MATH 341, Fall 2023, QUIZ 5 ANSWERS.

1. Find the general solution of the following ODEs:

1a. $y'' - 8y' + 20y = 0$

ANSWER:

The auxiliary equation is $r^2 - 8r + 20 = 0$, so $r = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$. A complex-valued solution is $e^{(4+2i)t} = e^{4t}(\cos 2t + i \sin 2t)$. Take real and imaginary parts to get two linearly independent solutions. The general solution is

$$c_1 e^{4t} \cos 2t + c_2 e^{4t} \sin 2t.$$

1b. $y''' + 2y'' - 2y' - 12y = 0$. *Hint: e^{2t} is a solution.*

ANSWER:

The auxiliary equation is $r^3 + 2r^2 - 2r - 12 = 0$. This looks hard to factor, but the hint tells us that $(r-2)$ is a factor, and that helps. We know right away that the other factor has to start with r^2 and end with 6, so we just need to work out that the middle term is $4r$. $r^3 + 2r^2 - 2r - 12 = (r-2)(r^2 + 4r + 6) = 0$. The new factor looks hard to factor too, but we can use the quadratic formula again.

$$r = 1 \quad \text{or} \quad r = \frac{-4 \pm \sqrt{16-24}}{2} = \frac{-4 \pm 2\sqrt{2}i}{2} = -2 \pm \sqrt{2}i.$$

The general solution is

$$c_1 e^{2t} + c_2 e^{-2t} \cos \sqrt{2}t + c_3 e^{-2t} \sin \sqrt{2}t.$$

1c. $y'' - 5y' + 4y = e^{3t}$

ANSWER:

First, the associated homogeneous ODE: The auxiliary equation is $r^2 - 5r + 4 = 0$ so $r = 1$ or 4 and the general solution to the homogeneous ODE is $y_c = c_1 e^t + c_2 e^{4t}$.

Next, starting from the term e^{3t} we start taking derivatives and find that we don't need any additional terms for y_p. We check e^{3t} against the solution to the homogeneous ODE and find that there is no problem, so we do not put in any powers of t. Write $y_p = Ae^{3t}$ and calculate $y_p' = 3Ae^{3t}$ and $y_p'' = 9Ae^{3t}$. Putting this into the ODE, we get

$$9Ae^{3t} - 15Ae^{3t} + 4Ae^{3t} = e^{3t} \quad \text{which simplifies to} \quad -2Ae^{3t} = e^{3t}.$$

We solve $A = -\frac{1}{2}$, so $y_p = -\frac{1}{2}e^{3t}$ and the general solution is

$$-\frac{1}{2}e^{3t} + c_1 e^t + c_2 e^{4t}.$$

1d. $y'' + 2y' - 3y = e^t$

ANSWER:

First, the associated homogeneous ODE: The auxiliary equation is $r^2 + 2r - 3 = 0$ so $r = 1$ or -3 and the general solution to the homogeneous ODE is $y_c = c_1 e^t + c_2 e^{-3t}$.

Normally, for the particular solution we would try $y_p = Ae^t$. However, e^t is a solution to the associated homogeneous ODE, so this won't work. Instead, we try $y_p = Ate^t$. Then $y_p' = Ate^t + Ae^t$ and $y_p'' = Ate^t + 2Ae^t$. So we solve:

$$\begin{aligned} (Ate^t + 2Ae^t) - 5(Ate^t + Ae^t) + 4Ate^t &= e^t. \\ (A - 5A + 4A)te^t + (2A - 5A)e^t = e^t. \end{aligned}$$

This gives two equations. Comparing coefficients of te^t on both sides gives $A - 5A + 4A = 0$, which gives no information. Comparing the coefficients of e^t on both sides gives $2A - 5A = 1$, so $A = -\frac{1}{3}$. Answer:

$$y_p = -\frac{1}{3}te^t.$$

The general solution is

$$y = -\frac{1}{3}te^t + c_1e^t + c_2e^{4t}.$$

2. Consider the ODE $y'' + 3y' + 2y = t\sin t$. Write down a correct **form** for a particular solution y_p to this ODE, with undetermined coefficients. **Do not** determine the coefficients, **Do** show your work.
ANSWER:
Taking the derivative of $t\sin t$, we get $t\cos t + \sin t$. Taking the derivative of $t\cos t$, we get $-t\sin t + \cos t$. The derivative of $\cos t$ is $-\sin t$ and the derivative of $\sin t$ is $\cos t$. So the collection of terms we need is $\{t\sin t, t\cos t, \sin t, \cos t\}$.

$$y_p = At\sin t + Bt\cos t + C\sin t + D\cos t.$$