MA 724
Homework 7, Comments and some solutions.

Lecture 6, Problem 1

I'll give 2 sample solutions.

Sample solution 1. Suppose X and X' are affine point configurations with the same set of minimal affine dependencies. We may as well assume that $\mathrm{aff}(X) = \mathbb{R}^d$, because otherwise we could restrict to the affine subspace $\mathrm{aff}(X)$. Thus X has a subset consisting of $d+1$ affinely independent vectors, and we may as well take them to be $\vec{x}_1, \ldots, \vec{x}_{d+1}$. (We could do this by renumbering X, and we would then renumber X' in the same way, to preserve the "same set of minimal affine dependencies" supposition.) We also may as well assume that $\vec{x}_{d+1} = \vec{0}$, because we could subtract $\vec{x}_{d+1}$ from every vector in X without changing the afine dependencies. Similarly, we may as well assume $\vec{x}'_{d+1} = \vec{0}$.

Define a linear map T that sends $\vec{x}_i$ to $\vec{x}'_i$ for all $i = 1, \ldots, d$. This map also sends $\vec{x}_{d+1}$ to $\vec{x}'_{d+1}$, because these are both $\vec{0}$. Suppose $i > d+1$. I claim that T sends $\vec{x}_i$ to $\vec{x}'_i$ as well. Since $\vec{x}_1, \ldots, \vec{x}_{d+1}$ are affinely independent, their affine hull is $\mathbb{R}^d$, so x_i is in their affine hull. Choose a minimal subset $\{\vec{x}_j : j \in S\}$ of $\{\vec{x}_1, \ldots, \vec{x}_{d+1}\}$ having $\vec{x}_i$ in its affine hull, and write $\vec{x}_i = \sum_{j\in S} \lambda_j \vec{x}_j$ with $\sum_{j\in S} \lambda_j = 1$. Then $x_i + \sum_{j\in S}(-\lambda_j)\vec{x}_j = 0$ is an affine dependence, and in fact, it's a minimal one: If we remove i from the set $S \cup \{i\}$ of indices, we don't get an affine dependence using the remaining indices, because $\{x_j : j \in S\}$ is affinely independent. For the same reason, and because S was chosen to be minimal, we can't remove any $j \in S$ from $\{x_j : j \in S\}$. Since this is a minimal dependence, the same dependence holds among the $\vec{x}'$, namely $x'_i + \sum_{j\in S}(-\lambda_j)\vec{x}'_j = 0$, or in other words, $\vec{x}'_i = \sum_{j\in S} \lambda_j \vec{x}'_j$. Since T is linear, $T(\vec{x}_i) = \sum_{j\in S} \lambda_j T(\vec{x}_j) = \sum_{j\in S} \lambda_j \vec{x}'_j = x'_i$. □

Why did I define a linear map that sends $\vec{x}_i$ to $\vec{x}'_i$ for all $i = 1, \ldots, d$ instead of an affine map that sends $\vec{x}_i$ to $\vec{x}'_i$ for all $i = 1, \ldots, d+1$? Well, an affine map is a linear map followed by a translation, but since I translated things to make $\vec{x}_{d+1} = \vec{x}'_{d+1} = \vec{0}$, for such an affine map, the translation would have to be $\vec{0}$ anyway. If you didn't move $\vec{x}_{d+1}$ and $\vec{x}'_{d+1}$ to $\vec{0}$, everything should come out the same (but you'll need to use the fact that the coefficients of an affine combination sum to 1, or the equivalent fact that the coefficients of an affine dependency sum to 0).

Sample solution 2. Let X be an affine point configuration. We first check that every affine dependence is a linear combination of minimal affine dependences. Let $\vec{z} \in \mathbb{R}^n$ be a nonzero affine dependence. If $\vec{z}$ is not a minimal dependence, let $\vec{z}\,'$ be a minimal nonzero affine dependence whose support is strictly contained in the support of $\vec{z}$. Then there exists $i \in [n]$ such that neither z_i nor z'_i is zero. Writing $c = z_i/z'_i$, the vector $\vec{z} - c\vec{z}\,'$ is a nonzero affine dependence with support smaller than the support of $\vec{z}$. Continuing in this manner with $\vec{z} - c\vec{z}\,'$, we eventually write $\vec{z}$ as a linear combination of minimal affine dependences.

Suppose X and X' are affine point configurations with the same set of minimal affine dependencies. By the paragraph above, X and X' have the same set of affine dependencies. We will assume that both affine point configurations affinely span $\mathbb{R}^d$, because otherwise we could restrict to an affine subspace. (The assertion that the points live in an affine hyperplane is itself a statement about affine dependencies, so both would be restricted to the same affine subspace.) Linearizing both as in Problem 3, we obtain two acyclic vector configurations $V = \left[\begin{smallmatrix}\mathbb{1}\\ X\end{smallmatrix}\right]$ and $V' = \left[\begin{smallmatrix}\mathbb{1}\\ X'\end{smallmatrix}\right]$. Problem 3 says that V and V' have the same set of linear dependencies.

Since X and X' both affinely span $\mathbb{R}^d$, V and V' both linearly span $\mathbb{R}^{d+1}$. Choose $d+1$ vectors in V that form a basis. We may as well take them to be $\vec{v}_1, \ldots, \vec{v}_{d+1}$. (We could do this by renumbering V, and we would then renumber V' in the same way, to preserve the fact that they have the same set of linear dependencies.) We can write every other vector in V as a linear combination of this basis, and these are linear dependecies. Thus the corresponding vectors in V' can be written as the same linear combinations of the $\vec{v}'_1, \ldots, \vec{v}'_{d+1}$. Thus if we map $\vec{v}_i$ to $\vec{v}'_i$ for $i = 1, \ldots, d+1$ and extend linearly, we get a linear isomorphism from V to V'. Since all the points in V and V' have 0^{th} coordinate 1, the linear map preserves the affine hyperplane where the points live. The linear map restricts to an affine map on this hyperplane, which in particular is an affine isomorphism from X to X'. □

Lecture 6, Problem 3

$$\text{a-Dep}(X) = \left\{\vec{z} \in \mathbb{R}^n : X\vec{z} = \vec{0},\ \mathbb{1}\vec{z} = 0\right\} = \left\{\vec{z} \in \mathbb{R}^n : \left[\begin{smallmatrix}\mathbb{1}\\X\end{smallmatrix}\right]\vec{z} = \vec{0}\right\} = \text{Dep}\left(\left[\begin{smallmatrix}\mathbb{1}\\X\end{smallmatrix}\right]\right)$$

$$\text{a-Val}(X) = \left\{\vec{c}X + z\mathbb{1} \in (\mathbb{R}^n)^* : \vec{c} \in (\mathbb{R}^d)^*,\ z \in \mathbb{R}\right\} = \left\{\vec{c}\left[\begin{smallmatrix}\mathbb{1}\\X\end{smallmatrix}\right] \in \mathbb{R}^n : \vec{c} \in (\mathbb{R}^{d+1})^*\right\} = \text{Val}\left(\left[\begin{smallmatrix}\mathbb{1}\\X\end{smallmatrix}\right]\right)$$

(I replaced $-z$ by $+z$ in the definition of a-Val(X) because z is an arbitrary real number and this seemed simpler. In the third term of the second equation, the new $\vec{c}$ is $[1\ \ (\text{old}\,\vec{c})]$.

Lecture 6, Problem 5

Sample proof 1 that a vector configuration is acyclic if and only if its dual is totally acyclic. There exists $\vec{c} \in (\mathbb{R}^r)^*$ with $\vec{c}V > \mathbb{0}$ (i.e. $\vec{c}\vec{v}_i > 0$ for all $i \in [n]$) if and only if there exists $\vec{c} \in (\mathbb{R}^r)^*$ with $\vec{c}V \geq \mathbb{1}$. This is (replacing $\vec{c}$ by $-\vec{c}$) if and only if there exists $\vec{c} \in (\mathbb{R}^r)^*$ with $\vec{c}V \leq -\mathbb{1}$. By Farkas I (written in dual form), there exists $\vec{c} \in (\mathbb{R}^r)^*$ with $\vec{c}V \leq -\mathbb{1}$ if and only if there does not exist $\vec{y} \in \mathbb{R}^n$ with $\vec{y} \geq \vec{0}$, $V\vec{y} = \vec{0}$, and $-\mathbb{1}\vec{y} < 0$. Thus the two characterizations are equivalent. □

Sample proof 2 that a vector configuration is acyclic if and only if its dual is totally acyclic. There exists $\vec{c} \in (\mathbb{R}^r)^*$ with $\vec{c}V > \mathbb{0}$ (i.e. $\vec{c}\vec{v}_i > 0$ for all $i \in [n]$) if and only if there exists $\vec{c}' \in (\mathbb{R}^{r+1})^*$ with $\vec{c}'\left[\begin{smallmatrix}\mathbb{1}\\V\end{smallmatrix}\right] \geq 0$ and $\vec{c}'\left[\begin{smallmatrix}1\\\vec{0}\end{smallmatrix}\right] < 0$. (The vector $\vec{c}'$ would be $[a\,\vec{c}]$, and $\vec{c}'\left[\begin{smallmatrix}1\\\vec{0}\end{smallmatrix}\right] < 0$ would say $a < 0$, while $\vec{c}'\left[\begin{smallmatrix}\mathbb{1}\\V\end{smallmatrix}\right] \geq 0$ would say each $\vec{v}_i$ has $a + \vec{c}\vec{v}_i \geq 0$, so that $\vec{c}\vec{v}_i > 0$.) By Farkas I, this is if and only if there does not exist $\vec{x} \geq 0$ with $\left[\begin{smallmatrix}\mathbb{1}\\V\end{smallmatrix}\right]\vec{x} = \left[\begin{smallmatrix}1\\\vec{0}\end{smallmatrix}\right]$. This is if and only if there does not exist $\vec{x} \geq 0$ with $V\vec{x} = \vec{0}$ and the sum of the coordinates of $\vec{x}$ equal to 1. By scaling, that is if and only if there does not exist $\vec{x} \geq 0$ with $\vec{x} \neq \vec{0}$ and $V\vec{x} = \vec{0}$. Thus the two characterizations are equivalent. □

The rest of the problem. The statement about the dual of an acyclic configuration follows immediately.

Finally, we want to find a vector configuration that admits a nonnegative dependence but no positive dependence. For example, take three vectors in $\mathbb{R}^2$, the column vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, so that $V = \left[\begin{smallmatrix}1 & -1 & 0\\0 & 0 & 1\end{smallmatrix}\right]$. Then there is a nonnegative dependence $\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right] \in \mathbb{R}^3$ but there is no $\vec{x} \in \mathbb{R}^3$ with $\vec{x} > \vec{0}$ such that $V\vec{x} = \vec{0}$ because the second entry of $V\vec{x}$ will always be x_2. □

The last point in the problem was possibly confusing because of the way Ziegler defines things: The definition of totally cyclic was given in terms of collections of row vectors. But you wanted to find a configuration of vectors that had a nonnegative dependence but no positive dependence, so your configuration needed to consist of either column vectors or row vectors once and for all, instead of changing your mind about rows/columns in the middle of the problem.

Lecture 6, Problem 8

As we discussed in class when we did the 3-polytopes with 5 vertices example, a 0-dimensional Gale diagram is the Gale diagram of a polytope if and only if it has at least 2 white vertices and at least 2 black vertices. Up to swapping white and black, it is then easy to see that the four possibilities listed are the only four polytopes.

You could do all of these mechanically by finding positive circuits and taking complements, then taking intersections of facets to get lower-dimensional faces. But for some cases, I used a shortcut, recalling that adding a gray vertex corresponds to taking a pyramid. In the other cases, the facets were simplices, so instead of taking all intersections, I could just take all subsets. In every case, it would be good to draw a Schlegel diagram, although I am not providing them here.

<u>2 black, 2 white, and 2 gray:</u> The only 2-polytope with 4 vertices is a quadrilateral, so this is a pyramid over a pyramid over a quadrilateral, or in other words a pyramid over an Egyptian pyramid. It has 6 facets, 13 ridges, 13 edges, and 6 vertices. It is not simplicial, because it has a facet that is an Egyptian pyramid. It is not simple, because it has a vertex contained in 5 facets.

<u>3 black, 2 white, and 1 gray:</u> As discussed in class, 3 black and 2 white is a bipyramid over a triangle, so with a gray point, it is a pyramid over a triangular bipyramid. It has 7 facets, 15 ridges, 14 edges, and 6 vertices. It is not simplicial, because it has a facet that is a bipyramid. It is not simple, because it has a vertex contained in 5 facets.

<u>3 black and 3 white:</u> Number the blacks $1, 2, 3$ and the whites $4, 5, 6$. The positive circuits are all pairs consisting of a black vertex and a white vertex, so the facets are all sets consisting of 2 elements from $\{1, 2, 3\}$ and 2 elements from $\{4, 5, 6\}$. Since they are all 3-dimensional and each has 4 vertices, each is a simplex, i.e. the polytope is simplicial. In particular, the lower-dimensional faces' vertex sets are all subsets of these 4-sets. We see that the 2-dimensional faces' vertex sets are all sets with either 2 elements from $\{1, 2, 3\}$ and 1 element from $\{4, 5, 6\}$ or 1 element from $\{1, 2, 3\}$ and 2 elements from $\{4, 5, 6\}$. There are $2 \cdot \binom{3}{2}\binom{3}{1} = 18$ of them. All $\binom{6}{2} = 15$ pairs of vertices occur as edges. In summary, this is a simplicial polytope with 9 facets, 18 ridges, 15 edges, and 6 vertices. (It is not simple, because to be simple and simplicial, it would have to be a simplex.) Since all pairs of vertices occur as edges, this is a neighborly polytope. Since the two previous polytopes are not simplicial and we will see that the last polytope is not neighborly, we conclude that this is the cyclic polytope.

<u>4 black and 2 white:</u> Number the blacks $1, 2, 3, 4$ and the whites $5, 6$. Again, the positive circuits are all pairs consisting of a black vertex and a white vertex, so this time the facets are all sets consisting of 3 elements from $\{1, 2, 3, 4\}$ and 1 element from $\{5, 6\}$. Again, these are all simplices, so the vertex sets of lower-dimensional faces are all of the subsets of the vertex sets of facets. All 3-element sets except those containing $\{5, 6\}$ occur for 2-faces, so there are $\binom{6}{3} - 4 = 16$ of them. All 2-element sets except $\{5, 6\}$ occur for edges, so there are $\binom{6}{2} - 1 = 14$. In summary, this is simplicial but not simple, has 8 facets, 16 ridges, 14 edges, and 6 vertices, and is not neighborly. This is the bipyramid over a 3-simplex. (One might see that by process of elimination. Alternatively, notice that there is exactly one pair of vertices that does not form an edge: $\{5, 6\}$. These are the bipyramid points. Now, notice that for every proper subset S of $\{1, 2, 3, 4\}$, there is a face $S \cup \{5\}$ and a face $S \cup \{6\}$.)

Lecture 6, Problem 9

I didn't assign this because it was much too long, if you drew Schlegel diagrams and computed vertex-facet matrices in every case. The first time I assigned it to a class, I thought that, since Ziegler gave it as a problem, just finding the Gale diagrams must be reasonable. I no longer think so. Let's give some indications of how you might start. Let P be the polytope described by the Gale diagram.

First, let's take care of gray vertices. Three gray vertices would be a pyramid over a pyramid over a pyramid over a 1-polytope with 4 vertices, but 1-polytopes have 2 vertices. More than three gets even more impossible. Thus there could be at most 2 gray vertices, in which case, P is a pyramid over a pyramid over a 2-polytope with 5 vertices. If there is 1 gray vertex, then P is a pyramid over a 3-polytope P' with 6 vertices (whose Gale diagram has no gray points). Determining the Gale diagram for P' is a good "warmup" if you were determined to do the whole problem.

Recall that the condition is that, if we pick any of the points and choose either direction of the line to be "positive," the number of black points on the positive side plus the number of white points on the negative side is at least 2. If you start trying to write down all the Gale diagrams on a line with 6 vertices (none gray), you will find a very large number! The problem is that each linear Gale diagram in $\mathbb{R}^2$ can be affinized in many ways, and we would list all of those separately and then be stuck with the task of figuring out which are combinatorially the same polytope. (I think this becomes reasonable if you read Grünbaum's book, Convex Polytopes, and learn how to cut down the symmetry. He writes his Gale diagrams on a circle, not a line, and had no black and white points.) Amazingly, there are only 6 of these (plus the pentagonal pyramid). See
http://www.mi.fu-berlin.de/math/groups/discgeom/ziegler/Preprintfiles/075PREPRINT.pdf.

The problem with no gray points is a step more complicated! If you're interested in the answer, it's also in the Grünbaum book.

Lecture 6, Problem 13

Here is the proof of what I think Ziegler meant to ask: (Prove that the positive co-circuits for the cyclic polytope coincide with the positive circuits of the affine point configuration shown.)

Proof of duality for positive (co-)circuits. Gale's Evenness condition tells us that every facet of $C_d(n)$ has d vertices and that a d-subset S is a facet if and only if for all $i < j$ not in S, the number $\#\{k \in S : i < k < j\}$ is even. Equivalently, a subset C is a positive co-circuit if and only if it has $n - d$ elements and for all $i < j$ in C, the number $\#\{k \notin C : i < k < j\}$ is even. Thus an affine point configuration with colored points is a Gale diagram for $C_d(n)$ if and only if it satisfies: C is a positive circuit if and only if it has $n-d$ elements and for all $i<j$ in C, the number $\#\{k \notin C : i<k<j\}$ is even. Applying this to our alternating black-white points on the moment curve in $\mathbb{R}^{n-d-2}$, we need to show that C is a positive circuit if and only if it has $n-d$ points and alternates black and white.

We begin by showing that given any $k+2$ points on the moment curve in $\mathbb{R}^k$, alternating black and white, the convex hull of the black points intersects the convex hull of the white points. If the points are $\vec{x}(t_i)$ for $t_1 < t_2 < \cdots < t_{k+2}$, this amounts to showing that there exist nonnegative constants $\lambda_1, \ldots, \lambda_{k+2}$ with $\lambda_1 + \lambda_3 + \cdots = 1$ and $\lambda_2 + \lambda_4 + \cdots = 1$ such that

$$\lambda_1 \vec{x}(t_1) + \lambda_3 \vec{x}(t_3) + \cdots = \lambda_4 \vec{x}(t_4) + \lambda_4 \vec{x}(t_4) + \cdots .$$

Equivalently, there exists $\vec{z} \in \mathbb{R}^{k+2}$ such that $\vec{z} \geq \mathbb{0}$ and

$$\begin{bmatrix} 1 & 0 & \cdots & * \\ 0 & 1 & \cdots & * \\ t_1 & -t_2 & \cdots & (-1)^k t_{k+2} \\ t_1^2 & -t_2^2 & \cdots & (-1)^k t_{k+2}^2 \\ \vdots & \vdots & & \vdots \\ t_1^k & -t_2^k & \cdots & (-1)^k t_{k+2}^k \end{bmatrix} \vec{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Where the first two rows of the matrix on left alternate 0 and 1 and the entries $*$ represent 0 or 1 as appropriate. By Farkas II, this is equivalent to the *non*existence of $\vec{c} = [a, b, c_1, \ldots, c_k] \in (\mathbb{R}^{k+2})^*$ such that $a+b<0$ (equivalently $-a>b$) and

$$\vec{c} \begin{bmatrix} 1 & 0 & \cdots & * \\ 0 & 1 & \cdots & * \\ t_1 & -t_2 & \cdots & (-1)^k t_{k+2} \\ t_1^2 & -t_2^2 & \cdots & (-1)^k t_{k+2}^2 \\ \vdots & \vdots & & \vdots \\ t_1^k & -t_2^k & \cdots & (-1)^k t_{k+2}^k \end{bmatrix} \geq \mathbb{0}.$$

But if such a $\vec{c}$ exists, then $c_1 t_i + c_2 t_i^2 + \cdots + c_k t_i^k \geq -a$ for $i \in [k+2]$ odd and $c_1 t_i + c_2 t_i^2 + \cdots + c_k t_i^k \leq b$ for $i \in [k+2]$ even. Since $-a > b$, we see that the polynomial $\frac{b-a}{2} + c_1 t_i + c_2 t_i^2 + \cdots + c_k t_i^k$ has at least $k+1$ zeros, which is impossible since its degree is k.

On the other hand, given any set of points on the moment curve not containing an alternating black-white subsequence of $k+2$ points, writing down the appropriate matrix, we find that we can find such a $\vec{c}$. (We find a polynomial with simple zeros between t_i and t_{i+1} whenever points i and $i+1$ change color and no other zeros. We choose, say, $a=-1$ and $b=0$ and scale the polynomial large enough so that it takes values $\geq -a = 1$ at some t_i and $\leq b = 0$ at other t_i as appropriate. The coefficients of the resulting polynomial are the c_i such that $\vec{c} = [a, b, c_1, \ldots, c_k]$.)

These considerations (with $k = n-d-2$) show the desired fact: C is a positive circuit if and only if it has $n-d$ points and alternates black and white. □

Here is the proof of what Ziegler actually asked. (Prove the two oriented matroids are dual.)

Proof that the oriented matroids are actually dual. We will find the signed co-circuits of the point configuration consisting of vertices of $C_d(n)$ (i.e. n points on the moment curve in $\mathbb{R}^d$) and show that they coincide with the signed circuits of the configuration consisting of n points on the moment curve in $\mathbb{R}^{n-d-2}$, alternating between black and white points.

Once we find the signed co-circuits of the point configuration consisting of vertices of $C_d(n)$, the proof that these are the cocircuits of the alternating black/white points consists of a sequence of rephrasings of what we are trying to prove, until it is just a fact about zeros of polynomials.

First, consider the n points on the moment curve in $\mathbb{R}^d$. We proved in class that any $d+1$ of them are affinely independent. Thus any d of them are affinely independent and span a hyperplane. In other words, for any d vertices, there is a signed circuit (or necessarily, two) whose zero positions are those d positions. To work out the signs on the remaining positions, we recall what we learned in the Gale Evenness proof: As t varies, the point $\vec{x}(t)$ on the moment curve passes through the hyperplane exactly when it passes through one of the d vertices that define the hyperplane. So the signed signed co-circuits are exactly the sequences in $\{+, -, 0\}^n$ described as follows:

(*) A sequence of $+$ entries, zero, a sequence of $-$ entries, zero, sequence of $+$ entries, zero, and so forth until there are d zeros and $d+1$ sequences. (Any of the sequences may be empty.) Or, the same thing starting with a sequence of $-$ entries.

We now prove that the signed circuits of n points on the moment curve in $\mathbb{R}^{n-d-2}$, alternating between black and white points, are the same as the signed co-circuits described above. Recall how we find circuits in a configuration of white and black points: We find circuits as usual, and then change the sign on all white points. Changing the sign on every other point in the descrition (*), we obtain the following description:

(**) A sequence with exactly d zeros and the nonzero entries alternating sign.

Thus we need to show that given n points on the moment curve in $\mathbb{R}^{n-d-2}$ (with no black/white coloring, or if you prefer, with all points black), a sequence in $\{+,-,0\}^n$ is a signed vector if and only if at most d entries are zero and the nonzero entries alternate sign.

By the definition of signed vectors and signed circuits, we need to prove the following: Given disjoint subsets S and T of the n points, the intersection $\operatorname{conv}(S) \cap \operatorname{conv}(T)$ is nonempty if and only if there exist $S' \subseteq S$ and $T' \subseteq T$ such that $|S'| + |T'| = n - d$ and S' and T' interleave. (This means that, listing the points in $S' \cup T'$ in order along the moment curve, we have an element of S', then an element of T', an element of S', of T, etc., or similarly starting with T'.) Equivalently, we need to prove that when we list the elements of $S \cup T$ in order, we switch from S to T or from T to S at least $n - d - 1$ times.

This is something we can prove without worrying about the original n points: We just need to show that given disjoint sets S and T of points on the moment curve in $\mathbb{R}^k$, the intersection $\operatorname{conv}(S) \cap \operatorname{conv}(T)$ is nonempty if and only if when we list the points of $S \cup T$ in order, we switch from S to T at least $k+1$ times.

Then the intersection $\operatorname{conv}(S) \cap \operatorname{conv}(T)$ is nonempty if and only if there exist a nonnegative constants $\lambda_{\vec{s}}$ for each $\vec{s} \in S$ and $\lambda_{\vec{t}}$ for each $\vec{t} \in T$ such that $\sum_{\vec{s}\in S} \lambda_{\vec{s}} = 1$ and $\sum_{\vec{t}\in T} \lambda_{\vec{t}} = 1$ and $\sum_{\vec{s}\in S} \lambda_{\vec{s}}\, \vec{s} = \sum_{\vec{t}\in T} \lambda_{\vec{t}} \vec{t}$. Equivalently, there exists $\vec{x} \in \mathbb{R}^k$ such that $\vec{x} \geq \mathbb{0}$ and

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_\ell & -u_1 & -u_2 & \cdots & -u_m \\ t_1^2 & t_2^2 & \cdots & t_\ell^2 & -u_1^2 & -u_2^2 & \cdots & -u_m^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_1^k & t_2^k & \cdots & t_\ell^k & -u_1^k & -u_2^k & \cdots & -u_m^k \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Here, $S = \{\vec{x}(t_1), \ldots, \vec{x}(t_\ell)\}$, $T = \{\vec{x}(u_1), \ldots, \vec{x}(u_m)\}$, and $\vec{x}$ would correspond to the vector of λ's.

By Farkas II, this is equivalent to the *non*existence of $\vec{c} = [a, b, c_1, \ldots, c_k] \in (\mathbb{R}^{k+2})^*$ such that $a + b < 0$ (equivalently $-a > b$) and

$$\vec{c} \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_\ell & -u_1 & -u_2 & \cdots & -u_m \\ t_1^2 & t_2^2 & \cdots & t_\ell^2 & -u_1^2 & -u_2^2 & \cdots & -u_m^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_1^k & t_2^k & \cdots & t_\ell^k & -u_1^k & -u_2^k & \cdots & -u_m^k \end{bmatrix} \geq \mathbb{0}.$$

So, we are trying to prove that such a $\vec{c}$ fails to exist if and only if when we list the points of $S \cup T$ in order, we switch from S to T at least $k+1$ times.

The condition on $\vec{c}$ is equivalent to $c_1 t_i + c_2 t_i^2 + \cdots + c_k t_i^k \geq -a$ for $i \in [\ell]$ and $c_1 u_j + c_2 u_j^2 + \cdots + c_k t_i^k \leq b$ for $i \in [m]$.

If such a $\vec{c}$ exists, then since $-a > b$, the number of zeros of the polynomial $\frac{b-a}{2} + c_1 t_i + c_2 t_i^2 + \cdots + c_{n-d-2} t_i^{n-d-2}$ is at least the number of times that we switch from S to T, when we list the points of $S \cup T$ in order. Since the polynomial has degree k, it has at most k zeros, so we switch from S to T at most k times.

On the other hand, if we switch from S to T exactly p times with $p \leq k$, we can choose points $\vec{x}(v_1), \ldots, \vec{x}(v_p)$ on the moment curve, strictly between the points of S and T every place where they switch. Take a polynomial with simple zeros at each $\vec{v}_i$, and scale it large enough so that it takes values $> -a$ for each t_i and $< b$ for each u_i. The coefficients of the resulting polynomial are the c_i such that $\vec{c} = [a, b, c_1, \ldots, c_k]$. □