MA/CSC 416
Homework 7, Comments and some solutions.

Information on scoring

Here is a list of which problems I graded, and out of how many points:

Problem	6ab	6c	8	13	17alt	20	10	14	15	18	19	Total
Points	4	4	4	4	4	4	2	2	2	2	2	34

Section 2.1, Problem 6

There were two ways to do this: By induction using the recursion for $S(m, n)$ or by direct combinatorial reasoning. The induction argument gets a little messy for part b, and I don't recommend it in that case. One thing that I hope all of you will appreciate is how Problem 2 of Section 1.8 plays an important role in Problems 6a and 6b of this assignment. Problem 2 of Section 1.8 basically tells you what the block sizes can be. Another thing I want to point out is that, in part c, if you want to count all subsets of an n-element set, you don't need to bother with summing binomial coefficients. The number of subsets of an n-element set is 2^n by a Fundamental Counting Principle argument that is much easier than summing binomial coefficients.

I'll start with samples of direct combinatorial arguments. (Notice, in this case, the combinatorial proofs are the easiest.)

Sample solution of Problem 6a. A partition of $[n+1]$ into n blocks must have $n-1$ singleton-blocks and one block of size 2. (Compare Problem 2c of Section 1.8.) Thus to make a partition of $[n+1]$ into n blocks, one only needs to choose which two elements will be together in a block. There are $\binom{n+1}{2}$ ways to do that. $\square$

Sample solution of Problem 6b. A partition of $[n+2]$ into n blocks must have either $n+1$ singleton-blocks and one block of size 3 or n singleton blocks and two blocks of size 2. (Compare Problem 2a of Section 1.8.) Thus to make a partition of $[n+2]$ into n blocks, one either chooses which three elements to be together in a block ($\binom{n+2}{3}$ ways) or chooses four elements to be in the two blocks of size 2 ($\binom{n+2}{4}$ ways) and then decides how to partition those four elements into two blocks of size 2 ($S(4, 2) = 3$ ways). Thus in all there are $\binom{n+2}{3} + 3\binom{n+2}{4}$ partitions of $[n+2]$ into n blocks. $\square$

Sample solution of Problem 6c. To partition $[n+1]$ into two blocks, we need only decide which elements of $[n]$ are in a block with the element $n+1$. As long as we don't put all of them in with $n+1$, we have a two-block partition. In other words, choose any proper subset of $[n]$ to form the block not containing the element $n+1$. There are $2^n - 1$ proper subsets of $[n]$ $\square$

Next, inductive proofs. I'm skipping the messy inductive argument for part b.

Sample solution by induction of Problem 6a. The case $n = 1$ is trivial, so assume that $n > 1$ and make the inductive assumption $S(n, n-1) = C(n, 2)$. Using this inductive assumption, we will prove that $S(n+1, n) = C(n+1, 2)$. We apply Theorem 2.1.20, then apply the inductive assumption, notice that $n = C(n, 1)$ and finally apply Pascal's relation:

$$\begin{aligned} S(n+1, n) &= S(n, n-1) + nS(n, n) \\ &= C(n, 2) + n \\ &= C(n, 2) + C(n, 1) \\ &= C(n+1, 2). \end{aligned}$$

This proves the statement by induction for all n. $\square$

Sample solution by induction of Problem 6c. The case $n = 1$ is trivial, so assume that $n > 1$ and make the inductive assumption $S(n, 2) = 2^{n-1} - 1$. Using this inductive assumption, we will prove that $S(n+1, 2) = 2^n - 1$. We apply Theorem 2.1.20, notice that $S(n, 1) = 1$ and apply the inductive assumption.

$$\begin{aligned} S(n+1,2) &= S(n,1)+2S(n,2) \\ &= 1+2(2^{n-1}-1) \\ &= 1+2^n-2 \\ &= 2^n-1. \end{aligned}$$

This proves the statement by induction for all n. $\square$

Section 2.1, Problem 8

This is not even the multinomial theorem, but it takes some thinking to get your brain around it. It's describing, in the language of finite functions, how we would expand the left side. Please make sure you see how this works.

Sample solution of Problem 13. To find a term in the expansion of

$$(x_1+x_2+\cdots+x_n)^m = (x_1+x_2+\cdots+x_n)(x_1+x_2+\cdots+x_n)\cdots(x_1+x_2+\cdots+x_n)$$

we choose a term x_i from each factor $(x_1+x_2+\cdots+x_n)$ and multiply together the string of terms we choose. Then we sum over all possible such sequences of choices.

Instead of choosing x_i, we can just choose the subscript i. So we can restate what it means to expand $(x_1+x_2+\cdots+x_n)^m$. Take all possible sequences of m subscripts, where the subscripts are between 1 and n. Call the chosen sequence of subscripts $(f(1), f(2), \ldots, f(m))$. Then write $x_{f(1)}x_{f(2)}\cdots x_{f(m)}$ and sum these monomials over all possible sequences of subscripts. That's exactly the right side of the equation. $\square$

Section 2.1, Problem 13

One approach is combinatorial. The right side is an alternative way to count $S(m+1,n+1)$. Induction also works, but the inductive proof has some subtleties, and I recommend the combinatorial approach. Another nice feature about the combinatorial approach is that we did a very similar proof in class. If you followed the book's hint, you had a "t" in the problem, which equals $m-k$ for the "k" in the statement of the problem.

Sample solution of Problem 13. The left side counts partitions of $[m+1]$ into $n+1$ blocks. The right side counts the same partitions in the following way: First, decide how many elements of $[m+1]$ are **not** in the same block as the element $m+1$, and call this number of elements k. Notice that k can be as large as m (if $m+1$ is alone in its block) or as small as n (if each of the n blocks not containing $m+1$ is a singleton). Given a choice of k, construct a partition by the following procedure:

Step 1: Choose a k-subset S of $[m]$. ($\binom{m}{k}$ ways).
Step 2: Choose a partition of S into n blocks. ($S(k,n)$ ways.)
Step 3: Put all of the elements of $[m]$ which are not in S into a block with $m+1$. (1 way.)

By the second counting principle and the fundamental counting principle,

$$S(m+1,n+1) = \sum_{k=n}^{m}\binom{m}{k}S(k,n). \qquad \square$$

Section 2.1, Problem 17alt

If you insist on 1 and 2 being in the same block then you may as well think of them as a single element "12" and consider partitions of the $(m-1)$-element set $\{12,3,4,\ldots,m\}$. (Think of "12" as "one-two," not "twelve.") So the answer to part a is $S(m-1,n)$.

On the other hand, let's try to make a partition of $[m]$ with n parts, with 1 and 2 not in the same block. There are two possibilities: One is to partition $\{2,3,\ldots,m\}$ into $n-1$ blocks and then make 1 a singleton-block. There are $S(m-1,n-1)$ ways to do that. The other is to partition $\{2,3,\ldots,m\}$ into n blocks, and then put 1 into any of the $n-1$ blocks which don't have the 2. There are $(n-1)S(m-1,n)$ ways to do that. In all, there are $S(m-1,n-1)+(n-1)S(m-1,n)$ partitions of $[m]$ having 1 and 2 in different blocks.

Combining parts a and b, the total number of partitions of $[m]$ into n blocks is $S(m,n) = S(m-1,n)+S(m-1,n-1)+(n-1)S(m-1,n)$, and this simplifies to $S(m,n) = S(m-1,n-1)+nS(m-1,n)$, (Theorem 2.1.20) which we proved in class. It shouldn't be too surprising that we got this formula, since our method for part b was basically the same as the proof of Theorem 2.1.20.

Section 2.1, Problem 20

Part a just requires that you understand what each side is. By definition, the blocks of a partition are nonempty, so requiring that all parts have at least one element is superfluous.

For part b, mimic the proof of Theorem 2.1.20. Let A be the set of partitions of $[m+1]$ with n parts, all of size at least k. Break A into two disjoint subsets B and C, and apply the second counting principle. The subset B is those partitions in A having the element $m+1$ in a block of size k. The subset C is those partitions in A having the element $m+1$ in a block of size greater than k.

The set B is counted by $\binom{m}{k-1}S(m-k+1, n-1)$. (Choose any $k-1$ elements of $[m]$ to form a k-element block with $m+1$ and then partition the remaining $m-k+1$ elements into $n-1$ blocks.) A set C is counted by $nS_k(m, n)$. (Take any partition of $[m]$ into n blocks, all of size at least k. Add the element $m+1$ to any block to obtain a partition in C.)

Comment on the problems in Section 2.2

For **every** problem in this section, the point was to look at the question in the right way. Once you did that, it became easy. Office hours and/or email questions to me could have been very helpful for some of you.

Section 2.2, Problem 10

The point was our discussion in class (also found in the book) that equivalence relations essentially "are" set partitions. An equivalence relation with k equivalence classes "is" a set partition with k blocks. Thus the answer is $S(n, k)$.

1. Section 2.2, Problem 14

There are at least two good ways to look at this. One way is to let u_i be the number of balls placed in each urn. Since there are a total of m balls, the question is just asking you to choose integers $u_1, \ldots, u_n$, all greater than or equal to zero, such that $u_1 + \cdots + u_n = m$. Corollary 1.6.9 tells us how many ways there are to do that.

Another good approach was to think of choosing an urn for each of the m balls. This is an unordered selection (with replacement) of m urns from the set of n urns.

2. Section 2.2, Problem 15

The first approach to Problem 14 can be modified for this problem. We're now choosing **positive** integers $u_1, \ldots, u_n$ with $u_1 + \cdots + u_n = m$ and applying Theorem 1.6.11. (These n-tuples of integers are just n-part compositions of m.)

Alternately, we can use Problem 14 directly to do this problem. To make sure that no urn is empty, first take n balls and place one in each urn. Then the remaining $m-n$ balls need to be placed in the unlabeled urns with no restrictions about empty urns. Problem 14 says there are $\binom{(m-n)+n-1}{m-n} = \binom{m-1}{n-1}$ ways to do that.

3. Section 2.2, Problems 18 and 19

If you have m unlabeled balls and n unlabeled urns, placements of the balls in the urns are integer partitions of m with at most n parts. (Given a placement, write down, in decreasing order, the numbers of balls occurring in each urn. Given an integer partition, for each part, put that many balls in some previously empty urn.)

By the second counting principle there are $p_1(m) + p_2(m) + \cdots p_n(m)$ integer partitions of m with at most n parts.

For Problem 19, note that for $m \leq n$, the restriction "with at most n parts" is superfluous. All integer partitions of m can occur, so the count is $p(m)$.

4. Section 2.2, Problems 24 and 25

24. a. $S(9,5) = 6951$. See class notes or Section 2.2 ("variations").
    b. $p_5(9) = 5$. This is similar to Exercises 2.2.18 and 2.2.19. Since empty urns are not allowed, you have an integer partition with exactly 5 parts.
    c. $\binom{8}{4} = 70$. See Exercise 2.2.15.
    d. $5!S(9,5) = 834120$. See class notes or Section 2.2 ("variations").
25. a. $S(9,1) + S(9,2) + S(9,3) + S(9,4) + S(9,5) = 18002$. See class notes or Section 2.2 ("variations").
    b. $p_1(9) + p_2(9) + p_3(9) + p_4(9) + p_5(9) = 23$. See Exercise 2.2.18.
    c. $\binom{13}{9} = 715$. See Exercise 2.2.14.
    d. $5^9 = 1953125$. See class notes or Section 2.2 ("variations").