MA/CSC 416
Homework 12, Comments and some solutions.

Section 5.1, Problem 4

Since the problem was to explain something that seems intuitively obvious, I wanted you to really explain it, instead of just relying on something else that seemed intuitively obvious. Here is one way to explain it completely rigorously.

Sample solution to Problem 4. Let r and k be any positive integers. Suppose I have any arrangement of pigeons into k holes such that every hole has fewer than $\lceil \frac{r}{k} \rceil$ pigeons. In other words, $p_i \leq \lceil \frac{r}{k} \rceil - 1$ for each $i \in [k]$, where p_i is the number of pigeons in hole i, But $\lceil \frac{r}{k} \rceil - 1$ is alway strictly less than $\frac{r}{k}$. (If k divides r then $\lceil \frac{r}{k} \rceil - 1$ is $\frac{r}{k} - 1$, and otherwise $\lceil \frac{r}{k} \rceil - 1$ is obtained from $\frac{r}{k}$ by rounding down.) So the total number of pigeons is $\sum_{i=1}^{k} p_i \; < \; k\frac{r}{k} \; = \; r$. This shows that if each hole has fewer than $\lceil \frac{r}{k} \rceil$ pigeons, then there are fewer than r pigeons total. □

Section 5.1, Problem 6

Sample solution to Problem 6. Apply Problem 4 with $r = 40$ and $k = 12$. (The pigeons are the people and the holes are the months.) □

Section 5.1, Problem 7

As suggested in the hint in the back of the book, factor each integer in S as a product of a power of 2 and an odd integer. (There is a unique factorization of this form for any integer: you have to factor out every power of 2 present.) Now think of the "pigeonholes" as the possible odd integers you can get in this factorization. There are n possibilities: $1, 3, \ldots, (2n-1)$. Since there are $n+1$ elements of S ("pigeons"), by the pigeonhole principle, two of them must have the same odd integer in their factorization. One of these (whichever has the smaller power of 2) exactly divides the other.

This was not part of the problem, but notice that subsets $S \subseteq [2n]$ need not have this property if they have fewer than $n+1$ elements. For example, take $S = \{n+1, n+2, \ldots, 2n\}$.

Section 5.1, Problem 8

I'm going to describe some things in words that would be more effectively described in pictures in your homework. (So, for example, you wouldn't have to name points.)

Call the triangle ABC. Let X be the midpoint of AB, let Y be the midpoint of AC and let Z be the midpoint of BC. Cut the triangle into 4 pieces by drawing 3 line segments: XY, XZ and YZ. Notice that if any two points are in the same one of these 4 triangles (even if they are on the boundary), their distance is at most 1. So to find a collection of points all distance > 1 apart, we need to put at most one point in each triangle. In particular, there is no way to choose 5 points all distance > 1 apart.

Section 5.2, Problem 2

Part a. Recall from class that the complement of a graph $G = (V, E)$ is a graph $G^c = (V, E')$ where $E' = \{\text{pairs } \{u,v\} \subseteq V : \{u,v\} \notin E\}$. In other words, every edge of G becomes a non-edge of G^c and every non-edge of G becomes an edge of G^c. In particular, an s-clique in G becomes an s-independent set in G^c and a t-independent set in becomes a t-clique in G^c. This is the symmetry that lets us exchange s and t in the Ramsey numbers.

Part b. By definition, $N(1,t)$ is the smallest integer n such that any graph with n vertices has either a 1-clique or a t-independent set. But a 1-clique is a vertex and as long as $n \geq 1$ the graph will have a vertex. If $n < 1$ (i.e. $n = 0$), the graph will not have a vertex and since $t \geq 1$, it will also not have a t-independent set. So $N(1,t) = 1$.

Part c. By definition, $N(2,t)$ is the smallest integer n such that any graph with n vertices has either a 2-clique or a t-independent set. But a 2-clique is an edge. A graph with n vertices and no edges will have a t-independent set if and only if $n \geq t$. So $N(2,t) = t$.

Section 5.2, Problem 3

Both parts are a special case of the general proof we did in class. The idea was to show me that you understood that proof by correctly explaining the proof in these special cases.

We can take it as given the fact (proved in class and in the book) that any graph on 6 vertices either has a 3-clique or a 3-independent set. Also, we can take it as given (or just obvious) that any graph on 4 vertices either has an edge or **is** a 4-independent set (by the special case $t = 4$ of Problem 2c).

Sample solution for 3a. To prove that $N(3, 4) \leq 10$, we need to show that any graph with 10 vertices has either a 3-clique or a 4-independent set. Let G be a graph with 10 vertices and let v be a vertex of G. Since G has 10 vertices, there must either be 4 or more vertices connected to v by edges or 6 or more vertices no connected to v. (Otherwise, v could have at most 9 vertices total.)

If there are 4 vertices connected to v then among those 6 vertices there is either an edge or the four vertices form a 4-independent set. In the latter case, we have found a 4-independent set in G. In the former case, if x and y are two of the four vertices and are connected by an edge then v, x, y form a 3-clique.

If there are 6 or more vertices not connected to v then among these 6 vertices there is either a 3-clique or a 3-independent set. If there is a 3-clique then we are done. If there is a 3-independent set, then adjoining v to that set gives a 4-independent set.

Thus in any case, G has either a 3-clique or a 4-independent set. $\square$

Part b is similar, but we have to refer to part a and to Problem 2a to know that any graph on 10 vertices either has a 3-clique or a 4-independent set and that any graph on 10 vertices either has a 3-independent set or a 4-clique.

Section 5.2, Problem 11

Either the two edges share a vertex or they do not.

Section 5.2, Problem 12

In general, why should there be exactly 2 graphs having n vertices and $\binom{n}{2} - 2$ edges? Because the complement is a graph with exactly two edges. Two graphs are isomorphic if and only if their complements are isomorphic. So you can use Problem 11.

Section 5.2, Problem 19

There might be many ways to do this. Here's one way: Both graphs have the same degree sequence, and in particular, each has exactly one vertex of degree 3. In the graph on the left, the vertex of degree 3 is connected to vertices of degree 4, 5 and 5. In the graph on the right, the vertex of degree 3 is connected to vertices of degree 4, 4 and 4. So they can't be isomorphic. Here's another way: Two graphs are isomorphic if and only if their complements are isomorphic. If you draw their complements, you'll see that they are clearly not isomorphic.

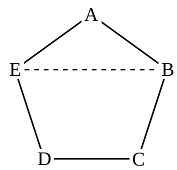
Section 5.3, Problem 1

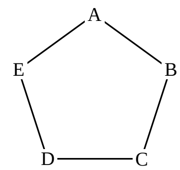
Sample direct approach to Problem 1a. I'll refer to the vertices as A, B. C, D and E, with A at the top, and then lettering them clockwise. First, choose a color for E (r choices). There are then $r - 1$ choices of a color for A and then $r - 2$ choices for a color for B. Now, to color D, I can either color it the same color as B (1 way), in which case I have $r - 1$ choices for C, or I can color it a different color than B ($r - 2$ ways) in which case I have $r - 2$ choices for C. The total number of colorings is

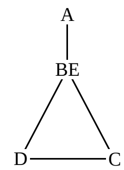
$$r(r-1)(r-2)\left[1 \cdot (r-1) + (r-2)(r-2)\right] = r(r-1)(r-2)(r^2 - 3r + 3). \qquad \square$$

The key to using chromatic reduction (AKA deletion/contraction) is to choose edges that turn your graph into graphs whose chromatic polynomials you already know.

Sample chromatic reduction approach to Problem 1a. I'll refer to the vertices as A, B. C, D and E, with A at the top, and then lettering them clockwise.

(1) $p\Bigg($ 

$, r\Bigg) = p\Bigg($ 

$, r\Bigg) - p\Bigg($ 

$, r\Bigg).$

The second term on the right side is $r(r-1)^2(r-2)$, by an example from class. The first term is $(r-1)^5 - (r-1) = (r-1)[(r-1)^4 - 1] = (r-1)(r^4 - 4r^3 + 6r^2 - 4r)$ by Problem 9a. Thus the chromatic polynomial is

$$(r-1)(r^4-4r^3+6r^2-4r)-r(r-1)^2(r-2) = (r-1)(r^4-5r^3+9r^2-6r) = r(r-1)(r-2)(r^2-3r+3). \ \square$$

There are many other ways we could have done this: at each step we have a lot of choices about which edges to delete/contract.

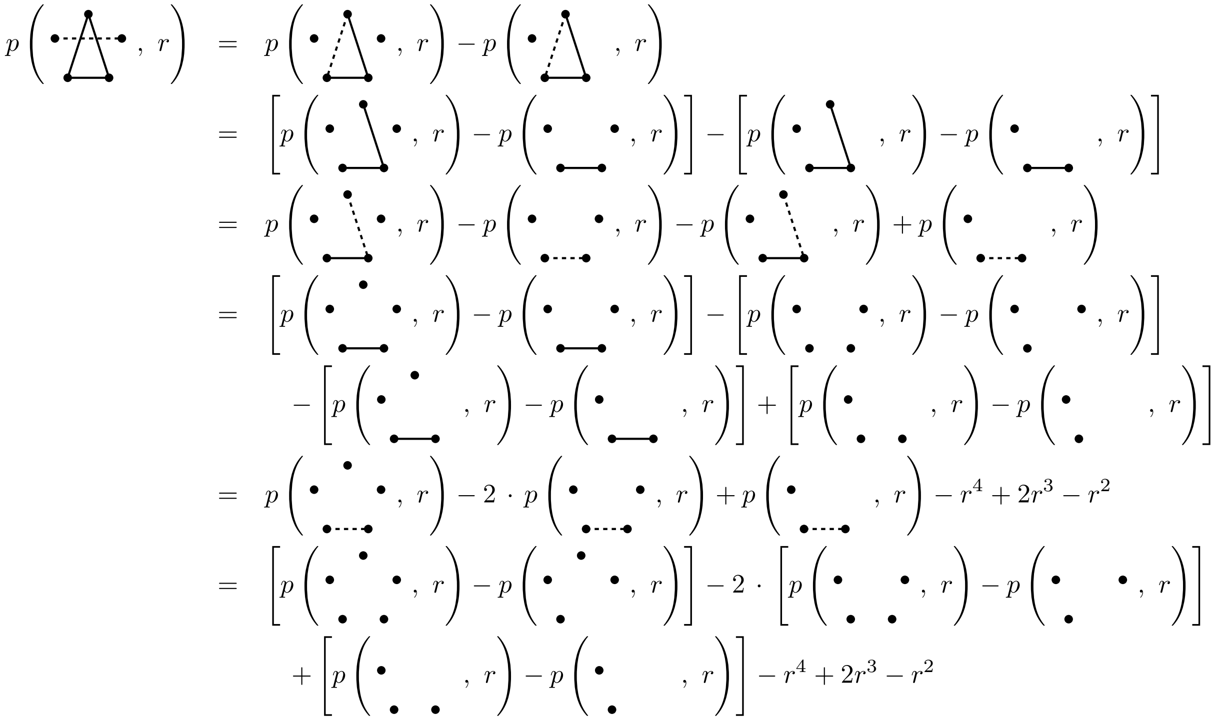
Sample proof of Problem 1b. Following the hint, we notice that the graph in 1b is isomorphic to the graph in 1a, so they have the same chromatic polynomial. $\square$

Sample direct approach to Problem 1c. Notice that we can break[1] this into two independent steps:
Step 1: Choose a coloring for vertices B and E.
Step 2: Choose a coloring for vertices A, C and D.
The number of ways to do Step 1 is $r(r-1)$ either by easy direct counting or by applying the theorem on the chromatic polynomial of trees. The number of ways to do Step 2 is $r(r-1)(r-2)$ by easy direct counting, or by Problem 9c. $\square$

For fun, let's also do part c by chromatic reduction, all the way to the "base case" where we have no edges. It will be tedious to keep track of vertex labelings, so I'll just write the unlabeled graphs, and I'll occasionally move vertices around so I can combine like terms. Each graph I draw, if it has any edges, will have one edge dotted, indicating the edge I'm going to delete/contract **in the next step**. When I get to a graph with no edges, I'll replace it in the next step by r^n, where n is the number of vertices of that graph. You'll need to think to follow this.

[1]Notice that this "breaking into two steps" is exactly the point of Theorem 5.3.11 in the book.

Sample chromatic reduction approach to Problem 1c.



$$
\begin{aligned}
&= \left[r^5 - r^4\right] - 2 \cdot \left[r^4 - r^3\right] + \left[r^3 - r^2\right] - r^4 + 2r^3 - r^2 \\
&= r^5 - 4r^4 + 5r^3 - 2r^2 \\
&= r^2(r-1)^2(r-2) \quad \square
\end{aligned}
$$

Section 5.3, Problem 3

We'll do part b first. This is a lot like problem 1, and you could do it either by chromatic reduction or directly. One nice shortcut: Notice that we can break this into two independent steps:
Step 1: Choose a coloring for the center vertex (r ways)
Step 2: Choose a coloring the remaining 5 vertices.
The number of ways to do Step 2 is $p(G', r-1)$, where G' is the subgraph (a 5-cycle) induced by the 5 outer vertices. (Why?) Now you can apply Problem 9c.

The book gives the answer to part b as $r(r-1)(r-2)(r-3)(r^2-4r+5)$. You should know how to get the answer to part a from there. If not, ask.

Section 5.3, Problem 9

Part c. This is by deletion/contraction, by Theorem 5.3.16 and by induction on n. I'll introduce some notation: P_n will be a "path" with n vertices. This is a tree with no "branches," or more specifically take $V = [n]$ and connect two vertices i and j if and only if $|j-i| = 1$. So P_n would look like •—•— $\cdots$ —•, with n vertices. The point is that P_n is a tree, so we know its chromatic polynomial.

Sample proof of Problem 9c. By direct counting, the chromatic polynomial of C_3 is

$$
\begin{aligned}
p(C_3, r) &= r(r-1)(r-2) \\
&= (r-1)(r^2-2r) \\
&= (r-1)\left[(r-1)^2 - 1\right] \\
&= (r-1)^3 - (r-1)
\end{aligned}
$$

That serves as a base case for induction. As an inductive assumption, assume that we have already shown that $p(C_{n-1}, r) = (r-1)^{n-1} + (-1)^{n-1}(r-1)$. Now for $n > 3$, choose any edge to delete/contract. Deleting gives a path P_n and contracting gives C_{n-1}. So, by our main deletion/contraction theorem

$$p(C_n, r) = p(P_{n-1}, r) - p(C_{n-1}, r).$$

Applying Theorem 5.3.16 and induction, we obtain

$$\begin{aligned}
p(C_n, r) &= r(r-1)^{n-1} - \left[(r-1)^{n-1} + (-1)^{n-1}(r-1)\right] \\
&= r(r-1)^{n-1} - (r-1)^{n-1} - (-1)^{n-1}(r-1) \\
&= r(r-1)^{n-1} - (r-1)^{n-1} + (-1)^{n}(r-1) \\
&= (r-1)(r-1)^{n-1} + (-1)^{n}(r-1) \\
&= (r-1)^{n} + (-1)^{n}(r-1) \quad \square
\end{aligned}$$

Part d. Bipartite means "having a proper 2-coloring." So we look at the formula for $p(C_n, r)$ in Part c and figure out whether specializing to $r = 2$ gives zero or not. If n is even:

$$p(C_n, 2) = (2-1)^n + (2-1) = 2$$

So there are exactly two 2-colorings of C_n for n even, and in particular C_n is bipartite for n even. If n is odd:

$$p(C_n, 2) = (2-1)^n - (2-1) = 0$$

So there are no 2-colorings of C_n for n odd. In other words, C_n is not bipartite for n odd.

Section 5.3, Problem 13

Sample proof of Problem 13. If $m = 0$ then $p(G, r) = r^n$ so $b_1 = 0$. This is the base case for induction on m. The inductive assumption is that the for any n and any $m' < m$, the chromatic polynomial is $(r^n - m' r^{n-1} +$[terms of degree $< n-1$]). Now for any edge e, by deletion/contraction $p(G, r) = p(G-e, r) - p(G/e, r)$. But G/e is a graph with $n-1$ vertices, so its chromatic polynomial is $(r^{n-1} +$ [terms of degree $< n-1$]) by induction, or by the "monic" assertion in Corollary 5.3.7. Since $G - e$ is a graph with n vertices and $m-1$ edges, by induction, its chromatic polynomial is $(r^n - (m-1)r^{n-1} +$ [terms of degree $< n-1$]). Thus

$$\begin{aligned}
p(G, r) &= p(G-e, r) - p(G/e, r) \\
&= (r^n - (m-1)r^{n-1} + [\text{terms of degree } < n-1]) - (r^{n-1} + [\text{terms of degree } < n-1]) \\
&= (r^n - m r^{n-1} + [\text{terms of degree } < n-1]). \quad \square
\end{aligned}$$

Section 5.3, Problem 21

One approach: Using the factorization $x^6 - 12x^5 + 54x^4 - 112x^3 + 105x^3 - 36x = x(x-1)^2(x-3)^2(x-4)$, we see that the zeros of this polynomial are 0, 1, 3 and 4. If this were a chromatic polynomial of some graph, then that graph would in particular have a 2-coloring but no 3 coloring. That's ridiculous. Why?

Another approach: find a positive integer such that this polynomial evaluates to a negative number. (For example, $x = 2$.) That makes no sense. (The number of ways to color some graph with 2 colors is negative?)