\}$ of P, contradicting $\vec{0} \in \operatorname{int}(P)$. Thus every entry of $\vec{z}$ is positive, so we can scale the rows of $A\vec{x} \leq z$ to get $A'\vec{x} \leq \vec{1}$ for some matrix A'. Thus $P = P(A', \vec{1})$.

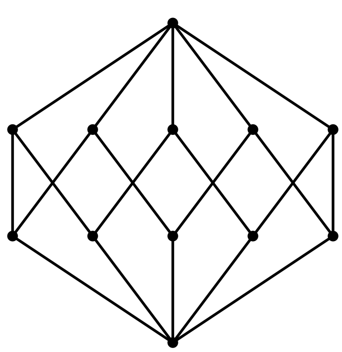
4. The entire point of this problem was that if you have a polytope, you can think of it as an $\mathcal{H}$-polytope or a $\mathcal{V}$-polytope, as you please.

Given two $\mathcal{H}$-polytopes P and Q, they are also $\mathcal{V}$-polytopes. Write $P = \operatorname{conv}(V)$ and $Q = \operatorname{conv}(W)$. The point is that $\operatorname{conv}(P \cup Q) = \operatorname{conv}(V \cup W)$, which is a $\mathcal{V}$-polytope (and thus also an $\mathcal{H}$ polytope).

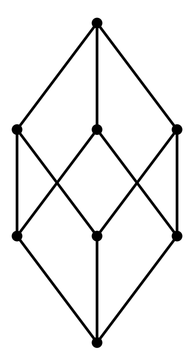
To see that $\operatorname{conv}(P \cup Q) = \operatorname{conv}(V \cup W)$, first note that $\operatorname{conv}(P \cup Q) \supseteq \operatorname{conv}(V \cup W)$ is immediate because $P \cup Q \supseteq V \cup W$. On the other hand, if $\vec{x} \in \operatorname{conv}(P \cup Q)$, then $\vec{x}$ can be written as a convex combination of points $\vec{y}_1, \ldots, \vec{y}_k$ where each $\vec{y}_i$ is either a convex combination of points in V or a convex combination of points in W. Thus $\vec{x}$ is a convex combination of points in $V \cup W$.



5a.



5b.



5c.

For part c, the polytope is the convex hull of 3 points on the moment curve. They're affinely independent (meaning "not collinear" since there are 3 of them), so their convex hull is a triangle.

6a. The statement $\vec{x} \in \mathrm{conv}(V)$ means precisely $\exists \vec{t} \in \mathbb{R}^n$ such that $\vec{t} \geq \vec{0}$, $\mathbb{1}\vec{t} = 1$, and $V\vec{t} = \vec{x}$. The statement that there exists an affine hyperplane separating $\vec{x}$ from V from v is precisely saying that there exists $\vec{a} \in (\mathbb{R}^d)^*$ and $z \in \mathbb{R}$ such that $\vec{a}\vec{v}_i \leq z$ for all $i \in [n]$ and $\vec{a}\vec{x} > z$.

Thus, the Lemma is that $\exists \vec{t} \in \mathbb{R}^n$ such that $\vec{t} \geq \vec{0}$, $\mathbb{1}\vec{t} = 1$, and $V\vec{t} = \vec{x}$ or $\exists \vec{a} \in (\mathbb{R}^d)^*$ and $z \in \mathbb{R}$ such that $\vec{a}\vec{v}_i \leq z$ for all $i \in [n]$ and $\vec{a}\vec{x} > z$, but not both. (The "but not both" part should be obvious.)

6b. We can rewrite the Lemma further as follows: $\exists \vec{t} \in \mathbb{R}^n$ such that $\vec{t} \geq \vec{0}$ and $\left[\begin{smallmatrix} V \\ \mathbb{1} \end{smallmatrix}\right]\vec{t} = \left[\begin{smallmatrix} \vec{x} \\ 1 \end{smallmatrix}\right]$ or $\exists \vec{b} \in (\mathbb{R}^{d+1})^*$ such that $\vec{b}\left[\begin{smallmatrix} V \\ \vec{1} \end{smallmatrix}\right] \geq \mathbb{0}$ and $\vec{b}\vec{x} < 0$, but not both. (The $\vec{b}$ in this restatement is $\left[\begin{smallmatrix} -\vec{a} & z \end{smallmatrix}\right]$ for $\vec{a}$ and z as in part a.) We see that the Lemma is exactly Farkas II with $A = \left[\begin{smallmatrix} V \\ \vec{1} \end{smallmatrix}\right]$, $m = d+1$, $d = n$, and $\vec{z} = \left[\begin{smallmatrix} \vec{x} \\ 1 \end{smallmatrix}\right]$.

7. *Note that this problem did not use any facts about the graph of a polytope. Just the definition of the graph and then facts from Lectures* ≤ 2.

7a. Because if the graph of a polytope is K_7, then the polytope has 7 vertices and thus has dimension at most 6.

7b. The 6-dimensional simplex.

7c. The cyclic polytope $C_7(5)$.

8. Suppose P has $\vec{0} \in \mathrm{int}(P)$. In Problem 3, we showed that $P = P(A, \vec{1})$. We proved in class (and in the book) that $P^{\triangle}$ is a polytope with $\mathbb{0} \in \mathrm{int}(P^{\triangle})$. But we also proved in class that $P^{\triangle}$ is the convex hull of the rows of A. (Compare Problem 1b.)

Conversely, suppose $P = P(A, \vec{1})$ for some matrix A such that the convex hull of the rows of A contains $\mathbb{0}$ in its interior. This convex hull of the rows of A is $P^{\triangle}$, and we proved in class that $(P^{\triangle})^{\triangle}$ has $\vec{0}$ in its interior. (In fact, we proved that dual of that fact.) But $P \subseteq (P^{\triangle})^{\triangle}$, so $\vec{0} \in \mathrm{int}(P)$.