MATH 341, Fall 2023, QUIZ 8 answers.

1. For each function below, **use the definition** to find the Laplace transform of the function, **and** answer the question: For what s is $\mathcal{L}\left(f(t)\right)$ defined?

a. $f(t) = 1$.

ANSWER:

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,\mathrm{dt} = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}, \qquad \text{as long as } s > 0.$$

The integral does not converge if $s \le 0$.

b. $f(t) = \begin{cases} 3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t > 1 \end{cases}$.

ANSWER:

$$\begin{aligned}\mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\,\mathrm{dt} = \int_0^1 3e^{-st}\,\mathrm{dt} + \int_1^\infty e^{-st}\,\mathrm{dt} &= \left[-\frac{3}{s}e^{-st}\right]_0^1 + \left[-\frac{1}{s}e^{-st}\right]_1^\infty \\ &= \left(\frac{-3e^{-s}}{s} + \frac{3}{s}\right) + \left(0 + \frac{e^{-s}}{s}\right) \\ &= \frac{3}{s} - \frac{2e^{-s}}{s}\end{aligned}$$

The second integral only converges when $s > 0$.

2. Is the function $f(t)$ shown below continuous on $[0, 3]$ or piecewise continuous on $[0, 3]$ or both or neither? **Sketch the graph.**

$$f(t) = \frac{t^2 - 1}{t - 1}$$

ANSWER: The function $f(t) = \frac{t^2-1}{t-1}$ equals $t + 1$ as long as $t \ne 1$ and is undefined if $t = 1$. This is not continuous on $[0, 3]$ but is piecewise continuous. The graph is a line with slope 1 and y-intercept 1, with an "empty circle" at $t = 1$ indicating that the function is not defined there.

3. Is the function $e^{\sin t}$ of exponential order? **Explain briefly.**
(Your explanation can be informal, as long as it is correct.)

ANSWER: The function $e^{\sin t}$ is of exponential order because it never gets bigger than e^1 and never gets smaller than e^{-1}. (So in particular it doesn't "get big faster than exponentially.")

I admit that "exponential order" is a deceptive-sounding term. You should think of it as "[at most] exponential order". Functions like $e^{\sin t}$ that *don't* grow exponentially because they *don't really grow at all* **are** of exponential order.

4. You are given (and do not need to prove) that $\mathcal{L}(t \sin t) = \dfrac{2s}{(s^2+1)^2}$.
Use the trick Transforms of Derivatives to find $\mathcal{L}(\sin t + t\cos t)$.

ANSWER: If $f(t) = t \sin t$, then $f'(t) = \sin t + t \cos t$. Since also $f(0) = 0$, Transforms of Derivatives says

$$\mathcal{L}(\sin t + t\cos t) = s \cdot \mathcal{L}(t \sin t) = \frac{2s^2}{(s^2+1)^2}.$$