MA 724
Homework 6, Comments and some solutions.

Lecture 3, Problem 0

(i) Stellar subdivision is closely related to the operator Pyr. Stellar subdivision of a facet F amounts to constructing the polytope $\mathrm{Pyr}(F)$ and "gluing" it to P along F. More formally, the faces of $\mathrm{st}(P,F)$ consist of all the faces of P except F together with the following additional faces: For each proper face G of F, there is a face $\mathrm{conv}(G \cup \{\vec{y}_F\})$ (a pyramid over G).
(ii) If P is simplicial, then F is a simplex, so every face G of F is a simplex, so every new face $\mathrm{conv}(G \cup \{\vec{y}_F\})$ of $\mathrm{st}(P,F)$ is a pyramid over a simplex, and thus a simplex. (Just think about the definition: A simplex is the convex hull of affinely independent points, and when we take the pyramid, we put in one more point to make a larger affinely independent set.) So $\mathrm{st}(P,F)$ is simplicial because P was simplicial and each new proper face is a simplex.

For $k = d$, there is one k-face of P and one k-face of $\mathrm{st}(P,F)$. For $k < d$, $\mathrm{st}(P,F)$ has a new k-face for each $(k-1)$-face of F. But since F is $(d-1)$-simplex (and thus has d vertices), the number of $(k-1)$-faces is $\binom{d}{k}$. So for $k < d-1$, $f_k(\mathrm{st}(P,F)) = f_k(P) + \binom{d}{k}$. For $k = d-1$, $f_k(\mathrm{st}(P,F)) = f_k(P) + d - 1$, because we lose F and only gain d new faces.
(iii) Another way to phrase this question (replacing P by $P^\triangle$ in Ziegler's formulation): What operation $\mathrm{st}^\triangle$ on $P^\triangle$ has $\mathrm{st}^\triangle(P^\triangle, F^\diamond) = \mathrm{st}(P,F)^\triangle$?

Since we're talking about polar duality, we should think a bit about the face lattice. This especially makes sense when you think—in the language of part (i)—of "constructing $\mathrm{Pyr}(F)$ and gluing it to P along F". How do we get $L(\mathrm{st}(P,F))$ from $L(P)$? We delete F and add an extra copy of the interval $[\emptyset, F)$—note that F is not included in this interval. The bottom of the new interval is the new vertex $\vec{y}_F$ and the other elements of the interval are $\mathrm{conv}(G \cup \{\vec{y}_F\})$ for $G \in [\emptyset, F)$. Each $\mathrm{conv}(G \cup \{\vec{y}_F\})$ is $G \vee \vec{y}_F$ in $L(\mathrm{st}(P,F))$.

What if we dualize this (by turning the face lattices upside down)? Then $L(\mathrm{st}^\triangle(P^\triangle, F^\diamond))$ is obtained from $L(P^\triangle)$ by deleting the vertex $F^\diamond$ and adding an extra copy of the interval $(F^\diamond, P]$ in such a way that each element of the new interval is $G \wedge (\vec{y}_F)^\diamond$ for a new facet $(\vec{y}_F)^\diamond$. We see that the operation polar to stellar subdivision is $\mathrm{st}^\triangle(P^\triangle, F^\diamond) = \left\{\vec{c} \in P^\triangle : \vec{c}\vec{z} \le a\right\}$ for some $\vec{z}$ and a chosen so that $\vec{c}\vec{z} \le a$ fails when $\vec{c}$ is the vertex $F^\diamond$ but is satisfied for every other vertex of $P^\triangle$. One might call this operation "shaving" the vertex $F^\diamond$ from $P^\triangle$. And of course, one could do a similar operation on a polytope P in space $\mathbb{R}^d$, which is how Ziegler phrased the question.
Comment: This makes sense with our intuition about duality: Stellar subdivision adds a vertex and the dual operation adds a new facet-defining hyperplane.

Lecture 3, Problem 3

One approach was to take the linear function $F_T(\vec{x})$ from the proof of Corollary 0.8. I don't think that's what Ziegler had in mind, though, and I asked you not to use Corollary 0.8. Here's what I think Ziegler had in mind:

Realize $C_d(n)$ as $C_d(t_1, \ldots, t_n) = \mathrm{conv}(\vec{x}(t_1), \ldots, \vec{x}(t_n))$ for $t_1 < \cdots < t_n$, where $\vec{x}(t)$ is the column vector with entries $t, t^2, \ldots, t^n$. Given i and j with $1 \le i < j \le n$, we need to find a linear inequality $\vec{a}\vec{x} \le \alpha$ that holds with equality for $\vec{x}(t_i)$ and $\vec{x}(t_j)$ but holds strictly for $\vec{x}(t_k)$ when $k \in ([n] \setminus \{i,j\}$. Taking $\vec{a} = (a_1, \ldots, a_d)$, this is equivalent to finding a polynomial $p(t) = a_1 t + \cdots a_d t^d$ such that the maximum of $p(t)$ on $\{t_1, \ldots, t_n\}$ is attained at t_i and t_j but not at any other points in $\{t_1, \ldots, t_n\}$. For $d < 4$, one can easily see that this is impossible, but for $d \ge 4$, it is easy: The polynomial $-(t-t_i)^2(t-t_j)^2$ is zero on t_i and t_j and negative on every other value of t. We are looking for a polynomial with constant term 0, so we take $p(t) = -(t-t_i)^2(t-t_j)^2 + t_i^2 t_j^2$. (Why are we looking for a polynomial with constant term 0? Look up higher in the paragraph: A linear functional applied to $\vec{x}(t)$ is precisely a polynomial with constant term 0.)

It surprised me at first that you can do this with a degree-4 polynomial for any $d \ge 4$. We can find these edges while ignoring dimensions 5 through d. But this should not be so surprising: The graph of $C_d(n)$ is complete for *any* $d \ge 4$ and furthermore if we project $C_d(n)$ to $\mathbb{R}^4$ by just ignoring the entries in positions 5 through d, we get exactly $C_4(n)$, with the same set of edges.

Lecture 3, Problem 4(i)

As mentioned in the statement of Problem 3, for $d \geq 4$, every cyclic polytope has a *complete* vertex-edge graph. If $n > d+1$, then the cyclic polytope $C_d(n)$ is not a simplex, because a d-simplex has $d+1$ vertices. So for every $d > 4$, we can make a 4-dimensional polytope whose graph is the same as the graph of the d-simplex.

If the graph of a d-simplex is to be dimensionally ambiguous, there would have to be a *lower*-dimensional polytope with the same graph. (No polytope with this graph could have dimension higher than d, because the affine hull of $d+1$ points cannot have dimension higher than d.) If $d < 4$, then we can rule out dimensional ambiguity for the simplex because we know very precisely what polytopes of dimensions -1, 0, 1, and 2 look like: Empty graphs, isolated points, line segments, and cycles. If $d = 4$, then the graph of the d-simplex is the complete graph K_5 with 5 vertices. Again, dimensions 0, 1, and 2 can't admit a polytope with this graph, and dimension 3 won't work because Steinitz' Theorem says the graph would have to be planar, but K_5 is not. (If you're not familiar with planarity and K_5 not being planar, then you could just as easily use the result you proved as Problem 0 of Lecture 0.)

Lecture 3, Problem 5

A d-polytope P in $\mathbb{R}^d$ is simple (by definition) if and only if each vertex $\vec{v}$ is contained in exactly d facets. The normals to the set of facets containing $\vec{v}$ must span $\mathbb{R}^d$, or else $\vec{v}$ and every vertex adjacent to $\vec{v}$ would be contained in a proper subspace of $\mathbb{R}^d$, and that would force P to be in the proper subspace. (Recall that we proved in class that P is contained in $\vec{v}+\mathrm{cone}(\{\vec{u}-\vec{v} : \vec{u} \in N(\vec{v})\})$, where $N(\vec{v})$ is the set of neighbors of $\vec{v}$.) Thus there are more than $\vec{d}$ facets containing $\vec{v}$ if and only if there is a linear dependence among facets of P containing $\vec{v}$. So: P is simple if and only if, for each vertex $\vec{v}$ of P, the set of normals of facets containing $\vec{v}$ is linearly independent.

Let $P = P(A, \vec{z})$. By Problem 14 of Lecture 2, the facet-defining inequalities of these facets are given by rows of $A\vec{x} \leq \vec{z}$. For each subset I of the set of indices of rows of A, let A_I be the submatrix of A consisting of the rows indexed by I, and let $\vec{z}_I$ be the corresponding subvector of $\vec{z}$. If P is not simple, then there exists a vertex $\vec{v}$ such that the set of normals of facets containing $\vec{v}$ is linearly dependent. In this case, we can choose a minimal linear dependence among $d+1$ of the normals. Let I be the first d indices in the support of the dependence and let J be the last d indices. By minimality, A_I and A_J are linearly independent, and since the corresponding facets all contain $\vec{v}$, the solution set to both $A_I\vec{x} = \vec{z}_I$ and $A_J\vec{x} = \vec{z}_J$ is $\{\vec{v}\}$.

For $\vec{z} = \vec{1}^{(\lambda)}$, we can thus avoid letting $P(A, \vec{z})$ be simple by making sure that, for every minimal linear dependence among $d+1$ rows of A with I and J as above, the solution sets to $A_I\vec{x} = \vec{z}_I$ and $A_J\vec{x} = \vec{z}_J$ don't coincide. The entries of A_I^{-1} and A_J^{-1} are constants, so this is a polynomial equation in λ. The equation is not tautologically true, because $\vec{z}_I$ has a smaller power of λ than occurs anywhere in $\vec{z}_J$. Taking all such equations for all minimal linear dependences among $d+1$ rows of A, we have finitely many polynomial equations in λ and we have to choose positive λ to make all of the equations false. We can thus choose λ to be positive, but smaller than the smallest positive zero of any of the equations, and we will make $P(A, \vec{1}^{(\lambda)})$ simple.

For λ small enough, the facets of the new polytope are in bijection with the facets of the old polytope. To see this, let F be a facet of $P(A, \vec{1})$ defined by the hyperplane $\{\vec{x} : \vec{a}\vec{x} = 1\}$ and let H' be the hyperplane $\{\vec{x} : \vec{a}\vec{x} = z\}$, where z is the appropriate entry of $\vec{1}^{(\lambda)}$. Choose a point $\vec{x}_F$ in the relative interior of the facet and let $\vec{x}'_F$ be the point in H' such that the line containing $\vec{x}_F$ and $\vec{x}'_F$ is perpendicular to H'. In particular, the distance $z-1$ between $\vec{x}_F$ and $\vec{x}'_F$ is a power of λ. Do this for every facet of $P(A, \vec{1})$. Now, there is some minimum distance that any $\vec{x}_F$ is from any other facet-defining hyperplane. As long as we choose λ small enough, the points $\vec{x}'_F$ will be in $P(A, \vec{1}^{(\lambda)})$ and will be contained in exactly one of the defining hyperplanes. Thus they will be in the relative interiors of facets, and we have shown that every row of $A\vec{x} \leq \vec{1}^{(\lambda)}$ defines a facet.

Why is the diameter of the graph of $P(A, \vec{1}^{(\lambda)})$ greater than or equal to the diameter of the graph of $P(A, \vec{1})$? Represent each edge of $P(A, \vec{1}^{(\lambda)})$ as an intersection of facets. Since $P(A, \vec{1}^{(\lambda)})$ is simple, it is an intersection of $d-1$ facets. Each vertex in the edge is the intersection of d facets (the $d-1$ facets containing the edge plus one more). There are two possibilities: Either the two sets of d facet-defining hyperplanes define the same point in $P(A, \vec{1})$ or they define different points.

So either the edge of $P(A,\vec{1}^{(\lambda)})$ corresponds to a vertex in $P(A,\vec{1})$ or it corresponds to an edge in $P(A,\vec{1})$. Given any vertex $\vec{v}$ of $P(A,\vec{1})$, we can choose (non-uniquely) a corresponding vertex $\vec{v}'$ in $P(A,\vec{1}^{(\lambda)})$, by finding any d facets of $P(A,\vec{1})$ containing $\vec{v}$ and intersecting the corresponding facets of $P(A,\vec{1}^{(\lambda)})$ to make $\vec{v}'$. Now given any two vertices of $P(A,\vec{1})$, we can look at two corresponding vertices in $P(A,\vec{1}^{(\lambda)})$, and make a path between them. The length of this path is bounded by the diameter of the graph of $P(A,\vec{1}^{(\lambda)})$. By contracting some edges to vertices, we get a path between the two vertices of $P(A,\vec{1})$. Thus all pairs of vertices of $P(A,\vec{1})$ are connected by a path whose length is bounded above by the diameter of the graph of $P(A,\vec{1}^{(\lambda)})$.

By the way, the part about "proving the Hirsch conjecture" can be ignored, since we now know that the Hirsch conjecture is false.