MATH 341, Fall 2023, QUIZ 4 ANSWERS

1. A tank initially holds 100 L of brine with 1 kg dissolved salt. A salt solution (concentration 0.3 kg/L) flows into a tank at a constant rate of 5 L/min. The solution flows out of the tank at a constant rate of 6 L/min. Assuming instantaneous, perfect mixing in the tank, write an IVP describing $S(t)$, the mass of salt in the tank at time t minutes. **Do not solve** the IVP, just write it down.

ANSWER: *(Your quiz might have had different numbers.)*
First, find the volume as a function of time:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 5 - 6 = -1, \text{ so } V = -t + C, \text{ and since } V(0) = 100, \text{ we find } C = 100, \text{ so } V = 100 - t.$$

No matter which copy of the quiz you had, it was always $V = 100 - t$.

Next write an ODE for $S(t)$:
The incoming flow rate (5 L/min) and incoming concentration (0.3 kg/L) mutliply together to give a positive contribution to $\frac{\mathrm{d}S}{\mathrm{d}t}$. Similarly the outgoing flow rate (6 L/min) and outgoing concentration (??) multiply together to give a negative contribution to $\frac{\mathrm{d}S}{\mathrm{d}t}$. Assuming perfect mixing, the outgoing concentration is just the mass of salt in the tank divided by the volume of solution in the tank: $\frac{S}{100-t}$. The ODE is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 5(0.3) - 6\frac{S}{100-t}, \text{ which simplifies to } \frac{\mathrm{d}S}{\mathrm{d}t} = 1.5 - \frac{6S}{100-t}.$$

You were not required to solve it, but could do so because it is first-order linear.

You were asked to write an IVP, not just an ODE. You need the initial condition (the amount of salt in the tank at time $t = 0$). So here is the final answer:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.5 - \frac{6S}{100-t} \qquad S(0) = 1.$$

For other quiz versions, the answers were:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.5 - \frac{4S}{100-t} \qquad S(0) = 3.$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.2 - \frac{3S}{100-t} \qquad S(0) = 4.$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.6 - \frac{5S}{100-t} \qquad S(0) = 2.$$

2. At what time t (in minutes) does the ODE in Problem 1 first fail to be a correct description of the mixing problem?

ANSWER: At time $t = 100$.
Two ways to think about this: (1) At time $t = 100$, the ODE has division by zero, or (2) When the tank hits empty, it's no longer possible for more water to flow out of the tank than flows in.

3. Find the general solution of the following ODEs: ***Note:*** *On a test, you should also expect to solve IVPs.*

ODE: $y'' - 2y' - 15y = 0$
$r^2 - 2r - 15 = 0$
$r = -3, 5$
GENERAL SOLUTION: $y = c_1 e^{-3t} + c_2 e^{5t}$

ODE: $y'' - 4y' - 12y = 0$
$r^2 - 4r - 12 = 0$
$r = -2, 6$
GENERAL SOLUTION: $y = c_1 e^{-2t} + c_2 e^{6t}$

ODE: $y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0$
$r = 1$
GENERAL SOLUTION: $y = c_1 e^{t} + c_2 t e^{t}$

ODE: $y'' - 4y' + 4y = 0$
$r^2 - 4r - 4 = 0$
$r = 2$
GENERAL SOLUTION: $y = c_1 e^{2t} + c_2 t e^{2t}$

4. Given functions

$$y_1(x) = \sin^2 x \qquad y_2(x) = \cos^2 x \qquad y_3(x) = 1$$

explain/show why y_1, y_2, and y_3 are **not** linearly independent.

ANSWER: The identity $\sin^2 x + \cos^2 x = 1$ can be rewritten $\sin^2 x + \cos^2 x - 1 = 0$.
This is $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$ for $c_1 = 1$, $c_2 = 1$, and $c_3 = -1$.