MA/CSC 416
Homework 8, Comments and some solutions.

1. Section 2.2, Problem 29

There are two ways to see that this is the multinomial coefficient $\binom{m}{r_1,\dots,r_n}$, i.e. the number of m-letter words in an alphabet with n different letters, with r_i copies of the i^{th} letter for each i.

One way is to use the fundamental counting principle to distribute the balls: First, choose r_1 (out of m) balls to go in U_1, then choose r_2 balls (out of the remaining $m - r_1$) for U_2, etc. You get exactly the right side of Theorem 1.5.1.

Another way is to make the m choices of urns and write down, for each ball, which urn you chose. The resulting sequence of urns is a word with the letter "U_i" occurring exactly r_i times for each i.

2. Section 2.2, Problem 31

The Bell number B_r is the total number of set partitions of $[r]$. So the problem is asking us to see why

"making an unordered factorization of n into integers greater than one"

is the same as

"making a set partition of some r-element set."

The natural choice of r-element set is $\{p_1, p_2, \dots, p_r\}$. Given a partition of $\{p_1, p_2, \dots, p_r\}$, we can multiply the elements within each block. Thus the blocks of the partition become the factors in a factorization. It is easy to see that this is a bijection. (That is, easy once you understand what I wrote. If you don't understand what I wrote, maybe I didn't write it clearly. Ask!)

Section 2.3, Problem 4

The point was that a permutation p fixes some i if and only if its inverse fixes i. Here's a proof:

Proof. Suppose $p(i) = i$. (In other words, suppose p fixes i.) Then $p^{-1}(i) = p^{-1}(p(i)) = i$ by definition of p^{-1}. In other words, p^{-1} fixes i.

Conversely, suppose $p^{-1}(i) = i$. (In other words, suppose p^{-1} fixes i.) Then $p(i) = p(p^{-1}(i)) = i$ by definition of p^{-1}. In other words, p fixes i. □

Section 2.3, Problem 5

Fallacious argument. By Problem 4, if p is a derangement, then so is p^{-1}. Thus the set of derangements consists of pairs $\{p, p^{-1}\}$. So the number of derangements is even. □

The fallacy is that Problem 4 does not show that derangements come in pairs. The "pair" $\{p, p^{-1}\}$ only has one element when p is its own inverse. And there are lots of examples of derangements p such that p is its own inverse. The smallest example is $p = 12 \in S_2$.

Section 2.3, Problem 12

Sample proof of Problem 12. The right side counts permutations according to the number of fixed points. More specifically, let k be the number of elements **not** fixed by a permutation p and break up the set of permutations into disjoint collections, according to k. You get:

$$n! = \sum_{k=0}^{n} (\text{the number of permutations in } S_n \text{ with exactly } k \text{ non-fixed points})$$

For $k = 0$ there is exactly one permutation $(12\cdots n)$ with k non-fixed points, and for $k = 1$ there are none. (Why?) For $k \geq 2$, we can count permutations in S_n with exactly k non-fixed points by a two-step process: first, pick k elements to be non-fixed and second, choose a derangement of those k permutations. The result is the desired formula $n! = 1 + \sum_{k=2}^{n} \binom{n}{k} D(k)$.

Another way to finish the proof: For *any* k, the number of permutations in S_n with exactly k non-fixed points is $\binom{n}{k} D(k)$, so $n! = \sum_{k=0}^{n} \binom{n}{k} D(k)$. Since $\binom{n}{0} = 1$ and $D(0) = 1$ and $D(1) = 0$, this is the desired formula $n! = 1 + \sum_{k=2}^{n} \binom{n}{k} D(k)$. (But why is $D(0) = 1$?) □

Section 2.3, Problem 18

You should have done this as a sum/difference with 15 terms. The point was not to just count the primes directly. Your terms looked like, for example, $|A_2 \cap A_5|$. You needed to recognize that $A_2 \cap A_5$ is the set of multiples of 10 strictly between 10 and 100, which you could calculate (without listing them all!) to have 8 elements.

This may not have been the most effective way to count primes between 10 and 100, but I can imagine this method working well by computer for larger ranges.

Additional Problem

The point is that increasing functions from $[r]$ to $[n]$ "are" r-element subsets of $[n]$. Then the book's version breaks up the sum according to r.