

PROBLEM 1

A little philosophy before I actually answer the question. Recall that a *simple graph* is a graph with no self-edges and no multiple edges between one pair of vertices. A 1-dimensional (geometric) simplicial complex is precisely a simple graph together with a specific “drawing” of the graph in some vector space with straight-line edges that don’t intersect each other except at endpoints. Kruskal-Katona answers the question “Does there exist a simple graph with f_1 edges and f_0 vertices?” (for given numbers f_0 and f_1).

Kruskal-Katona for $d = 2$ says that (f_{-1}, f_0, f_1) is the f -vector of a simplicial complex of dimension ≤ 1 if and only if $f_{-1} = 1$ and $f_0 \geq \partial_2(f_1)$. Thus, writing $f_1 - 1$ as $\binom{a_2}{2} + \binom{a_1}{1}$ for $a_2 > a_1 \geq 0$, the constraint is $f_0 \geq a_2 + 1$. Recalling how we compute a_2 and a_1 (taking the largest possible a_2 and then continuing), we can say this: $f_0 \geq a_2 + 1$, where a_2 is the largest number with $\binom{a_2}{2} \leq f_1 - 1$.

For each f_1 from 1 to 11, here is the constraint on f_0 :

f_1	
1	$f_0 \geq 2$ ($a_2 = 1$)
2	$f_0 \geq 3$ ($a_2 = 2$)
3	$f_0 \geq 3$ ($a_2 = 2$)
4	$f_0 \geq 4$ ($a_2 = 3$)
5	$f_0 \geq 4$ ($a_2 = 3$)
6	$f_0 \geq 4$ ($a_2 = 3$)
7	$f_0 \geq 5$ ($a_2 = 4$)
8	$f_0 \geq 5$ ($a_2 = 4$)
9	$f_0 \geq 5$ ($a_2 = 4$)
10	$f_0 \geq 5$ ($a_2 = 4$)
11	$f_0 \geq 6$ ($a_2 = 5$)

A simple explanation of Kruskal-Katona for graphs: It’s probably better to rephrase this way: $f_0 > a_2$, where a_2 is the largest number with $\binom{a_2}{2} < f_1$. The number $\binom{a_2}{2}$ is the number of edges in a complete graph with a_2 vertices. So $f_1 > \binom{a_2}{2}$, then you need strictly more than a_2 vertices to have that many edges. But since a_2 is the *largest* number with $\binom{a_2}{2} < f_1$, we see that $\binom{a_2+1}{2} \geq f_1$, so we know we can make a graph with $a_2 + 1$ vertices and f_1 edges (by removing 0 or more edges from a complete graph). Thus $f_0 > a_2$ is the only constraint.

PROBLEM 2

Kruskal-Katona for $d = 3$ says that (f_{-1}, f_0, f_1, f_2) is the f -vector of a simplicial complex of dimension ≤ 2 if and only if $f_{-1} = 1$ and $f_0 \geq \partial_2(f_1)$ and $f_1 \geq \partial_3(f_2)$. The inequality $f_0 \geq \partial_2(f_1)$ comes about because if we deleted all the 2-dimensional faces (i.e. triangles) from a 2-dimensional complex, we would get a 1-dimensional complex (a simple graph), and we already explained that inequality for graphs. For the inequality $f_1 \geq \partial_3(f_2)$, we find the largest a_3 such that $\binom{a_3}{3} \leq f_2 - 1$ and the largest a_2 such that $\binom{a_2}{2} \leq (f_2 - 1) - \binom{a_3}{3}$ and we require $f_1 \geq \binom{a_3}{2} + a_2 + 1$.

For each f_2 from 1 to 11, here is the constraint on f_1 :

f_2	
1	$f_1 \geq 3$ ($a_3 = 2, a_2 = 1$)
2	$f_1 \geq 5$ ($a_3 = 3, a_2 = 1$)
3	$f_1 \geq 6$ ($a_3 = 3, a_2 = 2$)
4	$f_1 \geq 6$ ($a_3 = 3, a_2 = 2$)
5	$f_1 \geq 8$ ($a_3 = 4, a_2 = 1$)
6	$f_1 \geq 9$ ($a_3 = 4, a_2 = 2$)
7	$f_1 \geq 9$ ($a_3 = 4, a_2 = 2$)
8	$f_1 \geq 10$ ($a_3 = 4, a_2 = 3$)
9	$f_1 \geq 10$ ($a_3 = 4, a_2 = 3$)
10	$f_1 \geq 10$ ($a_3 = 4, a_2 = 3$)
11	$f_1 \geq 12$ ($a_3 = 5, a_2 = 1$)

On the questions I asked you to think about but not turn in: Again, it's good to rephrase: We find the largest a_3 such that $\binom{a_3}{3} < f_2$ and the largest a_2 such that $\binom{a_2}{2} < f_2 - \binom{a_3}{3}$ and we require $f_1 > \binom{a_3}{2} + a_2$. Now, $\binom{a_3}{3}$ is the number of triangles in a simplex with a_3 vertices, and we can see that taking all the triangles in a simplex with a_3 vertices is the way to get $\binom{a_3}{3}$ triangles with the fewest vertices. A simplex with a_3 vertices has $\binom{a_3}{2}$ edges, and we also see that this is the fewest edges we could have for $\binom{a_3}{3}$ triangles. Since $f_2 > \binom{a_3}{3}$, then you need strictly more than $\binom{a_3}{3}$ edges to have that many triangles. It gets a little more complicated from here (and more complicated each time you raise the dimension, so you have to look at the actual proof of Kruskal-Katona, which doesn't treat each dimension separately). But basically, once you have all the triangles from a simplex S with a_3 vertices, you can add one more vertex and start making triangles with that vertex and edges of S . You're trying to get as many triangles as possible while making as few edges as possible, but this reduces to choosing as many edges of S as possible while using as few vertices of S as possible, so it reduces to the same question as in Problem 1: Find the largest a_2 such that $\binom{a_2}{2} < f_2 - \binom{a_3}{3}$ and require $f_1 > \binom{a_3}{2} + a_2$.

"Why does the minimum possible f_1 sometimes change more and sometimes less when f_2 increases by 1?"

- The constraint on f_1 sometimes changes by 2 when f_2 increases by 1. This happens when when the increase in f_2 causes a_3 to increase. In this case, the smaller value of f_2 meant that our complex could be all triangles in a simplex. When we add one more triangle, only one of its edges can be in the simplex, so we make two additional edges.
- The constraint on f_1 usually changes by 1 when f_2 increases by 1. This happens in cases where we can place a triangle that uses two edges that are already in the complex.
- The constraint on f_1 sometimes changes by 0 when f_2 increases by 1. This happens in cases where we can place a triangle all of whose edges are already in the complex.

PROBLEM 3

We start with a formula that is basically in the book, although since Ziegler does things "backwards" from what we did in class, it may be harder to see what he does. (He defines the h -vector in terms of sizes of restriction faces of a shelling and then proves that the h -polynomial and f -polynomial are related by $h(x) = f(x-1)$. We defined h -vectors in general using $h(x) = f(x-1)$ and proved that if the complex is shellable then the h -vector is given by sizes of restriction faces.)

Anyway, starting from $h(x) = f(x-1)$, we extract coefficients. To extract coefficients, we use the binomial theorem to write $f(x-1)$ as a double sum: (What do you do when you see a double sum? Reverse it.)

$$\begin{aligned} f(x-1) &= \sum_{i=0}^d f_{i-1} (x-1)^{d-i} \\ &= \sum_{i=0}^d f_{i-1} \sum_{j=0}^{d-i} (-1)^{d-i-j} \binom{d-i}{j} x^j \\ &= \sum_{j=0}^d x^j \sum_{i=0}^{d-j} (-1)^{d-i-j} \binom{d-i}{j} f_{i-1} \end{aligned}$$

Now, h_k is the coefficient of x^{d-k} in $f(x-1)$, which is $\sum_{i=0}^{d-(d-k)} (-1)^{d-i-(d-k)} \binom{d-i}{d-k} f_{i-1}$, which simplifies to Definition 8.18 (a "definition" because Ziegler and I do things backwards from each other):

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

Setting $k = 0$, we see that $h_0 = f_{-1} = 1$. Setting $k = 1$, we obtain $h_1 = -df_{-1} + f_0 = f_0 - d$. Setting $k = d$, we obtain

$$h_d = \sum_{i=0}^d (-1)^{d-i} \binom{d-i}{0} f_{i-1} = (-1)^{d-1} \sum_{i=0}^d (-1)^{i-1} f_{i-1} = (-1)^{d-1} \tilde{\chi}(\Delta).$$

That's everything but part c. To do part c, you could use the formula for f_{k-1} in the middle of page 249 and set $k = d - 1$. (In the book, the formula for f_{k-1} is proved for a shellable complex, but you could derive it from $f(x) = h(x + 1)$ by the same method we used above to get a formula for h_k .) Or, there is a speedier way to get part c: Just set $x = 0$ in the formula $f(x) = h(x + 1)$.

PROBLEM 4

Using the formula from Problem 3 for entries in the h -vector, we see that the Dehn-Sommerville equations for simplicial 3-polytopes are

$$\begin{array}{llll} h_0 = h_3 : & f_{-1} = -f_{-1} + f_0 - f_1 + f_2 & \text{simplified:} & 2f_{-1} + f_1 = f_0 + f_2 \\ h_1 = h_2 : & -3f_{-1} + f_0 = 3f_{-1} - 2f_0 + f_1 & \text{simplified:} & 3f_0 = 6f_{-1} + f_1 \end{array}$$

Recall that the directions said: "You should literally translate the equations $h_0 = h_3$ and $h_1 = h_2$ into equations relating entries of the f -vector, rather than just using Problem 18 from Lecture 18. (Why is that not the same thing?)"

Well, for one thing, if it was the same thing, then you would all have done Problem 18 easily using the formulas from the book. But let's look specifically at it. If we use Problem 18 from Lecture 18, we get four equations:

$$\begin{array}{llll} f_{-1} = -f_{-1} + f_0 - f_1 + f_2 & \text{simplified:} & 2f_{-1} + f_1 = f_0 + f_2 \\ f_0 = f_0 - 2f_1 + 3f_2 & \text{simplified:} & 2f_1 = 3f_2 \\ f_1 = -f_1 + 3f_2 & \text{simplified:} & 2f_1 = 3f_2 \\ f_2 = f_2 & \text{simplified:} & 0 = 0 \end{array}$$

Why are these different? The Dehn-Sommerville equations say that the f -vectors live in some subspace. These are two different ways to describe the same subspace. (Notice that if you multiply $2f_{-1} + f_1 = f_0 + f_2$ by 3, you can replace $3f_2$ by $2f_1$ to get $6f_{-1} + f_1 = 3f_0$.) We did two different kinds of algebraic manipulations to get the two descriptions, so it's not surprising that they are different.

PROBLEM 5

$$\begin{array}{llll} h_0 = h_4 : & f_{-1} = f_{-1} - f_0 + f_1 - f_2 + f_3 & \text{simplified:} & f_0 + f_2 = f_1 + f_3 \\ h_1 = h_3 : & -4f_{-1} + f_0 = -4f_{-1} + 3f_0 - 2f_1 + f_2 & \text{simplified:} & 2f_1 = 2f_0 + f_2 \end{array}$$