MATH 341 Fall 2023, TEST 2 answers

1a. $\mathcal{L}\left(e^{2t}\right) = \int_0^\infty e^{-st}e^{2t}dt = \int_0^\infty e^{(2-s)t}dt = \left[\frac{1}{2-s}e^{(2-s)t}\right]_0^\infty = \frac{1}{s-2}, \qquad s > 2.$

To the integral, use a substitution $u = (2-s)t$.

1b. $\mathcal{L}\left(t^4e^t\delta(t-2)\right) = \int_0^\infty e^{-st}t^4e^t\delta(t-2)dt = e^{-s\cdot 2}2^4e^2 = 16e^{-2s+2}$

2. No. The function has an asymptote at $t = 1$. (That's all you had to say, but I'll explain a bit more: If there is an asymptote, then the one-sided limits are infinite, so the function is not piecewise continuous.)

3. As discussed in class and in your reading, if $F(s)$ is the Laplace transform of a piecewise continuous function of exponential order then $\lim_{s\to\infty} F(s)$ is 0. But $\lim_{s\to\infty}(1 + e^{-s})$ is 1.

4. We transform the IVP and solve for $Y(s)$:

$$\begin{aligned} sY(s) - 0 - Y(s) &= \frac{1}{s-1} \\ (s-1)Y(s) &= \frac{1}{s-1} \\ Y(s) &= \frac{1}{(s-1)^2} \end{aligned}$$

Looking in the table, we see that $y(t) = te^t$.

5. Parts a–c are handled with the table and linearity.

$$\mathcal{L}\left(3e^{-t} - 7e^{6t}\right) = \frac{3}{s+1} - \frac{7}{s-6}.$$

$$\mathcal{L}\left(t^3 - 4t + 1\right) = \frac{6}{s^4} - \frac{4}{s^2} + \frac{1}{s}.$$

$$\mathcal{L}\left(\frac{1}{4}\sin 8t\right) = \frac{2}{s^2+64}.$$

Part d can be done in at least two ways. One way: $\mathcal{L}\left(e^t\sin t\right) = \frac{1}{(s-1)^2+1}$ (table). So Differentiation of Transforms says that

$$\mathcal{L}\left(te^t\sin t\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{(s-1)^2+1} = \frac{2(s-1)}{[(s-1)^2+1]^2}.$$

The other way: $\mathcal{L}\left(t\sin t\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left(\sin t\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = \frac{2s}{[s^2+1]^2}$ (Differentiation of Transforms and table). If we say $F(s) = \frac{2s}{[s^2+1]^2}$, then Translation on the s-Axis says that

$$\mathcal{L}\left(te^t\sin t\right) = F(s-1) = \frac{2(s-1)}{[(s-1)^2+1]^2}.$$

Part e can also be done in two ways. One way is to actually do the integral and then do the transform. That's a pain! But here is a much easier way: The left side fits into the trick Transforms of Integrals with $f(t) = t\cos t$. Since $\mathcal{L}\left(e^t\cos t\right) = \frac{s-1}{(s-1)^2+1} = \frac{s-1}{s^2-2s+2}$ (table), Transforms of Integrals says that

$$\mathcal{L}\left(\int_0^t e^\tau\cos\tau\ \mathrm{d}\tau\right) = \frac{1}{s}\cdot\frac{s-1}{s^2-2s+2} = \frac{s-1}{s(s^2-2s+2)} = \frac{s-1}{s^3-2s^2+2s}.$$

Part f is the Laplace Transform of the convolution of t^5 and $\sin t$. So the answer is $\mathcal{L}\left(t^5\right) \cdot \mathcal{L}(\sin t)$ (meaning ordinary multiplication!), and by the table, that's

$$\frac{120}{s^6} \cdot \frac{1}{s^2+1} = \frac{120}{s^6(s^2+1)}.$$

The key to Part g is to notice that $f(t) = u(t-1)g(t-1)$, where $g(t)$ is $e^t \cos t$. Then Translation on the t-Axis says that

$$\mathcal{L}\left(f(t)\right) = e^{-s}G(s) = e^{-s}\frac{s-1}{(s-1)^2+1} = \frac{e^{-s}(s-1)}{(s-1)^2+1}$$

6.a. $\mathcal{L}^{-1}\left(\frac{s+1}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \cos t + \sin t.$

6.b. $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$, so Translation on the t-Axis says that $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s-1}\right) = u(t-3)e^{t-3}$.

6.c. There are at least two ways to do this. One is to rewrite $\frac{1}{s(s+1)}$ as $\frac{1}{s} - \frac{1}{s+1}$ (partial fractions), so $\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) = 1 - e^{-t}$. The other way is to see from the table that $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$ is e^{-t}. Then Transforms of integrals says that $\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right)$ is $\int_0^t e^{-\tau} d\tau = 1 - e^{-t}$.

6.d. Set $F(s) = \arctan s - \frac{\pi}{2}$, so we want to find $f(t)$. Then $F'(s) = \frac{1}{s^2+1}$. The table says $\mathcal{L}^{-1}\left(F'(s)\right) = \sin t$. Differentiation of Transforms tells us that $\mathcal{L}^{-1}\left(F'(s)\right) = -tf(t)$. So $-tf(t) = \sin t$, and $f(t) = \frac{-\sin t}{t}$. You could check this by using Integration of Transforms to compute $\mathcal{L}\left(\frac{-\sin t}{t}\right)$.

7. We write $\frac{s+5}{(s+2)(s-1)(s+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1}$ and calculate $A = 1$, $B = 1$, and $C = -2$. If you ask me, I can show you how these were calculated.

8.a. Since $\mathcal{L}\left(t^3\right) = \dfrac{6}{s^4}$, $\quad \mathcal{L}^{-1}\left(\dfrac{1}{s^4}\right) = \dfrac{t^3}{6}$.

8.b.

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) &= t * t \\ = \int_0^t \tau \cdot (t-\tau)\, \mathrm{d}\tau = \int_0^t t\tau - \tau^2\, \mathrm{d}\tau = \left[\frac{\tau^2}{2} - \frac{\tau^3}{3}\right]_0^t &= \frac{t^3}{6}. \end{aligned}$$

9. $s^2Y(s) + Y(s) = e^{-\pi s} + e^{-2\pi s}$.
$Y(s) = \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$.
$y(t) = u(t-\pi)\sin(t-\pi) + u(t-2\pi)\sin(t-2\pi)$. (By Translation on the t-Axis.)
Recall that $\sin(t-\pi) = -\sin t$ and $\sin(t-2\pi) = \sin t$.

So $y(t) = \begin{cases} 0 & \text{if } t < \pi \\ \sin(t-\pi) & \text{if } \pi \le t < 2\pi \\ \sin(t-\pi) + \sin(t-2\pi) & \text{if } 2\pi \le t \end{cases} = \begin{cases} 0 & \text{if } t < \pi \\ -\sin t & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases}$

From there, you can sketch it. (Between π and 2π, it looks like one "“bump” of a sinusoidal wave, starting at zero and ending at zero. And of course, everywhere else, it's zero. This ODE is modeling a mass on a spring that is just sitting there, and then you kick it. As soon as it comes back to equilibrium, you kick it again, and that stops it.)

10. There are two good ways to do this, although the second involves some subtlety in determining a hidden second initial condition.

Transform method:
The equation transforms to

$$sF(s) - 1 + 2F(s) + 2\frac{F(s)}{s} = 0.$$

We simplify

$$F(s)\left[s + 2 + \frac{2}{s}\right] = 1,$$

$$F(s)\left[\frac{s^2 + 2s + 2}{s}\right] = 1,$$

so that

$$F(s) = \frac{s}{s^2 + 2s + 2}.$$

Rewrite

$$F(s) = \frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1},$$

and use the table to find

$$f(t) = e^{-t}\cos t - e^{-t}\sin t.$$

Differentiation method:
Differentiate both sides of the equation (keeping in mind the fundamental theorem of calculus):

$$f''(t) + 2f'(t) + 2f(t) = 0.$$

This is now an equation I could have given you outside the Challenge section of the test. But where is your second initial condition? If you plug in $t = 0$ to the original equation, you get $f'(0) - 2f(0) = 0$, and since you know $f(0) = 1$, you can solve for $f'(0) = -2$. That will lead you to

$$f(t) = e^{-t}\cos t - e^{-t}\sin t.$$

11. By Integration of Transforms,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\int_s^\infty \frac{2}{(\sigma^2+1)^2}d\sigma\right) &= \frac{1}{t}\mathcal{L}^{-1}\left(\frac{2}{(s^2+1)^2}\right) \\
&= \frac{1}{t}(\sin t - t\cos t) \\
&= \frac{\sin t}{t} - \cos t
\end{aligned}$$

Thus the integral we want to evaluate is the Laplace transform of $\frac{\sin t}{t} - \cos t$. We have to use Integration of transforms to transform $\frac{\sin t}{t}$. The answer is:

$$\begin{aligned}
\mathcal{L}\left(\frac{\sin t}{t} - \cos t\right) &= \int_s^\infty \frac{1}{\sigma^2+1}d\sigma - \frac{s}{s^2+1} \\
&= [\arctan\sigma]_s^\infty - \frac{s}{s^2+1} \\
&= \frac{\pi}{2} - \arctan s - \frac{s}{s^2+1}
\end{aligned}$$

Laplace transform is a powerful technique! (I don't know a better way to do this integral, besides finding it on a fairly detailed table.)

12. Rewrite $f(t)$ as $u(t-1)\cdot(t-1)$. Here $t-1$ is $g(t-1)$ for $g(t)=t$. So Translation on the $t-Axis$ says that $\mathcal{L}\left(f(t)\right) = \dfrac{e^{-s}}{s^2}$. (As usual, call this $F(s)$.) Definitely $f(0)=0$. Also, $f(t)$ is just a horizontal line near $t=0$, so $f'(0)=0$ also. Thus Transforms of Derivatives says that $\mathcal{L}\left(f''(t)\right) = s^2F(s) - s\cdot 0 - 0 = s^2F(s) = e^{-s}$. Now the table says that $f''(t)$ is $\delta(t-1)$. Why does this make sense?

13. Show that the Laplace transform of $\quad f(t) = e^{-\frac{1}{2}t^2} \quad$ is $\quad F(s) = e^{\frac{1}{2}s^2}\displaystyle\int_s^\infty e^{-\frac{1}{2}\sigma^2}\,\mathrm{d}\sigma.$

We compute $f'(t) = -te^{-\frac{1}{2}t^2}$. Since $f(0)=1$, Transforms of Derivatives says that $\mathcal{L}\left(f'(t)\right) = sF(s)-1$. On the other hand, since $f'(t) = -tf(t)$, Differentiation of Transforms says that $\mathcal{L}\left(f'(t)\right) = F'(s)$. We set these two expressions for $\mathcal{L}\left(f'(t)\right)$ equal to obtain an ODE: $F'(s) = sF(s)-1$. This is first-order linear, and we rearrange in the usual way: $F'(s) - sF(s) = -1$. We solve like usual ($\mu(s) = e^{-\frac{1}{2}s^2}$) and get as far as $e^{-\frac{1}{2}s^2}F(s) = -\int e^{-\frac{1}{2}s^2}\,\mathrm{d}s$, but we don't know that antiderivative. (It can't be written using the functions we usually use.) We can't leave it as an indefinite integral, because then we only know $F(s)$ up to a constant. The suggested formula for $F(s)$ takes one possibility for this antiderivative. (Putting the s as the lower limit of integration took away the minus sign.)

Let's check that this is the right one by showing that $\lim_{s\to\infty} F(s) = 0$ for this formula for $F(s)$. (Remember that Laplace transforms of piecewise linear functions of exponential order have that property.)

$$\lim_{s\to\infty} e^{\frac{1}{2}s^2}\int_s^\infty e^{-\frac{1}{2}\sigma^2}\,\mathrm{d}\sigma = \lim_{s\to\infty}\frac{\int_s^\infty e^{-\frac{1}{2}\sigma^2}\,\mathrm{d}\sigma}{e^{-\frac{1}{2}s^2}}$$

This is a $\frac{0}{0}$ indefinite form, so we use l'Hôpital's Rule (differentiate top and bottom) to rewrite the limit as

$$\lim_{s\to\infty}\frac{-e^{-\frac{1}{2}s^2}}{-se^{-\frac{1}{2}s^2}} = \lim_{s\to\infty}\frac{1}{s} = 0.$$

Finally, in case there were so many steps that you're worried that we made a mistake, let's check that the think we're calling $F(s)$ really satisfies the ODE: We compute the derivative of $F(s) = e^{\frac{1}{2}s^2}\int_s^\infty e^{-\frac{1}{2}\sigma^2}\,\mathrm{d}\sigma$ using the Product rule and the Fundamental Theorem of Calculus:

$$F'(s) = e^{\frac{1}{2}s^2}(-e^{-\frac{1}{2}s^2}) + se^{\frac{1}{2}s^2}\int_s^\infty e^{-\frac{1}{2}\sigma^2}\,\mathrm{d}\sigma = -1 + sF(s).$$