MATH 341 Fall 2023, POULTRY QUIZ answers

1. $\vec{x}\,' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}$. Eigenvalues: $\det \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix} = 0$, so $(3-\lambda)(-1-\lambda)+8=0$, so $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$.

For $\lambda = 1+2i$: $\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. One can check that these two equations say the same thing. We'll use the first: $(2-2i)v_1 - 2v_2 = 0$, so $v_2 = (1-i)v_1$, and we'll choose $\vec{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$. A complex solution is $\begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1+2i)t}$, which equals $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) (e^t \cos 2t + i e^t \sin 2t)$, and simplifies to

$$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \cos 2t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t \sin 2t \right) + i \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t \cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \sin 2t \right)$$

which equals

$$\begin{bmatrix} e^t \cos 2t \\ e^t \cos 2t + e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ -e^t \cos 2t + e^t \sin 2t \end{bmatrix}$$

We use the real and imaginary parts of this to write the general solution:

$$\vec{x} = c_1 \begin{bmatrix} e^t \cos 2t \\ e^t \cos 2t + e^t \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} e^t \sin 2t \\ -e^t \cos 2t + e^t \sin 2t \end{bmatrix}$$

2. Evaluating the general solution at $t = 0$, the initial condition says $\begin{bmatrix} -1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

That is two equations $-1 = c_1$ and $3 = c_2 - c_2$. We solve to get $c_1 = -1$ and $c_2 = -4$.

The solution is $\vec{x} = -\begin{bmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} - 4 \begin{bmatrix} \sin 2t \\ -\cos 2t + 2\sin 2t \end{bmatrix}$, but it would be nice to simplify and write $\vec{x} = \begin{bmatrix} -\cos 2t - 4\sin 2t \\ 3\cos 2t - 7\sin 2t \end{bmatrix}$

3. Find the critical points for the system below. Write your answer as a list of xy-pairs, like "(3, 4), (−2, 5), and (0, 0)". (That's not really the answer!)

$$\begin{cases} x' &= x - y \\ y' &= x^2 + y^2 - 2 \end{cases}$$

Answer: Critical points are where $x - y = 0$ and $x^2 + y^2 - 2 = 0$. The first equation says $x = y$, so replacing x by y in the second equation, we get $y^2 + y^2 = 2$, so $y^2 = 1$, so $y = \pm 1$. For $y = 1$, we get $x = 1$ and for $y = -1$, we get $x = -1$. The critical points are $(1, 1)$ and $(-1, -1)$.

Note: The answer $x = \pm 1$ and $y = \pm 1$ is not correct! It gives four possibilities: $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. But only two of those are critical points, because the first equation says $x = y$.

4. Solve the phase plane equation for the system $\begin{cases} x' &= y \\ y' &= y^2 + ye^x \end{cases}$

Answer: Don't memorize the phase plane equation! Remember it by understanding it! The slope of the tangent line to a trajectory is $\frac{\mathrm{d}y}{\mathrm{d}x}$, which equals $\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$. That's the chain rule (or just imagine "canceling the dt's"). The autonomous system tells you what $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$ are. In this case $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + ye^x$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = y$. So the phase plane equation is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+ye^x}{y}$, which simplifies to $\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^x$.

This is a linear first-order ODE, and should be rewritten as $\frac{\mathrm{d}y}{\mathrm{d}x} - y = e^x$. The integrating factor is $\mu = e^{\int -1 \mathrm{d}x} = e^{-x}$. Multiply by μ as usual: $e^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - ye^{-x} = 1$. $ye^{-x} = x + C$. $y = xe^x + Ce^x$.

5. When and where is your final exam for this class?

ANSWER: **Make sure you find out!!!** Look at the class website if you don't know.