MATH 341, Fall 2023, QUIZ 3 Answers

1a. Find the general solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5x^4}{6y^5+2y}$.

ANSWER:
Separate: $(6y^5+2y)\,\mathrm{d}y = 5x^4\,\mathrm{d}x$
Integrate: $\int (6y^5+2y)\,\mathrm{d}y = \int 5x^4\,\mathrm{d}x$
$y^6 + y^2 = x^5 + C$
It is inadvisable (and might be mathematically impossible) to solve for y.

Your quiz might have had slightly different numbers.

1b. Find the general solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x^2} = e^{\frac{1}{x}}$
ANSWER:
$P(x) = \frac{1}{x^2}$, so $\mu = e^{\int \frac{1}{x^2}\,\mathrm{d}x} = e^{-\frac{1}{x}}$.
Multiply through by μ: $e^{-\frac{1}{x}}\dfrac{\mathrm{d}y}{\mathrm{d}x} + e^{-\frac{1}{x}}\dfrac{y}{x^2} = 1$.

Now integrate, using the insight that the left hand side is the derivative of $y\mu$.

$$y \cdot e^{-\frac{1}{x}} = x + C.$$

Divide through by $e^{-\frac{1}{x}}$, or in other words multiply through by $e^{\frac{1}{x}}$.

$$y = e^{\frac{1}{x}}(x + C).$$

2. Write down any one solution to the IVP below and explain (briefly!) where your answer came from.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2}\left(y^2-1\right)^4, \qquad y(0) = -1.$$

ANSWER: This ODE is separable, even though we can't do the integrals. When we separate, we get "extra" solutions because we divide through by $(y^2-1)^4$ in solving. These are the constant solutions $y \equiv 1$ and $y \equiv -1$. One of these is a solution to the IVP: $y \equiv -1$.

3. Suppose you wrote an ODE in the differential form shown below. Is this differential form exact?

$$(2xy + y^3)\,\mathrm{d}x + (x^2 + 3y)\,\mathrm{d}y = 0$$

ANSWER: If your form was $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$, then we are trying to figure out whether M is $\frac{\partial F}{\partial x}$ and N is $\frac{\partial F}{\partial y}$ for some F. We need to test whether $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$. All versions of the quiz had $M = 2xy + y^3$, so $\frac{\partial M}{\partial y} = 2x + 3y^2$.
One version of the quiz had $N = x^2 + 3xy^2$ so $\frac{\partial N}{\partial x} = 2x + 3y^2$. For that version of the quiz, the form is exact.
The other version of the quiz had $N = x^2+3y$ so $\frac{\partial N}{\partial x} = 2x$. For that version of the quiz, the form is not exact.

A thought: It's easier when I'm writing this solution to name M and N. But when you're doing the problem, all you really need to remember is that you want $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$, so just take $\frac{\partial}{\partial y}$ of the thing in front of the $\mathrm{d}x$ and take $\frac{\partial}{\partial x}$ of the thing in front of the $\mathrm{d}y$, and see if they are equal.

4. The following differential form of an ODE is exact. Find the general solution to the ODE.

$$(2x + a)\,\mathrm{d}x + (2y + b)\,\mathrm{d}y = 0$$

Your quiz had specific numbers for a and b.

ANSWER: First, we need to find F so that $2x+a$ is $\frac{\partial F}{\partial x}$ and $2y+b$ is $\frac{\partial F}{\partial y}$. There were two options for doing this, just like in class. I'll give one option here. Since $\frac{\partial F}{\partial x} = 2x+a$, we integrate and find $F = x^2+ax+g(y)$. We know $\frac{\partial F}{\partial y}$ is $2y+b$ and we can calculate $\frac{\partial F}{\partial y} = g'(y)$, so $g'(y) = 2y+b$. Integrating, we get $g(y) = y^2+by$. (Why no constant of integration?) So $F = x^2 + ax + y^2 + by$.

We're not done yet! The question asked for solutions to an ODE, not "find F." The solutions are level curves of F: $x^2 + ax + y^2 + by = C$ for constants C.

Warning: Some of you did something shorter that worked on this problem, but only because this problem is simple. In general, you have to do the kind of integration that's explained above.