MA/CSC 416
Homework 10, Comments and some solutions.

Section 2.5, Problem 11b

Like part a from the last assignment, this was also a specialization of Theorem 2.5.4, although this time, it took a little more thought to see what to do. Let's talk through the thought process (although this is much more than I would ever want **you** to write):

Looking at the right side, it is apparent that we want a version of Theorem 2.5.4 with n replacing m and a different dummy variable for the summation:

$$\sum_{i=1}^{n} s(n,i)x^i = x(x+1)(x+2)\cdots(x+n-1).$$

It also looks like we want to set $x = n$, but not quite. If we set $x = n$, we get a sum that is missing the $\pm$ signs we want. **How do we get signs to alternate in a sum of powers of x?** Set x to be something negative! If we set $x = -n$, we get:

$$\sum_{i=1}^{n} s(n,i)(-n)^i = (-n)(-n+1)(-n+2)\cdots(-n+n-1),$$

Notice that the signs in the sum are now wrong when n is odd. So multiply through by $(-1)^n$ (and separate out the signs from the powers of n):

$$\sum_{i=1}^{n} (-1)^{n-i} s(n,i)n^i = (-1)^n \cdot (-n)(-n+1)(-n+2)\cdots(-1),$$

Here, I also used the fact that $(-1)^{n+i} = (-1)^{n-i}$ for any i. Now the summation is exactly the right side of what we want. So I guess we're hoping that $(-1)^n \cdot (-n)(-n+1)(-n+2)\cdots(-1) = n!$. Since there are n factors in $(-n)(-n+1)(-n+2)\cdots(-1)$ and n factors of (-1) in $(-1)^n$, we give one (-1) to each factor of $(-n)(-n+1)(-n+2)\cdots(-1)$. That gives $(n)(n-1)(n-2)\cdots(1)$, which equals $n!$. So, we're done, and it's time to write this concisely to turn in. "Concisely" could be something as short as: "Set $m = n$ and $x = -n$ in Theorem 2.5.4 and then multiply by $(-1)^n$."

I am a little annoyed because Equation (2.34) provided you an opportunity to do the problem without understanding what was going on. It is true that you can get the desired equation by setting $x = m = n$ in (2.34). But if you don't understand where (2.34) came from, you don't understand where the desired equation comes from. Here's the point: You can get (2.34) by replacing x by $-x$ in Theorem 2.5.4, and then multiplying through by $(-1)^n$.

Section 2.5, Problem 14

Sample solution. The number $s(m+1, n+1)$ counts permutations in S_{m+1} with exactly $n+1$ cycles. Given such a permutation p, let k be the number of elements of $[m]$ **not** contained in the same cycle of p as the element $m+1$. The number k can by as large as m or as small as n (but not smaller, since the k elements form n cycles). The Second Counting Principle implies that

$$s(m+1,n+1) = \sum_{k=n}^{m} \left|\{p \in S_{m+1} : \text{p has } n+1 \text{ cycles and } m+1 \text{ is in a cycle of size } m-k+1\}\right|.$$

To calculate the term for a given k, we apply the Fundamental Counting Principle. First, choose k elements of $[m]$ to not share a cycle with $m+1$ ($\binom{m}{k}$ ways). Next, choose a permutation of those k elements with exactly n cycles ($s(k,n)$ ways). Finally, make a cycle of the $m-k+1$ remaining elements of $[m+1]$. As discussed in class, there are $(t-1)!$ permutations with one cycle of a t-element set. Thus there are $(m-k)!$ ways to make a cycle from these $m-k+1$ elements.

$$s(m+1,n+1) = \sum_{k=n}^{m} \binom{m}{k} s(k,n)(m-k)!. \qquad \square$$

Section 2.5, Problem 15

We will do this in class.

1. Additional problem

All of these were immediate from thinking about the definition of E_r. (The only wrinkle: The empty sum is zero.) Please ask me about these if you still have questions.

Section 4.2, Problem 2

The book and your notes have lots of examples of manipulations of generating functions similar to what you would need to do these. You'll want to make sure you understand these before the final. To make things simpler, I'll refer to the generating function $g(x)$ from part a as $g_a(x)$, the generating function $g(x)$ from part b as $g_b(x)$, etc.

There were other ways to do many of these problems.

Part a. We did this in class many times.

$$g_a(x) = \sum_{n\geq 0} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

Part b.

$$g_b(x) = \sum_{n\geq 1} x^n = x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \cdots) = \frac{x}{1-x}$$

or

$$g_b(x) = \left(\sum_{n\geq 0} x^n\right) - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

Part c.

$$g_c(x) = \sum_{n\geq 2} x^n = x^2 + x^3 + x^4 + \cdots = x^2(1 + x + x^2 + \cdots) = \frac{x^2}{1-x}$$

or

$$g_c(x) = \left(\sum_{n\geq 0} x^n\right) - 1 - x = \frac{1}{1-x} - 1 - x = \frac{x^2}{1-x}$$

Part d. This touches on an important theme: substitutions (or "plugging in") in generating functions. And in particular, how do you get an alternating sign? Substitute negative-something for x. (See comments on Section 2.5, Problem 11b.)

$$g_d(x) = \sum_{n\geq 0} (-1)^n x^n = \sum_{n\geq 0} (-x)^n = g_a(-x) = \frac{1}{1+x}$$

Part e. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}x} g_a(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n\geq 0} x^n = \sum_{n\geq 0} \frac{\mathrm{d}}{\mathrm{d}x} x^n = \sum_{n\geq 0} n x^{n-1} = \sum_{n\geq 0} (n+1) x^n$$

So

$$g_e(x) = \frac{\mathrm{d}}{\mathrm{d}x} g_a(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} = \frac{1}{(1-x)^2}.$$

Part f.

$$g_f(x) = \sum_{n\geq 0} n x^n = \sum_{n\geq 0} (n+1) x^n - \sum_{n\geq 0} x^n = g_e(x) - g_a(x) = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

Part g.

$$g_g(x) = \sum_{n\geq 0} (-1)^n n x^n = \sum_{n\geq 0} n(-x)^n = g_f(-x) = \frac{-x}{(1+x)^2}$$

Section 4.2, Problems 3a and 4a

You can get these by doing exactly what I did in examples in class (or what Merris did in the book), with different numbers. Make sure you know how to do this.

The answer to 3a is in the back of the book: $\dfrac{1-x}{1-3x-2x^2}$.

For 4a, I get $a_n = 4 \cdot 3^n + 3 \cdot (-2)^n$.