MATH 341, Fall 2023, QUIZ 11 ANSWERS

1. Write $\vec{x}\,' = \begin{bmatrix} -3 & 6 \\ 10 & 2 \end{bmatrix} \vec{x} + (\cos t) \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ as a system of (scalar) ODEs.

ANSWER:

1. $\begin{array}{rcrcrcr} x_1' & = & -3x_1 & + & 6x_2 & - & 4\cos t \\ x_2' & = & 10x_1 & + & 2x_2 & + & \cos t \end{array}$.

2. Consider the matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$.

a. Find the eigenvalues of $A$.

ANSWER:

$\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)-8 = \lambda^2-4\lambda-5 = (\lambda-5)(\lambda+1) = 0.$

So $\lambda = -1, 5$.

b. For each eigenvalue, find a (nonzero!) eigenvector associated to each eigenvalue.
*On a test, I wouldn't remind you that the zero vector doesn't count as an eigenvector.*

ANSWER:

<u>For $\lambda = -1$:</u> $\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and both of these equations say $v_1 + v_2 = 0$.

So $v_2 = -v_1$, and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ works.

<u>For $\lambda = 5$:</u> $\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and both of these equations say $-2v_1 + 4v_2 = 0$.

So $v_1 = 2v_2$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ works.

*If your equations force you to take $\vec{v} = \vec{0}$, that's not an eigenvector, and that tells you that you made a mistake and your supposed eigenvalue isn't really an eigenvalue.*

c. Write the general solution to the linear system $\vec{x}\,' = A\vec{x}$.
*On a test, I might just ask part c without breaking it down into steps like this.*
*I might start with a system not in matrix form and ask for solutions not in vector form.*

ANSWER:

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}.$$

If I had started with a something not in matrix form, it would have been $\begin{array}{rcrcr} x_1' & = & 3x_1 & + & 4x_2 \\ x_2' & = & 2x_1 & + & x_2 \end{array}$

and you would have needed to write your solutions as $\begin{array}{rcrcr} x_1 & = & c_1e^{-t} & + & 2c_2e^{5t} \\ x_2 & = & -c_1e^{-t} & + & c_2e^{5t} \end{array}$.

3. Find the solution to the IVP $\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}\,, \quad \vec{x}\,(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

*To save you time on the quiz: The eigenvalues of the matrix are* 3 *and* $-1$*.*
*An eigenvector for* $\lambda = 3$ *is* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and an eigenvector for* $\lambda = -1$ *is* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

ANSWER:

Since we know eigenvalues and eigenvectors (and everything is real), we can write down the general solution immediately:

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$$

The initial condition then says

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

This is
$c_1 - c_2 = -1$
$c_1 + c_2 = 5.$

We solve to get $c_1 = 2$ and $c_2 = 3$, so the solution to the IVP is

$$\vec{x}(t) = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$$

If we had been asked to give the solutions not in matrix form, we would write

$$\begin{aligned} x_1 &= 2e^{3t} - 3e^{-t} \\ x_2 &= 2e^{3t} + 3e^{-t}. \end{aligned}$$