MATH 341, Fall 2023, TEST 1 ANSWERS.

1. Determine if $y = xe^x$ is a solution to the differential equation $y'\,y = ye^x + x^2e^{2x}$.

ANSWER: Calculate $y' = xe^x + e^x$ and plug in: $(xe^x + e^x)xe^x \stackrel{?}{=} xe^x \cdot e^x + x^2e^{2x}$. Yes!

COMMENT: If you tried some method of solving this ODE, you were wasting your time (and trying to do something hard that we haven't learned). You were given the proposed solution. Just test it.

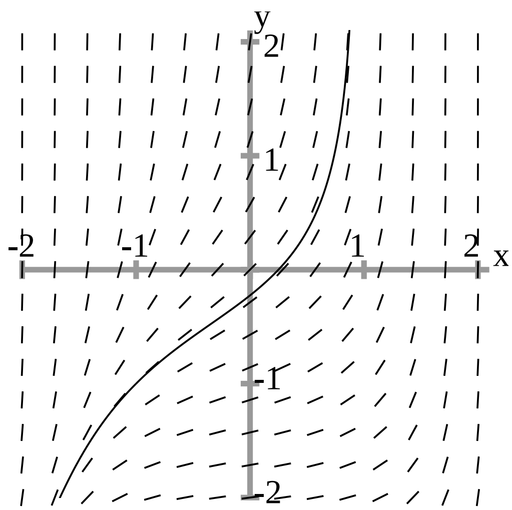
2. Consider the direction field drawn below.

a. Sketch (right on the picture shown) a reasonable approximation to the solution to the differential equation whose direction field is shown, subject to the initial condition $y(-1) = -1$.

b. Of the six ODEs below, circle the one whose direction field is shown, and cross out all the others.

ANSWER:

2a.



2b. There were many ways to eliminate wrong choices. Here's how I did it: Consider the slope mark at $(0,0)$, which looks like it represents a slope of about 1. Plug in $x = 0$ and $y = 0$ to the choices and see if you get something not close to 1. You can eliminate $y' = x\sin(y)$, $y' = y^2$ and $y' = \sin(x)$ because they all give zero. Now, for $y' = \cos(x)$, the slope marks shouldn't change when you change y (i.e. move up or down) without changing x. That's certainly not what happens in the picture. Looking at the two remaining choices, think about $y' = 1 - x^2y$. Its direction field should show slope-zero at $(1,1)$, but this doesn't match the picture. The answer is $y' = e^{x^2+y}$.

3. Consider the following differential equation.

$$y' + \frac{1}{x^2}y = e^{\frac{1}{x}}.$$

a. Find the general solution.

b. Find the solution to the initial value problem with the above ODE and y(1)=0.

ANSWER:

3a. This is a first-order linear ODE, with $P(x) = \frac{1}{x^2}$ so $\mu(x) = e^{\frac{-1}{x}}$.

Multiplying through by $\mu(x)$ and integrating both sides (remembering that we reverse-engineered the product rule on the left side), we get to

$$ye^{\frac{-1}{x}} = x + C \quad \text{so} \quad y = xe^{\frac{1}{x}} + Ce^{\frac{1}{x}}$$

Don't forget that we have to divide **everything**, including the "C" by $\mu(x)$.

3b. Set $x = 1$ and $y = 0$ in the general solution you found, and solve for C:

$$0 = e + Ce \qquad \text{so} \qquad Ce = -e \qquad \text{so} \qquad C = -1.$$

The answer was

$$y = xe^{\frac{1}{x}} - e^{\frac{1}{x}}.$$

4. Consider the ODE $y^3 y' = (x+1)^2$.
a. Find a one-parameter family of solutions to the ODE. It is okay to leave your solution in implicit form.
b. Recall that the Existence and Uniqueness Theorem for First-order ODEs says:

> Consider the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \quad y(x_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $R = \{(x,y) : a < x < b,\ c < y < d\}$ that contains the point (x_0, y_0) then the IVP has a unique solution in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

Does the Existence and Uniqueness Theorem guarantee a unique solution to the IVP $y^3 y' = (x+1)^2, \quad y(3) = 0$? Why or why not?
ANSWER:
4a. This is a separable ODE, and in fact, it's already separated for you (except for moving the dx to the right side). Separate and integrate:

$$\int y^3 \,\mathrm{d}y = \int (x+1)^2 \,\mathrm{d}y.$$

$$\frac{y^4}{4} = \frac{(x+1)^3}{3} + C.$$

You can stop there or simplify further.
4b. No, it doesn't. The function $f(x,y) = \frac{(x+1)^3}{y^3}$ isn't even defined, much less continuous, at $(3,0)$.

5. I claim that the ODE with differential form $3x^2y^3 \,\mathrm{d}x + (3x^3y^2 + ay)\,\mathrm{d}y = 0$ describes the level sets of some F. (Your a was -18, 14, -6 or 10.)
a. **Without finding** F, check that my claim is correct.
b. Write the general solution for the ODE.
ANSWER:
5a. (First, the thought process:)
An ODE describes the level sets of a function $F(x,y)$ if the ODE can be written in differential form as $\frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y = 0$. The function $F(x,y)$ would have $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$.
The part in front of dx must be $\frac{\partial F}{\partial x}$ and the part in front of dy must be $\frac{\partial F}{\partial y}$. So to check $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$, I'm checking

$$\frac{\partial}{\partial y}(\text{the part in front of d}x) \stackrel{?}{=} \frac{\partial}{\partial x}(\text{the part in front of d}y).$$

(Now what you write:)
Check that $\frac{\partial}{\partial y}(3x^2y^3) \stackrel{?}{=} \frac{\partial}{\partial x}(3x^3y^2 + ay)$. Both come out to $9x^2y^2$, so the answer is yes.

5b. Thinking as described in part a, you know that $\frac{\partial F}{\partial x} = 3x^2y^3$. Integrating $\int \mathrm{d}x$, we find that $F = x^3y^3 + g(y)$ where $g(y)$ is an arbitrary function of y. But we also know that $\frac{\partial F}{\partial y} = 3x^3y^2 + ay$ and plugging in $F = x^3y^3 + g(y)$ to $\frac{\partial F}{\partial y} = 3x^3y^2 + ay$, we get $3x^3y^2 + g'(y) = 3x^3y^2 + ay$, so $g'(y) = ay$. Solving this simple ODE, $g(y) = \frac{a}{2}x^2 + C$. The final answer is $F = x^3y^3 + \frac{a}{2}x^2$. (We don't need to bother with the C.) The point now is to remember that that the solutions are level curves of some function, and we just found the function.

$$\text{ANSWER:} \quad x^3y^3 + \frac{a}{2}x^2 = C \quad \text{for constants } C.$$

6. A tank initially holds 600 L of brine with 20 kg of salt. A salt solution (concentration 0.3 kg/L) flows into a tank at a constant rate of 9 L/min. The solution flows out of the tank at a constant rate of 7 L/min. Assuming instantaneous, perfect mixing in the tank, write an IVP describing $S(t)$, the mass of salt in the tank at time t minutes. **Do not solve** the IVP, just write it down.

ANSWER:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2.7 - \frac{7S}{600 + 2t} \qquad S(0) = 20.$$

What you think but don't have to write:
The 2.7 is 0.3 kg/L times 9 L/min (the salt flowing in). The $\frac{7S}{600+2t}$ is the salt flowing out: 7 L/min times the concentration in the tank, which is S divided by volume. (You solved a separate, easy ODE to find that the volume at t minutes is $600 + 2t$.
(Your numbers might have been different.)

7. Find the general solution to each of the following ODEs. Use t as an independent variable.

7a. $y'' + 2y' - 15y = 0$
$r^2 + 2r - 15 = 0$, $r = -5, 3$, $y(t) = c_1e^{-5t} + c_2e^{3t}$.

7b. $y'' + 8y' + 16y = 0$
$r^2 + 8r + 16 = 0$, $r = -4$ (repeated), $y(t) = c_1e^{-4t} + c_2te^{-4t}$.

7c. $y''' + 4y'' - 2y' - 20y = 0$ *(Hint: e^{2t} is a solution.)*
$r^3 + 4r^2 - 2r - 20 = 0$. Since e^{2t} is a solution, we know $r - 2$ is a factor.
$(r - 2)(r^2 + 6r + 10) = 0$, so $r = 2$ or $r = \frac{-6\pm\sqrt{36-40}}{2} = -3 \pm i$.

$$y(x) = c_1e^{2t} + c_2e^{-3t}\cos t + c_3e^{-3t}\sin t.$$

7c. *Sorry, this is the one that had the typo. On the test it was $11y'$.*
$y''' - 6y'' + 13y' - 10y = 0$ *(Hint: e^{2t} is a solution.)* $r^3 - 6r^2 + 13r - 10 = 0$. Since e^{2t} is a solution, we know $r - 2$ is a factor.
$(r - 2)(r^2 - 4r + 5) = 0$, so $r = 2$ or $r = \frac{4\pm\sqrt{16-20}}{2} = 2 \pm i$.

$$y(x) = c_1e^{2t} + c_2e^{2t}\cos t + c_3e^{2t}\sin t.$$

7d. $y'' - y = t^2$
The auxiliary equation is $r^2-1 = 0$ so the solution to the associated homogeneous ODE is $y(t) = c_1e^t+c_2e^{-t}$. We find the collection of terms t^2, t, 1 by differentiating t^2 repeatedly, and we write $y_p = At^2 + Bt + C$. So $y_p'' = 2A$ and we substitute into the ODE to get $2A - (At^2 + Bt + C) = t^2$. This gives three equations $-A = 1$ (coefficient of t^2) and $-B = 0$ (coefficient of t) and $2A - C = 0$ (constant term). We solve $A = -1$ and $B = 0$ and $C = -2$, so a particular solution is $y_p = -t^2 - 2$. The general solution is

$$y = -t^2 - 2 + c_1e^t + c_2e^{-t}.$$

8. Solve the IVP $y'' + y = 1 \quad y(0) = a \quad y'(0) = 1$
(Your a was 2, 4, 6, or 8.)

ANSWER:
The auxiliary equation is $r^2 + 1 = 0$ so the solution to the associated homogeneous ODE is $y(t) = c_1 \sin t + c_2 \cos t$. (Danger: Don't use the initial conditions at this point to find c_1 and c_2. The initial conditions are for the nonhomogeneous ODE $y'' + y = 1$, **not** for the associated homogeneous ODE $y'' + y = 0$.)

To find a particular solution, a good initial guess is $y_p = A$ and this doesn't conflict with the solution to the associated homogeneous ODE, so we don't revise it. Calculate $y_p'' = 0$, and substitute into $y'' + y = 1$, to get $A = 1$. A particular solution is $y_p = 1$, so the general solution is $y = 1 + c_1 \sin t + c_2 \cos t$.

Now, $y(0) = a$ says $a = 1 + c_2$, so $c_2 = a - 1$. Also, $y' = c_1 \cos t - c_2 \sin t$, so $y'(0) = 1$ says $1 = c_1$.
The solution to the IVP is $y = 1 + \sin t + (a - 1) \cos t$.

9. For each of the following equations, write down the **form** of the particular solution which you would find using the Method of Undetermined Coefficients. You **do not** need to determine the coefficients. To save you time, I have solved each auxiliary equation for you.

$y'' - 4y' + 4y = e^t$	
$r = 2$ (repeated)	$y_p = Ae^t$
$y'' - 6y' + 9y = e^t$	
$r = 3$ (repeated)	$y_p = Ae^t$
$y'' + 4y' - 5y = t^2 \cos t$	
$r = 1, -5$	$y_p = At^2 \cos t + Bt^2 \sin t + Ct \cos t + Dt \sin t + E \cos t + F \sin t$
$y'' - 6y' + 9y = e^{3t}$	
$r = 3$ (repeated)	$y_p = At^2 e^{3t}$
$y'' - 4y' + 4y = e^{2t}$	
$r = 2$ (repeated)	$y_p = At^2 e^{2t}$

10. Consider the second-order ODE $y'' = yy'$. Note that we have not taught you a method that applies to this ODE. Use your creativity!

a. Find a one-parameter family of solutions.
ANSWER: The constant functions $y \equiv C$.

b. Find a solution not represented in the one-parameter family.
ANSWER: Maybe you found another one, but here's what I found: Try a function of the form $y = ax^n$ for a constant a and a constant n. Plugging into the ODE we get:

$$an(n-1)x^{n-2} = ax^n(anx^{n-1}).$$

For this to be an equality you need two things: first, the exponent for x has to be the same on both sides, and second, the constant needs to be the same on both sides. The exponents give the equation $n - 2 = 2n - 1$ with solution $n = -1$. The constant gives the equation (using the fact that $n = -1$): $a(-1)(-2) = a^2(-1)$ with solutions $a = 0$ and $a = -2$. But $a = 0$ would give the constant solution $y \equiv 0$, which was in the one-parameter family, so take a=-2. The solution is $y = \frac{-2}{x}$.

c. Find a two-parameter family of solutions.
ANSWER: I don't know the answer to this question. (It was a challenge question!) Our intuition tells us that there ought to be some 2-parameter family of solutions, but it's possible that it's not something we can write down. Did you find one? By the way, this is **not** a linear ODE, so you **can't** just write down something like $y = C_1 + C_2 \frac{-2}{x}$.

11. This problem is about sines and cosines of imaginary numbers.
As usual, i is the imaginary constant $\sqrt{-1}$.

a. Prove that $\cos(i\theta) = \frac{1}{2}(e^{\theta} + e^{-\theta})$.
ANSWER:
We know that $e^{i\theta} = \cos\theta + i\sin\theta$. Also, since cosine is an even function and sine is an odd function, $e^{-i\theta} = \cos\theta - i\sin\theta$. If we add these two formulas together, the sines cancel and we get $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, or $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. Substituting $i\theta$ for θ and using $i^2 = -1$, that's

$$\cos(i\theta) = \frac{1}{2}(e^{-\theta} + e^{\theta}).$$

Interestingly, if θ is a real number, then $\cos(i\theta)$ is also a real number. Sometimes, people call this the "hyperbolic cosine" $\cosh\theta = \frac{1}{2}(e^{-\theta} + e^{\theta})$.

b. Give a formula for $\sin(i\theta)$.
ANSWER:
This time, subtract the two equations: Taking $e^{i\theta} = \cos\theta + i\sin\theta$ minus $e^{-i\theta} = \cos\theta - i\sin\theta$, the cosines cancel, and we get $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ or $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, or better, $\sin\theta = \frac{-i}{2}(e^{i\theta} - e^{-i\theta})$. Again substituting $i\theta$ for θ, that's $\sin(i\theta) = \frac{-i}{2}(e^{-\theta} - e^{\theta})$, or

$$\sin(i\theta) = \frac{i}{2}(e^{\theta} - e^{-\theta}).$$

Interestingly, if θ is a real number, then $\sin(i\theta)$ is an imaginary number. It is i times the "hyperbolic sine" $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$.

12. Find the general solution to $ty'' + 2y' + ty' + y = 0$. (Hint: Write this as $[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (2+t)\frac{\mathrm{d}}{\mathrm{d}t} + 1]\,y = 0$. Can you "factor" the linear operator?)
ANSWER:
This is a hard problem. I have no idea if I would have figured this out if I were taking the test...I made the problem up backwards, starting from the answer. (But if you managed to factor the linear operator, the work you did afterwards was like something I showed you in class.) Anyway:

You can factor $[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (2+t)\frac{\mathrm{d}}{\mathrm{d}t} + 1]\,y = 0$ as $\left[\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\left[t\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\,y = 0$.

What does that mean? If you apply the linear operator $\left[t\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]$ to y you get $t\,\dfrac{\mathrm{d}y}{\mathrm{d}t} + y$. If you then apply the linear operator $\left[\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]$ to $t\,\dfrac{\mathrm{d}y}{\mathrm{d}t} + y$ (and if you don't forget to use product rule), you get $(t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t}) + \frac{\mathrm{d}y}{\mathrm{d}t} + t\,\dfrac{\mathrm{d}y}{\mathrm{d}t} + y$. That's the same as if you apply $[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (2+t)\frac{\mathrm{d}}{\mathrm{d}t} + 1]$ to y.

Now let z be $\left[t\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\,y$, so the equation is $\left[\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\,z = 0$. This has general solution $z = c_1e^{-t}$. But the equation that defines z ("Let z be $\left[t\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\,y$") is an ODE that determines y: It's $\left[t\frac{\mathrm{d}}{\mathrm{d}t} + 1\right]\,y = z$. In other words, to find y, we solve the first-order linear ODE $t\frac{\mathrm{d}y}{\mathrm{d}t} + y = c_1e^{-t}$. The general solution is $y = c_1\dfrac{e^{-t}}{t} + c_2\dfrac{1}{t}$.