Chapter 1, Problem 3

I think Ziegler was looking for a constructive solution to this problem, i.e. given $C = \text{cone}(W)$, determine $V$ and $Y$ so that the set $\{ \vec{x} \in \mathbb{R}^d : \left( \begin{array}{c} 1 \\ \vec{x} \end{array} \right) \in C \}$ equals $\text{conv}(V) + \text{cone}(Y)$. I don’t see how to do that, but I admit I have not put hours and hours into it. But we can do this problem just by appealing to the results in Lecture 1, and that’s the point I was trying to get across when I assigned this. (But, good job to one of you for finding a nice way to do this!) Here goes:

By the Main Theorem, $C$ is also an $H$-polyhedron, so $\left\{ \left( \begin{array}{c} 1 \\ \vec{x} \end{array} \right) \in \mathbb{R}^{d+1} : \left( \begin{array}{c} 1 \\ \vec{x} \end{array} \right) \in C \right\}$ is also an $H$-polyhedron. (Take the defining inequalities for $C$ this. (But, good job to one of you for finding a nice way to do this!) Here goes:

Each vector in $V$ and $Y$ has 1 as it’s 0th coordinate, so we can write $\left\{ \vec{x} \in \mathbb{R}^d : \left( \begin{array}{c} 1 \\ \vec{x} \end{array} \right) \in C \right\}$ as a $V$-polyhedron in $\mathbb{R}^d$ by deleting all the 0th coordinates from the vectors in $V$ and the vectors in $Y$.

Chapter 1, Problem 4

Proposition 1. Given an $m \times d$ matrix $A$ and $\vec{z} \in \mathbb{R}^m$, then

Either there exists a vector $\vec{x} \in \mathbb{R}^d$ with $A\vec{x} \leq \vec{z}$ and $\vec{x} \geq \vec{0}$, or there exists $\vec{c} \in (\mathbb{R}^n)^*$ with $\vec{c} \geq \vec{0}$, $\vec{c}A = \vec{0}$, and $\vec{c}\vec{z} < 0$, but not both.

Proof. 

$\exists \vec{x} \in \mathbb{R}^d$ with $A\vec{x} \leq \vec{z}$, $-\vec{x} \leq \vec{0} \iff \exists \vec{x} \in \mathbb{R}^d$ with $\left( \begin{array}{c} -I \\ A \end{array} \right) \vec{x} \leq \left( \begin{array}{c} \vec{0} \\ \vec{z} \end{array} \right)$

(by Farkas I) $\iff \vec{z} (\vec{b}, \vec{c}) \in (\mathbb{R}^{d+n})^*$ with $(\vec{b}, \vec{c}) \geq \vec{0}$, $(\vec{b}, \vec{c}) \left( \begin{array}{c} -I \\ A \end{array} \right) = \vec{0}$, $(\vec{b}, \vec{c}) \left( \begin{array}{c} \vec{0} \\ \vec{z} \end{array} \right) < 0$

$\iff \vec{z} (\vec{b}, \vec{c}) \in (\mathbb{R}^{d+n})^*$ with $(\vec{b}, \vec{c}) \geq \vec{0}$, $-\vec{b} + \vec{c}A = \vec{0}$, $\vec{c}\vec{z} < 0$  

$\iff \vec{z} \vec{c} \in (\mathbb{R}^n)^*$ with $\vec{c} \geq \vec{0}$, $\vec{c}A = \vec{0}$, $\vec{c}\vec{z} < 0$  

This makes sense: either there is a solution with nonnegative coefficients or there is a positive linear combination of the defining inequalities which is obviously false for vectors with nonnegative coefficients.

Chapter 3, Problem 5

Proof. Suppose $A \in \mathbb{R}^{m \times d}$, $B \in \mathbb{R}^{m \times d}$, and $C \in \mathbb{R}^{p \times d}$, so that $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{w} \in \mathbb{R}^p$.

$\exists \vec{x} \in \mathbb{R}^d$ with $A\vec{x} = \vec{u}$, $B\vec{x} \geq \vec{v}$, $C\vec{x} \leq \vec{w} \iff \exists \vec{x} \in \mathbb{R}^d$ with $\left( \begin{array}{c} A \\ -A \\ -B \\ C \end{array} \right) \vec{x} \leq \left( \begin{array}{c} \vec{u} \\ -\vec{u} \\ -\vec{v} \\ \vec{w} \end{array} \right)$

(by Farkas I) $\iff \vec{z} (\vec{a}_1, \vec{a}_2, -\vec{b}, \vec{c}) \in (\mathbb{R}^{2m+n+p})^*$ with $(\vec{a}_1, \vec{a}_2, -\vec{b}, \vec{c}) \geq \vec{0}$, $(\vec{a}_1, \vec{a}_2, -\vec{b}, \vec{c}) \left( \begin{array}{c} A \\ -A \\ -B \\ C \end{array} \right) = \vec{0}$, $(\vec{a}_1, \vec{a}_2, -\vec{b}, \vec{c}) \left( \begin{array}{c} \vec{u} \\ -\vec{u} \\ -\vec{v} \\ \vec{w} \end{array} \right) < 0$

(setting $\vec{a} = \vec{a}_1 - \vec{a}_2$) $\iff \vec{z} (\vec{a}, \vec{b}, \vec{c}) \in (\mathbb{R}^{m+n+p})^*$ with $\vec{b} \leq \vec{0}$, $\vec{c} \geq \vec{0}$, $\vec{a}A + \vec{b}B + \vec{c}C = \vec{0}$, $\vec{a}\vec{u} + \vec{b}\vec{v} + \vec{c}\vec{w} < 0$
There were several ways to argue this. Here are three. In every case, the “easy” direction is exactly the same.

**Sample solution 1.** By definition, \( \text{lin}(P) \) equals \( \{ \vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P, t \in \mathbb{R} \} \). Fix \( x_0 \in P \) and let \( U \) be the set \( \{ \vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R} \} \). The inclusion \( \text{lin}(P) \subseteq U \) is immediate. Suppose \( \vec{y} \in U \). Given any point \( \vec{x} \in P \) and any \( t \in \mathbb{R} \), let \( L \) be the line segment \([\vec{x}_0, \vec{x} + t\vec{y}]\). All points on \( L \) are of the form \((1 - \lambda)\vec{x}_0 + \lambda(\vec{x} + t\vec{y})\) for \( 0 \leq \lambda \leq 1 \). For each \( \lambda \) with \( 0 \leq \lambda < 1 \), the point \( \vec{z} = (1 - \lambda)\vec{x}_0 + \lambda(\vec{x} + t\vec{y}) \) equals \((1 - \lambda)(\vec{x}_0 + t'\vec{y}) + \lambda\vec{x} \), where \( t' = t \frac{1}{1-\lambda} \). Since \( P \) is convex and since \( \vec{x} \) and \( \vec{x}_0 + t'\vec{y} \) are both in \( P \), we see that \( \vec{z} \in P \). Thus all of \( L \), except possibly \( \vec{x} + t\vec{y} \) is in \( P \). But \( P \) is closed, so \( \vec{x} + t\vec{y} \in P \) as well. We see that \( \vec{y} \in \text{lin}(P) \). □

**Sample solution 2.** By definition, \( \text{lin}(P) \) equals \( \{ \vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P, t \in \mathbb{R} \} \). Fix \( x_0 \in P \) and let \( U \) be the set \( \{ \vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R} \} \). The inclusion \( \text{lin}(P) \subseteq U \) is immediate.

On the other hand, suppose \( \vec{y} \not\in \text{lin}(P) \). Then there exists \( \vec{x} \in P \) and \( t \in \mathbb{R} \) such that \( \vec{x} + t\vec{y} \not\in P \). Without loss of generality (perhaps by replacing \( \vec{y} \) with \(-\vec{y} \), which we can do because if \( \vec{y} \in U \) then \(-\vec{y} \in U \)), we can assume that \( t > 0 \). Writing \( P = P(A, \vec{z}) \), there is a row \( \vec{a}_i \vec{x} \leq z_i \) of the inequalities \( A\vec{x} \leq \vec{z} \) with \( \vec{a}_i(\vec{x} + t\vec{y}) > z_i \). Since \( \vec{x} \in P \), we have \( \vec{a}_i \vec{x} \leq z_i \), and we conclude that \( \vec{a}_i(\vec{y}) > 0 \), and since \( t > 0 \), we see that \( \vec{a}_i(\vec{x} + t\vec{y}) > z_i \), so \( \vec{y} \not\in U \). □

Here is a nice way to explain it that I learned from a student.

**Sample solution 3.** By definition, \( \text{lin}(P) \) equals \( \{ \vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P, t \in \mathbb{R} \} \). Fix \( x_0 \in P \) and let \( U \) be the set \( \{ \vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R} \} \). The inclusion \( \text{lin}(P) \subseteq U \) is immediate.

Conversely, suppose \( \vec{y} \in U \). That is, \( \vec{x}_0 + t\vec{y} \in P \) for all \( t \in \mathbb{R} \). Writing \( P = P(A, \vec{z}) \), notice that \( A\vec{y} = 0 \). If \( A\vec{y} \) has some positive entry (say in position \( i \)), then we can choose \( t \) large enough to make \( \vec{a}_i(x_0 + t\vec{y}) \leq z_i \) false, and if \( A\vec{y} \) has a negative entry in position \( i \) then we can choose \( t \) negative with large enough absolute value to make \( \vec{a}_i(x_0 + t\vec{y}) \leq z_i \) false. Thus for all \( \vec{x} \in P \), we have \( A(\vec{x} + t\vec{y}) = A\vec{x} \leq \vec{z} \), so \( \vec{x} + t\vec{y} \in P \). □

**Additional Problem 2**

Let \( P \) be a polyhedron and let \( F \) be a nonempty face of \( P \). We are to show that \( \text{lin}(F) = \text{lin}(P) \). Choosing \( x_0 \in F \) and using the identities \( \text{lin}(P) = \{ \vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R} \} \) and \( \text{lin}(F) = \{ \vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in F \text{ for all } t \in \mathbb{R} \} \) from Additional Problem 1, we see that \( \text{lin}(F) \subseteq \text{lin}(P) \). On the other hand, suppose \( \vec{y} \not\in \text{lin}(P) \). We already know \( 0 \in \text{lin}(F) \), so assume \( \vec{y} \neq 0 \). Let \( H \) be a hyperplane defining \( F \) as a face of \( P \). Since \( H \) is associated to a valid inequality for \( P \), all of \( P \) is contained on one side of \( H \). The line \( \{ \vec{x}_0 + t\vec{y} : t \in \mathbb{R} \} \) must be contained in the hyperplane \( H \). Otherwise, the line contains points on both sides of \( H \), so the line contains points not in \( P \). That would contradict the supposition that \( \vec{y} \in \text{lin}(P) \). Thus \( \vec{x}_0 + t\vec{y} \) is in \( P \cap H \) for all \( t \). But \( P \cap H = F \), and we have shown that \( \vec{y} \in \text{lin}(F) \).