MA 724
Homework 8, Comments and some solutions.

Lecture 7, Problem 1

We need to show that the two fans are the same collection of cones. Since $P^{\triangle\triangle} = P$, is enough to show that the face fan of P is the normal fan of $P^{\triangle}$.

The face fan of P is the collection of all cones cone(F) such that F is a face of P. This is the set of all positive linear combinations of finite sets of points in F. But we can write it more simply, because F is a convex set. If $\vec{y} = \sum \lambda_i \vec{x}_i$ is such a combination and $\lambda = \sum \lambda_i$ is nonzero, then $\vec{y}' = \frac{1}{\lambda}\vec{y}$ is a convex combination of points in F. So $\vec{y}$ is a positive multiple of a point in F. We see that $\text{cone}(F) = \{t\vec{x} : \vec{x} \in F, t \geq 0\}$.

The polar polytope $P^{\triangle}$ has faces $F^{\diamond} = \{\vec{c} \in (\mathbb{R}^d)^* : \vec{c}\vec{x} \leq 1 \,\forall \vec{x} \in P, \vec{c}\vec{x} = 1 \,\forall \vec{x} \in F\}$. The cone in $\mathcal{N}(P^{\triangle})$ corresponding to $F^{\diamond}$ is $N_{F^{\diamond}} = \left\{\vec{x} \in \mathbb{R}^d : F^{\diamond} \subseteq \{\vec{c} \in P^{\triangle} : \vec{c}\vec{x} = \max\{\vec{b}\vec{x} : \vec{b} \in P^{\triangle}\}\}\right\}$. Putting the definition of $F^{\diamond}$ into it, you get

$$N_{F^{\diamond}} = \left\{\vec{x} \in \mathbb{R}^d : \text{ if } \vec{c}\vec{y} \leq 1 \,\forall \vec{y} \in P \text{ and } \vec{c}\vec{y} = 1 \,\forall \vec{y} \in F \text{ then } \vec{c}\vec{x} = \max\{\vec{b}\vec{x} : \vec{b} \in P^{\triangle}\}\}\right\}.$$

The assertion that $\text{cone}(F) \subseteq N_{F^{\diamond}}$ is equivalent to the following: If $\vec{x} \in F$ and $t \geq 0$ and $\vec{c}\vec{y} \leq 1$ for all $\vec{y} \in P$ and $\vec{c}\vec{y} = 1$ for all $\vec{y} \in F$, then $\vec{c}t\vec{x} = \max\{\vec{b}t\vec{x} : \vec{b} \in P^{\triangle}\}$. These hypotheses imply that $\vec{c}t\vec{x} = t$, and furthermore, since $\vec{x} \in P$, we know that $\vec{b}\vec{x} \leq 1$ for all $\vec{b} \in P^{\triangle}$, so t is $\max\{\vec{b}t\vec{x} : \vec{b} \in P^{\triangle}\}$.

On the other hand, suppose $\vec{x} \in N_{F^{\diamond}}$. If $\vec{x} = \vec{0}$, then since cone(F) is a cone, $\vec{x} \in \text{cone}(F)$, so assume $\vec{x} \neq \vec{0}$. Since P is compact and contains $\vec{0}$, there exists a maximal $t \geq 0$ such that $t\vec{x} \in P$. Since $N_{F^{\diamond}}$ is a cone, $t\vec{x} \in N_{F^{\diamond}}$. We will show that $t\vec{x}$ is in F. Let G be a facet of P containing F and let $\vec{c}\vec{y} \leq 1$ be the facet-defining inequality for this facet. (As discussed before, we can write the inequality in this way because $\vec{0} \in \text{int}(P)$.) Now, $F \subseteq G$, so $\vec{c}\vec{y} = 1$ for all $\vec{y} \in F$. Since $t\vec{x} \in N_{F^{\diamond}}$, we known that $\vec{c}t\vec{x} = \max\{\vec{b}t\vec{x} : \vec{b} \in P^{\triangle}\}$. By the definition of $P^{\triangle}$, this maximum is at most 1. But since $t\vec{x}$ is on the boundary of P, it is contained in some facet, so there is some facet-defining inequality $\vec{b}\vec{y} \leq 1$ that is satisfied with equality at $\vec{y} = t\vec{x}$. Thus the maximum is 1. So $\vec{c}t\vec{x} = 1$, i.e. $t\vec{x}$ is on the facet G. This is true for all the facets G containing F, so $t\vec{x} \in F$. We have showed that $N_{F^{\diamond}} \subseteq \text{cone}(F)$.

Lecture 7, Problem 2

My way of checking all the possibilities involved *a lot* of cases. Did you find a better approach? I think there are better approaches.

I'm going to start with Theorem 7.20, which we did not discuss in class, but which was in the reading. That says that every zonotope has at least one simple vertex. (Actually it says $2n$, but we just need *one*.) So, what can the zonotope look like at that vertex? We may as well take that vertex to be the origin, and represent the three edges coming out of it by the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. The other two line segments are represented by nonzero vectors $\vec{x}$ and $\vec{y}$ with nonnegative coefficients. By the symmetry of permuting the standard basis vectors and swapping $\vec{x}$ and $\vec{y}$, the following are all the possibilities.

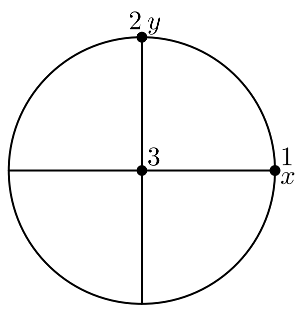
Case 1: $\vec{x}$ and $\vec{y}$ are both positive scalar multiples of $\vec{e}_1$, $\vec{e}_2$, or $\vec{e}_3$. This is a cube.
Case 2: $\vec{y}$ is a positive scalar multiple of $\vec{e}_1$, $\vec{e}_2$, or $\vec{e}_3$ but $\vec{x}$ is not.
Case 2a: $\vec{x}$ is in the strictly positive span of $\vec{e}_1$ and $\vec{e}_2$. This is a hexagonal prism.
Case 2b: $\vec{x}$ is in the strictly positive span of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$. This is the rhombic dodecahedron.
Case 3: Neither of them is a scalar multiple of $\vec{e}_1$, $\vec{e}_2$, or $\vec{e}_3$. In this case, we may as well assume that $\vec{x}$ and $\vec{y}$ are not parallel. (If they are parallel, the possibilities are exactly as in Case 2.)
Case 3a: $\vec{x}$ and $\vec{y}$ are both in the strictly positive span of $\vec{e}_1$ and $\vec{e}_2$. This is an octagonal prism.
Case 3b: $\vec{x}$ is in the strictly positive span of $\vec{e}_1$ and $\vec{e}_2$ and $\vec{y}$ is in the strictly positive span of $\vec{e}_2$ and $\vec{e}_3$. This is a dodecahedron with 4 hexagons and 8 rhombi.
Case 3c: $\vec{x}$ is in the strictly positive span of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.
Case 3c(i): $\vec{y}$ is in the strictly positive span of $\vec{e}_1$ and $\vec{e}_2$, such that $\vec{e}_3$, $\vec{x}$ and $\vec{y}$ are coplanar. This is a dodecahedron with 4 hexagons and 8 rhombi, isomorphic to the other one.
Case 3c(ii): $\vec{y}$ is in the strictly positive span of $\vec{e}_1$ and $\vec{e}_2$, such that $\vec{e}_3$, $\vec{x}$ and $\vec{y}$ are not coplanar. This is a zonotope with two hexagons and 14 rhombi.

Case 3c(iii): $\vec{y}$ is in the strictly positive span of $\vec{e}_1$, $\vec{e}_2$, and $\vec{x}$, such that $\vec{e}_3$, $\vec{x}$ and $\vec{y}$ are coplanar. This is a zonotope with two hexagons and 14 rhombi, isomorphic to the other one.
Case 3c(iv): $\vec{y}$ is in the strictly positive span of $\vec{e}_1$, $\vec{e}_2$, and $\vec{x}$, such that $\vec{e}_3$, $\vec{x}$ and $\vec{y}$ are not coplanar. This is a simple zonotope with 20 rhombi.

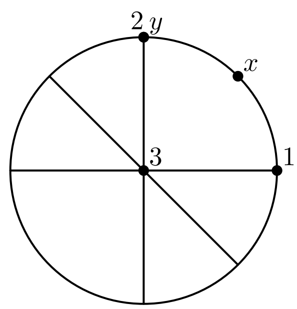
I got seven possibilities. My favorite way to draw the pictures is as hyperplane arrangements. (Drawing the zonotopes themselves can get very difficult.) Recall that the normal fans of zonotopes are defined by the hyperplane arrangement consisting of a normal hyperplane to each vector.

Each picture shows the intersection of the hyperplane arrangement with the top half ($x_3 \geq 0$) of a unit sphere. To get the graph of the zonotope, you could put a vertex in each spherical triangle and connect vertices for adjacent triangles. (You would also have to fill in what happens on the bottom half of the sphere, but antipodal symmetry gives you that.) In each picture, each e_i is labeled as i and $\vec{x}$ and $\vec{y}$ are labeled without the arrow above.

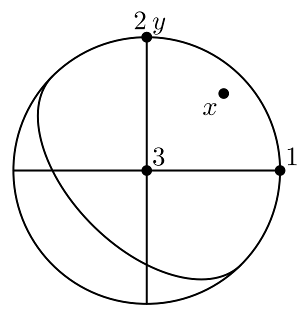
From here, you can find the counts of vertices, simple vertices, etc.



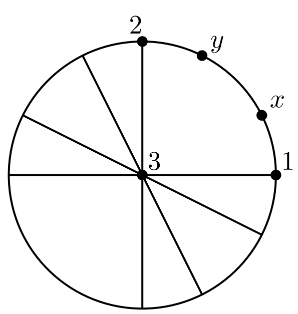
Case 1
(many places for $\vec{x}$ and $\vec{y}$)
cube



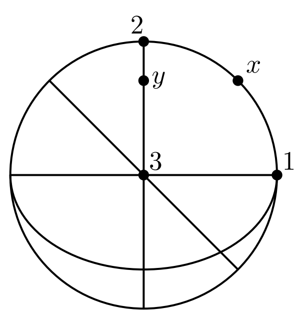
Case 2a
(3 places for $\vec{y}$)
hexagonal prism



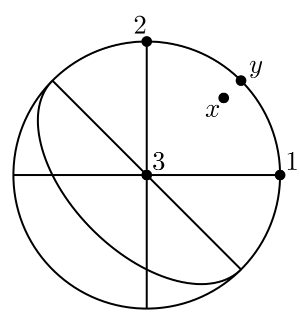
Case 2b
(3 places for $\vec{y}$)
rhombic dodecahedron



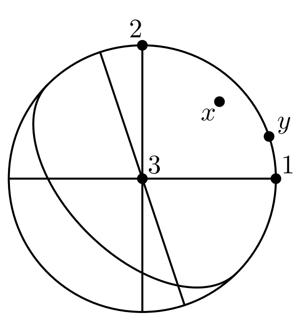
Case 3a
octagonal prism



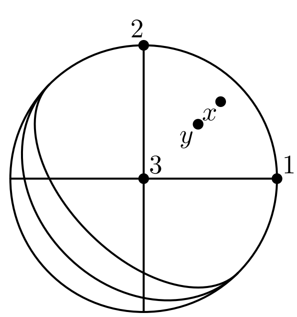
Case 3b
4 hexagons, 8 rhombi



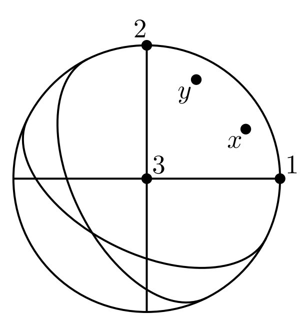
Case 3c(i)
4 hexagons, 8 rhombi



Case 3c(ii)
2 hexagons, 14 rhombi



Case 3c(iii)
2 hexagons, 14 rhombi



Case 3c(iv)
20 rhombi

Comment on computing h-vectors. I noticed after TEX-ing this all up, that I had done all the h-vectors by a different version of Stanley's trick than what we did in class (which is also in the book). I think it's pretty plain to see that it's the same thing, so I'm going to leave it. I don't care how you compute h-vectors, but you should make sure you have a quick way to do it.

Lecture 8, Problem 13

Part (i). Check Kruskal-Katona:

$f_{-1} = 1$: Yup.

$f_0 \geq \partial_2(f_1)$:

$47 - 1 = \binom{10}{2} + \binom{1}{1}$, so $\partial_2(47) = \binom{10}{1} + \binom{1}{0} = 11$. Yup.

$f_1 \geq \partial_3(f_2)$:

$52 - 1 = \binom{7}{3} + \binom{6}{2} + \binom{1}{1}$, so $\partial_3(52) = \binom{7}{2} + \binom{6}{1} + \binom{1}{0} = 28$. Yup.

$f_2 \geq \partial_4(f_3)$:

$38 - 1 = \binom{7}{4} + \binom{3}{3} + \binom{2}{2} + \binom{0}{1}$, so $\partial_4(38) = \binom{7}{3} + \binom{3}{2} + \binom{2}{1} + \binom{0}{0} = 41$. Yup.

$f_3 \geq \partial_5(f_4)$:

$12 - 1 = \binom{6}{5} + \binom{5}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1}$, so $\partial_5(12) = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} + \binom{1}{1} + \binom{0}{0} = 28$. Yup.

This passes Kruskal-Katona, so it's the f-vector of a simplicial complex.

Part (ii). I interpret "shellable complex" as "shellable **simplicial** complex." Compute the h-vector:

$$
\begin{array}{ccccccccccccc}
 & & & & & 12 & & ? & & & & & \\
 & & & & 38 & & ? & & ? & & & & \\
 & & & 52 & & ? & & ? & & ? & & & \\
 & & 47 & & ? & & ? & & ? & & ? & & \\
 & 23 & & ? & & ? & & ? & & ? & & ? & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccccccc}
 & & & & & 12 & & 1 & & & & & \\
 & & & & 38 & & 11 & & -12 & & & & \\
 & & & 52 & & 27 & & 23 & & 39 & & & \\
 & & 47 & & 25 & & 4 & & -16 & & -35 & & \\
 & 23 & & 22 & & 21 & & 20 & & 19 & & 18 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Shellable simplicial complexes have nonnegative h-vectors, so this is not shellable. (**By the way:** If we hadn't found negative entries in the h-vector, I **don't** think we could have concluded that there exists a shellable complex with this f-vector.)

Part (iii). Simplicial polytopes are shellable! So by Part (ii), this is not the f-vector of a simplicial polytope. (**However:** What if we hadn't found negative entries in the h-vector? Then we would have checked Dehn-Sommerville, and if that failed, we would know that this is not the f-vector of a simplicial polytope.)

Lecture 8, Problem 18

The h-vector is defined by $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$. Write $h(x)$ for $\sum_{i=0}^{d} h_i x^{d-i}$. Then Dehn-Sommerville says that $x^d h(x^{-1}) = h(x)$. Thus

$$x^d \sum_{i=0}^{d} f_{i-1}(x^{-1}-1)^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

Rewriting:

$$\sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

Substituting $x+1$ for x:

$$\sum_{i=0}^{d} f_{i-1}(x+1)^i(-1)^{d-i} = \sum_{i=0}^{d} f_{i-1}x^{d-i}.$$

Thus f_{k-1} is the coefficient of x^{d-k} on the right side. We use the binomial theorem inside the summand to rewrite the left side:

$$\sum_{i=0}^{d} f_{i-1}(-1)^{d-i}\sum_{j=0}^{i}\binom{i}{j}x^j$$

Reversing the sum:

$$\sum_{j=0}^{d} x^j \sum_{i=j}^{d} f_{i-1}(-1)^{d-i}\binom{i}{j}$$

The coefficient of x^{d-k} is $\sum_{i=k}^{d} f_{i-1}(-1)^{d-i}\binom{i}{k}$, as desired.

If $k=d-1$, this says $f_{d-2} = f_{d-2}(-1)^1\binom{d-1}{d-1} + f_{d-1}(-1)^0\binom{d}{d-1}$, which simplifies to $2f_{d-2} = df_{d-1}$.

Lecture 8, Problem 28

For a 3-polytope, the Euler-Poincare formula says $-1+f_0-f_1+f_2=1$. This, with the requirement that $f_{-1}=1$, defines an affine plane in the 4-dimensional space of f-vectors. We easily see that any vector of the form $(1,4,6,4)+a(0,1,1,0)+b(0,0,1,1)$ is in that plane. Since $(0,1,1,0)$ and $(0,0,1,1)$ are linearly independent, the vectors in that form span the plane. (There is a subtle point though: Do vectors of that form with *integers* a and b cover all the *integer* vectors in that plane? Yes: Given any integer point (f_{-1},f_0,f_1,f_2) in that plane, we can choose $a=f_0-4$ and $b=f_2-4$, and the other entry must be right, because the point is in the plane.)

We know that a 3-polytope cannot have fewer than 4 vertices or fewer than 4 facets, so we get $a\geq 0$ and $b\geq 0$.

For the simplicial case, we compute the h-vector:

$$\begin{array}{ccccccc} & & & 4+b & & ? & \\ & & 6+a+b & & ? & & ? \\ & 4+a & & ? & & ? & & ? \\ 1 & & 1 & & 1 & & 1 & & 1 \end{array} \longrightarrow \begin{array}{ccccccc} & & & 4+b & & 1 & \\ & & 6+a+b & & 3+b & & 1+b-a \\ & 4+a & & 3+a & & 2+a & & 1+a \\ 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$

Dehn-Sommerville says $1+b-a=1+a$, so $b=2a$, and in particular the f-vector is $(1,4+a,6+3a,4+2a)$.

Here is my complicated way of seeing that $b\leq 2a$ and $a\leq 2b$: A cyclic polytope is in particular simplicial, and since it maximizes f-vectors with a given number of vertices (Theorem 8.23), in particular any 3-polytope with $4+a$ vertices must have at most $4+2a$ facets, and thus since the number of facets is $4+b$, we have $b\leq 2a$ for general polytopes. Passing to the polar dual polytope switches the roles of a and b, so also $2b\geq a$.

Here is a simpler way that I learned from some of your papers (good job!): Since every edge is in exactly 2 facets and every facet has at least 3 edges, $2f_1\geq 3f_2$, so that $12+2a+2b\geq 12+3b$, and thus $2a\geq b$. Since every edge has exactly 2 vertices and every vertex is in at least 3 edges, $2f_1\geq 3f_0$, so that $12+2a+2b\geq 12+3a$, and thus $2b\geq a$.

It remains to show that, for any vector $(1,4,6,4)+a(0,1,1,0)+b(0,0,1,1)$ with $2a\geq b\geq 0$ and $2b\geq a\geq 0$, there is a 3-polytope with that f-vector. This is tricky in some sense, but also straightforward once you see what you have to do. The point is to consider some operations that give the polytope more faces.

One is stellar subdivision of triangular facets, as in Lecture 3, Problem 0 in Homework 7. If a polytope has a triangular facet, find a point *beyond* that facet and take the convex hull of the simplex and that new point. We lose one facet, but create 1 new vertex, 3 new edges, and 3 new (triangular) facets. In effect, we increase a by 1 and increase b by 2. Importantly, the result is a polytope that continues to have a triangular facet, and also has a simple vertex (a vertex incident to exactly 3 edges).

The other operation is the dual of stellar subdivision, "shaving a simple vertex", also discussed in Lecture 3, Problem 0 in Homework 7. If a polytope has a simple vertex, find a hyperplane H that separates that vertex from all other vertices. Replace the polytope by its intersection with the halfspace defined by H (the one containing all the other vertices). We lose one vertex, but create 3 new vertices, 3 new edges, and 1 new (triangular) facet. That is increasing a by 2 and increasing b by 1. The result is a polytope that continues to have a simple vertex and now has a triangular facet.

So, what do we do? For convenience, let's think of vectors $[a\,b]$ in $\mathbb{Z}^2$. We want to be able to find a polytope for every $[a\,b]$ with $2a \geq b \geq 0$ and $2b \geq a \geq 0$. We find a finite set C of vectors $[a\,b]$ such that every vector $[a\,b]$ with $2a \geq b \geq 0$ and $2b \geq a \geq 0$ can be obtained by adding integer multiples of $[1\,2]$ and $[2\,1]$. This set is $\{[0\,0], [1\,1], [2\,2]\}$.

If we can find polytopes corresponding to these three vectors $[a\,b]$ and if they each have at least one simple vertex and at least one triangular facet, then we can get every possible $[a\,b]$ using iterated stellar subdivision of triangular facets and shaving of simple vertices.

We can:

- $[0\,0]$ gives f-vector $(1, 4, 6, 4)$, which is the tetrahedron.
- $[1\,1]$ gives f-vector $(1, 5, 8, 5)$, which is the Egyptian pyramid.
- $[2\,2]$ gives f-vector $(1, 6, 10, 6)$. There is a suitable polytope with this f-vector, but it is not one with a name that I knew. Take a triangular prism and "nudge" one of the vertices so that one of the quadrilateral faces "breaks" into two triangles but the other two quadrilaterals don't. Since Wikipedia has everything, you can find both a picture and a name for this on the page entitled Hexahedron.

Additional Problem 1

The tetrahedron has f-vector $(1, 4, 6, 4)$. Its h-vector is $(1, 1, 1, 1)$.

$$\begin{array}{ccccccccc} &&&4&&?&&& \\ &&6&&?&&?&& \\ &4&&?&&?&&?& \\ 1&&1&&1&&1&&1 \end{array} \longrightarrow \begin{array}{ccccccccc} &&&4&&1&&& \\ &&6&&3&&1&& \\ &4&&3&&2&&1& \\ 1&&1&&1&&1&&1 \end{array}$$

The octahedron has f-vector $(1, 6, 12, 8)$. Its h-vector is $(1, 3, 3, 1)$.

$$\begin{array}{ccccccccc} &&&8&&?&&& \\ &&12&&?&&?&& \\ &6&&?&&?&&?& \\ 1&&1&&1&&1&&1 \end{array} \longrightarrow \begin{array}{ccccccccc} &&&8&&1&&& \\ &&12&&7&&3&& \\ &6&&5&&4&&3& \\ 1&&1&&1&&1&&1 \end{array}$$

The icosahedron has f-vector $(1, 12, 30, 20)$. Its h-vector is $(1, 9, 9, 1)$.

$$\begin{array}{ccccccccc} &&&20&&?&&& \\ &&30&&?&&?&& \\ &12&&?&&?&&?& \\ 1&&1&&1&&1&&1 \end{array} \longrightarrow \begin{array}{ccccccccc} &&&20&&1&&& \\ &&30&&19&&9&& \\ &12&&11&&10&&9& \\ 1&&1&&1&&1&&1 \end{array}$$

Additional Problem 2

The 0 dimensional permutohedron has f-vector (1) and h-vector (1).
The 1 dimensional permutohedron has f-vector $(1, 2)$ and h-vector $(1, 1)$.
The 2 dimensional permutohedron (a hexagon) has f-vector $(1, 6, 6)$ and h-vector $(1, 4, 1)$.

$$\begin{array}{ccccccc} &&6&&?&& \\ &6&&?&&?& \\ 1&&1&&1&&1 \end{array} \longrightarrow \begin{array}{ccccccc} &&6&&1&& \\ &6&&5&&4& \\ 1&&1&&1&&1 \end{array}$$

The 3 dimensional permutohedron (a hexagon) has f-vector $(1, 24, 36, 14)$. Up to now (because of low dimensions) these polytopes have been isomorphic to their duals. Now we have to dualize. The dual has f-vector $(1, 14, 36, 24)$ and h-vector $(1, 11, 11, 1)$.

$$\begin{array}{ccccccccc} &&&24&&?&&& \\ &&36&&?&&?&& \\ &14&&?&&?&&?& \\ 1&&1&&1&&1&&1 \end{array} \longrightarrow \begin{array}{ccccccccc} &&&24&&1&&& \\ &&36&&23&&11&& \\ &14&&13&&12&&11& \\ 1&&1&&1&&1&&1 \end{array}$$

In general, the h-vector of the $(n-1)$-dimensional permutohedron has $h_k = A(n,k)$, where $A(n,k)$ is the Eulerian number, the number of permutations of $[n]$ with exactly k ascents (adjacent entries that are in correct numerical order). Here is one way to see why:

Recall that the weak order is a partial order (in fact a lattice) on permutations whose Hasse diagram (if we think of it as an undirected graph) coincides with the graph of the permutohedron. Thus we can think of the weak order as a partial order on the facets of the dual to the permutohedron, and two facets share a ridge if and only if they are related by a cover in the weak order. By results of your last homework, the face fan of the dual permutohedron is the normal fan of the permutohedron, and that's the fan cut out by the hyperplanes $x_i = x_j$ for $i \neq j$. Thus we can think of the weak order as a partial order on the cones of that fan. As we discussed in class, we can figure out which direction a cover relation between adjacent cones goes by a linear functional that has its maximum in the region $x_1 \leq x_2 \leq \cdots \leq x_n$.

The upshot: If we order the maximal cones as a linear extension of the weak order, every maximal cone's "old faces" are a union of facets, specifically those facets whose normals have positive inner product with the linear functional. Thus this linear order is a shelling (on the fan or on the boundary complex).

The restriction face of a cone is the intersections of the facets of the cone whose normals have negative inner product with the linear functional. The dimension of the restriction face is the number of ways to go up from the cone in the weak order. That's the number of ascents of the corresponding permutation.