Comment on computing $h$-vectors. I noticed after TEX-ing this all up, that I had done all the $h$-vectors by a different version of Stanley’s trick than what we did in class (which is also in the book). I think it’s pretty plain to see that it’s the same thing, so I’m going to leave it. I don’t care how you compute $h$-vectors, but you should make sure you have a quick way to do it.

Lecture 8, Problem 13

Part (i). Check Kruskal-Katona:

$f_{-1} = 1$: Yup.

$f_0 \geq \partial_2(f_1)$:

$47 - 1 = \binom{10}{2} + \binom{1}{0}$, so $\partial_2(47) = \binom{10}{1} + \binom{1}{0} = 11$. Yup.

$f_1 \geq \partial_3(f_2)$:

$52 - 1 = \binom{7}{3} + \binom{6}{2} + \binom{5}{1}$, so $\partial_3(52) = \binom{7}{2} + \binom{6}{1} + \binom{5}{0} = 28$. Yup.

$f_2 \geq \partial_4(f_3)$:

$38 - 1 = \binom{10}{4} + \binom{9}{3} + \binom{8}{2} + \binom{7}{1}$, so $\partial_4(38) = \binom{10}{3} + \binom{9}{2} + \binom{8}{1} + \binom{7}{0} = 41$. Yup.

$f_3 \geq \partial_5(f_4)$:

$12 - 1 = \binom{6}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0}$, so $\partial_5(12) = \binom{6}{4} + \binom{5}{3} + \binom{4}{2} + \binom{3}{1} + \binom{2}{0} = 28$. Yup.

This passes Kruskal-Katona, so it’s the $f$-vector of a simplicial complex.

Part (ii). I interpret “shellable complex” as “shellable simplicial complex.” Compute the $h$-vector:

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Shellable simplicial complexes have nonnegative $h$-vectors, so this is not shellable. (By the way: If we hadn’t found negative entries in the $h$-vector, I don’t think we could have concluded that there exists a shellable complex with this $f$-vector.)

Part (iii). Simplicial polytopes are shellable! So by Part (ii), this is not the $f$-vector of a simplicial polytope. (However: What if we hadn’t found negative entries in the $h$-vector? Then we would have checked Dehn-Sommerville, and if that failed, we would know that this is not the $f$-vector of a simplicial polytope.)
Substituting \( x + 1 \) for \( x \):
\[
\sum_{i=0}^{d} f_{i-1}(x + 1)^i(-1)^{d-i} = \sum_{i=0}^{d} f_{i-1}x^{d-i}.
\]
Thus \( f_{d-1} \) is the coefficient of \( x^{d-k} \) on the right side. We use the binomial theorem inside the summand to rewrite the left side:
\[
\sum_{i=0}^{d} f_{i-1}(-1)^{d-i} \sum_{j=0}^{i} \binom{i}{j} x^j
\]
Reversing the sum:
\[
\sum_{j=0}^{d} \sum_{i=j}^{d} f_{i-1}(-1)^{d-i} \binom{i}{j} x^j
\]
The coefficient of \( x^{d-k} \) is \( \sum_{i=k}^{d} f_{i-1}(-1)^{d-i} \binom{d}{i} \), as desired.

If \( k = d-1 \), this says \( f_{d-2} = f_{d-2}(-1)^{d-1} \binom{d-1}{d-1} + f_{d-1}(-1)^{0} \binom{d}{d-1} \), which simplifies to \( 2f_{d-2} = df_{d-1} \).

**Lecture 8, Problem 28**

For a 3-polytope, the Euler-Poincare formula says \(-1 + f_0 - f_1 + f_2 = 1\). This, with the requirement that \( f_{-1} = 1 \), defines an affine plane in the 4-dimensional space of \( f \)-vectors. We easily see that any vector of the form \( (1, 4, 6, 4) + a(0, 1, 1, 0) + b(0, 0, 1, 1) \) is in that plane. Since \((0, 1, 1, 0)\) and \((0, 0, 1, 1)\) are linearly independent, the vectors in that form span the plane. (There is a subtle point though: Do vectors of that form with integers \( a \) and \( b \) cover all the integer vectors in that plane? Yes: Given any integer point \((f_{-1}, f_0, f_1, f_2)\) in that plane, we can choose \( a = f_0 - 4 \) and \( b = f_2 - 4 \), and the other entry must be right, because the point is in the plane.)

We know that a 3-polytope cannot have fewer than 4 vertices or fewer than 4 facets, so we get \( a \geq 0 \) and \( b \geq 0 \).

For the simplicial case, we compute the \( h \)-vector:
\[
\begin{align*}
4 + b & \quad ? & \quad ? & \quad ? & \quad \rightarrow & \quad 6 + a + b & \quad 3 + b & \quad 2 + a & \quad 1 + b - a \\
4 + a & \quad ? & \quad ? & \quad ? & \quad \rightarrow & \quad 4 + a & \quad 3 + a & \quad 2 + a & \quad 1 + a \\
1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 1
\end{align*}
\]
Dehn-Sommerville says \( 1 + b - a = 1 + a \), so \( b = 2a \), and in particular the \( f \)-vector is \((1, 4 + a, 6 + 3a, 4 + 2a)\).

Here is my complicated way of seeing that \( b \leq 2a \) and \( a \leq 2b \): A cyclic polytope is in particular simplicial, and since it maximizes \( f \)-vectors with a given number of vertices (Theorem 8.23), in particular any 3-polytope with \( 4 + a \) vertices must have at most \( 4 + 2a \) facets, and thus since the number of facets is \( 4 + b \), we have \( b \leq 2a \) for general polytopes. Passing to the polar dual polytope switches the roles of \( a \) and \( b \), so also \( 2b \geq a \).

Here is a simpler way that I learned from some of your papers (good job!): Since every edge is in exactly 2 facets and every facet has at least 3 edges, \( 2f_1 \geq 3f_2 \), so that \( 12 + 2a + 2b \geq 12 + 3b \), and thus \( 2a \geq b \). Since every edge has exactly 2 vertices and every vertex is in at least 3 edges, \( 2f_1 \geq 3f_0 \), so that \( 12 + 2a + 2b \geq 12 + 3a \), and thus \( 2b \geq a \).

It remains to show that, for any vector \((1, 4, 6, 4) + a(0, 1, 1, 0) + b(0, 0, 1, 1)\) with \( 2a \geq b \geq 0 \) and \( 2b \geq a \geq 0 \), there is a 3-polytope with that \( f \)-vector. This is tricky in some sense, but also straightforward once you see what you have to do. The point is to consider some operations that give the polytope more faces.

One is stellar subdivision of triangular facets, as in Lecture 3, Problem 0 in Homework 7. If a polytope has a triangular facet, find a point beyond that facet and take the convex hull of the simplex and that new point. We lose one facet, but create 1 new vertex, 3 new edges, and 3 new (triangular) facets. In effect, we increase \( a \) by 1 and increase \( b \) by 2. Importantly, the result is a polytope that continues to have a triangular facet, and also has a simple vertex (a vertex incident to exactly 3 edges).

The other operation is the dual of stellar subdivision, “shaving a simple vertex”, also discussed in Lecture 3, Problem 0 in Homework 7. If a polytope has a simple vertex, find a hyperplane \( H \) that separates that vertex from all other vertices. Replace the polytope by its intersection with the halfspace defined by \( H \) (the one containing all the other vertices). We lose one vertex, but create 3
new vertices, 3 new edges, and 1 new (triangular) facet. That is increasing $a$ by 2 and increasing $b$ by 1. The result is a polytope that continues to have a simple vertex and now has a triangular facet.

So, what do we do? For convenience, let’s think of vectors $[a, b]$ in $\mathbb{Z}^2$. We want to be able to find a polytope for every $[a, b]$ with $2a \geq b \geq 0$ and $2b \geq a \geq 0$. We find a finite set $C$ of vectors $[a, b]$ such that every vector $[a, b]$ with $2a \geq b \geq 0$ and $2b \geq a \geq 0$ can be obtained by adding integer multiples of $[1, 2]$ and $[2, 1]$. This set is $\{[0, 0], [1, 1], [2, 2]\}$.

If we can find polytopes corresponding to these three vectors $[a, b]$ and if they each have at least one simple vertex and at least one triangular facet, then we can get every possible $[a, b]$ using iterated stellar subdivision of triangular facets and shaving of simple vertices.

We can:

- $[0, 0]$ gives $f$-vector $(1, 4, 6, 4)$, which is the tetrahedron.
- $[1, 1]$ gives $f$-vector $(1, 5, 8, 5)$, which is the Egyptian pyramid.
- $[2, 2]$ gives $f$-vector $(1, 6, 10, 6)$. There is a suitable polytope with this $f$-vector, but it is not one with a name that I knew. Take a triangular prism and “nudge” one of the vertices so that one of the quadrilateral faces “breaks” into two triangles but the other two quadrilaterals don’t. Since Wikipedia has everything, you can find both a picture and a name for this on the page entitled [Hexahedron].

**Additional Problem 1**

The tetrahedron has $f$-vector $(1, 4, 6, 4)$. Its $h$-vector is $(1, 1, 1, 1)$.

\[
\begin{array}{cccc}
4 & ? & ? & ? \\
6 & ? & ? & ? \\
4 & ? & ? & ? \\
1 & 1 & 1 & 1
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
4 & 3 & 2 & 1 \\
6 & 3 & 1 & 1 \\
4 & 3 & 2 & 1 \\
1 & 1 & 1 & 1
\end{array}
\]

The octahedron has $f$-vector $(1, 6, 12, 8)$. Its $h$-vector is $(1, 3, 3, 1)$.

\[
\begin{array}{cccc}
8 & ? & ? & ? \\
12 & ? & ? & ? \\
6 & ? & ? & ? \\
1 & 1 & 1 & 1
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
8 & 7 & 3 & 1 \\
12 & 7 & 3 & 1 \\
6 & 5 & 4 & 3 \\
1 & 1 & 1 & 1
\end{array}
\]

The icosahedron has $f$-vector $(1, 12, 30, 20)$. Its $h$-vector is $(1, 9, 9, 1)$.

\[
\begin{array}{cccc}
20 & ? & ? & ? \\
30 & ? & ? & ? \\
12 & ? & ? & ? \\
1 & 1 & 1 & 1
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
20 & 19 & 11 & 9 \\
30 & 19 & 11 & 9 \\
12 & 11 & 10 & 9 \\
1 & 1 & 1 & 1
\end{array}
\]

**Additional Problem 2**

The 0 dimensional permutohedron has $f$-vector $(1)$ and $h$-vector $(1)$.

The 1 dimensional permutohedron has $f$-vector $(1, 2)$ and $h$-vector $(1, 1)$.

The 2 dimensional permutohedron (a hexagon) has $f$-vector $(1, 6, 6)$ and $h$-vector $(1, 4, 1)$.

\[
\begin{array}{cccc}
6 & ? & ? & ? \\
6 & ? & ? & ? \\
1 & 1 & 1 & 1
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
6 & 5 & 4 & 1 \\
6 & 5 & 4 & 1 \\
1 & 1 & 1 & 1
\end{array}
\]

The 3 dimensional permutohedron (a hexagon) has $f$-vector $(1, 24, 36, 14)$. Up to now (because of low dimensions) these polytopes have been isomorphic to their duals. Now we have to dualize. The dual has $f$-vector $(1, 14, 36, 24)$ and $h$-vector $(1, 11, 11, 1)$.

\[
\begin{array}{cccc}
24 & ? & ? & ? \\
36 & ? & ? & ? \\
14 & ? & ? & ? \\
1 & 1 & 1 & 1
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
24 & 23 & 11 & 1 \\
36 & 23 & 11 & 1 \\
14 & 13 & 12 & 11 \\
1 & 1 & 1 & 1
\end{array}
\]
In general, the $h$-vector of the $(n-1)$-dimensional permutohedron has $h_k = A(n, k)$, where $A(n, k)$ is the Eulerian number, the number of permutations of $[n]$ with exactly $k$ ascents (adjacent entries that are in correct numerical order). Here is one way to see why:

Recall that the weak order is a partial order (in fact a lattice) on permutations whose Hasse diagram (if we think of it as an undirected graph) coincides with the graph of the permutohedron. Thus we can think of the weak order as a partial order on the facets of the dual to the permutohedron, and two facets share a ridge if and only if they are related by a cover in the weak order. By results of your last homework, the face fan of the dual permutohedron is the normal fan of the permutohedron, and that’s the fan cut out by the hyperplanes $x_i = x_j$ for $i \neq j$. Thus we can think of the weak order as a partial order on the cones of that fan. As we discussed in class, we can figure out which direction a cover relation between adjacent cones goes by a linear functional that has its maximum in the region $x_1 \leq x_2 \leq \cdots \leq x_n$.

The upshot: If we order the maximal cones as a linear extension of the weak order, every maximal cone’s “old faces” are a union of facets, specifically those facets whose normals have positive inner product with the linear functional. Thus this linear order is a shelling (on the fan or on the boundary complex).

The restriction face of a cone is the intersections of the facets of the cone whose normals have negative inner product with the linear functional. The dimension of the restriction face is the number of ways to go up from the cone in the weak order. That’s the number of ascents of the corresponding permutation.