MA/CSC 416 Homework 9, Comments and some solutions.

I graded Section 2.3, Problems 13a and 13bd, Section 2.4, Problem 19a, and Section 2.5 11a for 4 points each. Because there were a lot of more "computational" problems (as opposed to writing proofs) that I didn't grade, I gave 12 points for having attempted those, so the whole assignment was out of 28. Please don't let the fact that I didn't grade these problems stop you from making sure you know how to do them.

Section 2.3, Problem 13

Part a. This is a lot like the other recursions we have proved. It was a little tricky, but there was a good hint in the book. The second sample proof below uses the hint more directly.

Sample proof 1 of Problem 13a. A derangement has no 1-cycles. Thus the set of derangements is the disjoint union

$$\{\text{derangements such that } n \text{ is in a 2-cycle}\} \cup \{\text{derangements such that } n \text{ is in a larger cycle}\}.$$

Derangements such that n is in a 2-cycle are counted by $(n-1)D_(n-2)$, because there are $n-1$ ways to choose the other element of the 2-cycle and $D(n-2)$ ways to arrange the remaining $n-2$ elements into cycles larger than 1. Given a derangement p such that n is in a cycle larger than 2, deleting n from its cycle gives a derangement p' of $\{1, 2, \ldots, n-1\}$. And given a derangement p' of $\{1, 2, \ldots, n-1\}$, there are $n-1$ derangements of $\{1, 2, \ldots, n\}$ with n in a cycle larger than 2, one for each choice of placing n *after* an element of a cycle in p'. Thus, derangements such that n is in a larger cycle are counted by $(n-1)D(n-1)$. We see that $D(n) = (n-1)(D(n-1) + D(n-2))$. □

Sample proof 2 of Problem 13a. By FCP, what we need to show is that for each $k \in [n-1]$, there are exactly $D(n-1) + D(n-2)$ derangements p with $p(n) = k$. The set of derangements p with $p(n) = k$ is the disjoint union

$$\{\text{derangements } p \text{ with } p(n) = k \text{ and } p(k) = n\} \cup \{\text{derangements } p \text{ with } p(n) = k \text{ and } p(k) \neq n\}.$$

Derangements with $p(n) = k$ and $p(k) = n$ are counted by $D(n-2)$, because once we have decided that $p(n) = k$ and $p(k) = n$, it only remains to permute the remaining $n-2$ elements without fixing any. Also, there is a bijection from the set of derangements of $[n-1]$ to the set of derangements p with $p(n) = k$ and $p(k) \neq n$: Given a derangement p' of $[n-1]$, place n in position k and place $p'(k)$ (the element that was in position k) at the end to obtain a derangement p with $p(n) = k$ and $p(k) \neq n$. The inverse map is to remove the last element of the one-line notation of p and use it to replace n (which was in position k), to obtain the one-line notation of p'. □

By the way, this also proves that $D(n)$ is always a multiple of $n-1$. Is that obvious from the $\frac{n!}{e}$ formula?

Part b. This can be done by induction using part a.

Sample proof of Problem 13b. If $n = 2$, then the equation says $1 = 2 \cdot 0 + 1$. If $n > 2$, then assume as an inductive hypothesis that we have already proved the corresponding formula for $D(n-1)$:

$$D(n-1) = (n-1)D(n-2) + (-1)^{n-1}, \text{ or equivalently } (n-1)D(n-2) = D(n-1) - (-1)^{n-1}.$$

Thus

$$\begin{aligned} D(n) &= (n-1)(D(n-1) + D(n-2)) \\ &= (n-1)D(n-1) + (n-1)D(n-2)) \\ &= (n-1)D(n-1) + D(n-1) + (-1)^n \\ &= nD(n-1) - (-1)^{n-1} \\ &= nD(n-1) + (-1)^n \end{aligned}$$

The first equality is Part a, the third equality uses the inductive hypothesis, and the other equalities are just manipulations. □

Part d. Also induction, using part b.

Sample proof of Problem 13d. If $n = 1$ then the desired equation is just $1![\frac{1}{0!} - \frac{1}{1!}] = 1[1-1] = 0$. If $n > 1$, then assume as an inductive hypothesis that

$$D(n-1) = (n-1)!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right].$$

Thus

$$\begin{aligned}
D(n) &= nD(n-1) + (-1)^n \\
&= n(n-1)!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right] + (-1)^n \\
&= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right] + n!\frac{(-1)^n}{n!} \\
&= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right]
\end{aligned}$$

The first equality is Part b and the other equalities are just manipulations. □

Section 2.4, Problem 3bdf

Answers are in the back of the book, except 3f: (15432) is 51234.

Section 2.4, Problem 4fh

The point here was: just reverse every cycle.
Part f: $p^{-1} = (148)(2657)(3)$.
Part h: $p^{-1} = (1247)(3685)$.

Section 2.4, Problem 8

The point was to see that this was only asking for the seven integer partitions of 5. These appear in Example 1.8.6

Section 2.4, Problem 9

Answers are in the back. Here's a comment on one of the answers:

How many permutations are there with cycle type $[2, 2, 1]$? Choose 2 elements for the first cycle ($\binom{5}{2} = 10$ ways) and then choose 2 elements for the second cycle ($\binom{3}{2} = 3$ ways). So the total number is 30, right?

No. There only 15 of them. The above procedure counts every one of them exactly twice. It counts the number of ways to make two 2-cycles from 5 elements, where the order on the 2-cycles matters. To get the right count, divide by the number of possible orders of the 2-cycles. The answer is $\frac{30}{2!} = 15$.

Note that in later problems, a similar overcount occurred. For example, in Section 2.5, Problem 4a, you counted the number of permutations with cycle type $[2, 2, 2, 1]$. The correct answer is $\binom{7}{2}\binom{5}{2}\binom{3}{2}\frac{1}{3!}$.

Section 2.4, Problem 13

There are $\binom{m}{2}$ of them: Once you have chosen which two elements will form the 2-element cycle, there is only one way to make the 2-cycle and put the other elements into 1-element cycles.

You could also do this by just quoting Problem 14: $P(m, 2)/2 = \binom{m}{2}$.

Section 2.4, Problem 14

There are $\binom{m}{k}$ ways to choose the elements for the k-element cycle. Then there are $(k-1)!$ ways to make those k elements into a cycle. Finally, there is only one way to put the remaining elements into 1-element cycles. The answer is $\frac{\binom{m}{k}}{(k-1)!} = \frac{P(n,k)}{k}$.

Section 2.4, Problem 19a

For any permutation $p \in S_m$, the sum $c_1(p) + 2c_2(p) + 3c_3(p) + \cdots + mc_m(p)$ counts the elements of $[m]$ according to the size of cycle they occur in in p. To see that, suppose you want to choose an element of $[m]$ that occurs in an i-cycle of p. First, choose an i-cycle ($c_i(p)$ choices) and then choose an element of that cycle (m choices). Thus for each i there are $ic_i(p)$ elements of $[m]$ occurring in i-cycles of p. Now apply the Second Counting Principle.

Section 2.4, Problem 19b

We will do this in class.

Section 2.5, Problems 3-4

Answers to Problem 3 are in the book. For both of these, look at the comments to Section 2.4, Problem 9. Please make sure you master these problems. Talk to me if you're still confused.

Section 2.5, Problem 11a

Sample combinatorial proof. The number $m!$ counts permutations in S_m. The right side counts permutations in S_m according to the number of cycles. A permutation in S_m can have anywhere from 1 to m cycles and, by definition, $s(m, k)$ is the number of permutations in S_m with exactly k cycles. Applying the Second Counting Principle, the sum $\sum_{k=1}^{m} s(m, k)$ must equal $m!$. □

The generating function proof was a straightforward application of Theorem 2.5.4, and highlights the idea of specializing a polynomial (generating function!) identity to get a numerical identity (as discussed several times already in the course).

Sample generating function proof. Theorem 2.5.4 says

$$\sum_{n=1}^{m} s(m, n)x^n = x(x + 1)(x + 2)\cdots(x + m - 1).$$

If you specialize x to 1, you get part a. □

Section 3.1, Problem 4

It was a helpful exercise to actually do part f and part g in the two different ways they suggest.
Part a: $f \circ g = (3, 6, 1, 2, 5, 4)$.
Part b: $g \circ f = (2, 1, 4, 5, 3, 6)$.
Part c: $f^{-1} = (1, 5, 2, 4, 6, 3)$.
Part d: $f = (1)(2365)(4)$.
Part e: $g = (123)(456)$.
Part f: $f \circ g = (13)(264)(5)$.
Part g: $f \circ g = (13)(264)(5)$.