MATH 341, Fall 2023, QUIZ 10 ANSWERS.

1. If I write the following system in matrix form as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, what is the matrix A?

$$\begin{aligned} x_1' &= x_1 - 4x_3 \\ x_2' &= 3x_2 - x_3 \\ x_3' &= x_3 \end{aligned} \qquad \longrightarrow \qquad A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1' &= x_1 - 5x_3 \\ x_2' &= 4x_2 - x_3 \\ x_3' &= x_2 \end{aligned} \qquad \longrightarrow \qquad A = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1' &= x_1 - 6x_3 \\ x_2' &= 5x_2 - x_3 \\ x_3' &= x_3 \end{aligned} \qquad \longrightarrow \qquad A = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1' &= x_1 - 3x_3 \\ x_2' &= 6x_2 - x_3 \\ x_3' &= x_2 \end{aligned} \qquad \longrightarrow \qquad A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 6 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Write the following 3rd-order ODE as a system of first-order ODEs. You'll be defining new variables, so please say how those relate to the given ODE.

$$y''' - ty'' + t^2y' - y = 0 \qquad \longrightarrow \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= ty_3 - t^2y_2 + y_1 \end{aligned} \qquad \text{where } y_1 = y,\ y_2 = y',\text{ and } y_3 = y''.$$

$$y''' - t^2y'' + t^3y' - y = 0 \qquad \longrightarrow \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= t^2y_3 - t^3y_2 + y_1 \end{aligned} \qquad \text{where } y_1 = y,\ y_2 = y',\text{ and } y_3 = y''.$$

$$y''' - t^3y'' + t^2y' - y = 0 \qquad \longrightarrow \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= t^3y_3 - t^2y_2 + y_1 \end{aligned} \qquad \text{where } y_1 = y,\ y_2 = y',\text{ and } y_3 = y''.$$

$$y''' - t^2y'' + ty' - y = 0 \qquad \longrightarrow \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= t^2y_3 - ty_2 + y_1 \end{aligned} \qquad \text{where } y_1 = y,\ y_2 = y',\text{ and } y_3 = y''.$$

3.

$$\det\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = 1\cdot(-1)\cdot 1 + 0\cdot 2\cdot 2 + (-1)\cdot 1\cdot 1 - 1\cdot 2\cdot 1 - 0\cdot 1\cdot 1 - (-1)\cdot(-1)\cdot 2 = -6$$

$$\det\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = 0\cdot(-1)\cdot 1 + 1\cdot 2\cdot 2 + (-1)\cdot 1\cdot 1 - 0\cdot 2\cdot 1 - 1\cdot 1\cdot 1 - (-1)\cdot(-1)\cdot 2 = 0$$

$$\det\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ -1 & 1 & -2 \end{bmatrix} = 1\cdot 0\cdot(-2) + (-1)\cdot 1\cdot(-1) + 2\cdot 1\cdot 1 - 1\cdot 1\cdot 1 - (-1)\cdot 1\cdot(-2) - 2\cdot 0\cdot(-1) = 0$$

$$\det\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} = 1\cdot 1\cdot(-2) + (-1)\cdot 0\cdot 1 + 2\cdot 1\cdot 1 - 1\cdot 0\cdot 1 - (-1)\cdot 1\cdot(-2) - 2\cdot 1\cdot 1 = -4$$

4. Find all values of r for which $\det(A - rI) = 0$.

$A = \left[\begin{smallmatrix} 1 & 1 \\ 8 & 3 \end{smallmatrix}\right]$: $\quad \det(A - rI) = \det\left[\begin{smallmatrix} 1-r & 1 \\ 8 & 3-r \end{smallmatrix}\right] = (1-r)(3-r) - 8 = r^2 - 4r - 5 = (r-5)(r+1) = 0 \quad r = -1, 5.$

$A = \left[\begin{smallmatrix} 2 & 4 \\ 5 & 3 \end{smallmatrix}\right]$: $\quad \det(A - rI) = \det\left[\begin{smallmatrix} 2-r & 4 \\ 5 & 3-r \end{smallmatrix}\right] = (2-r)(3-r) - 20 = r^2 - 5r - 14 = (r-7)(r+2) = 0 \quad r = -2, 7.$

$A = \left[\begin{smallmatrix} 5 & 3 \\ 2 & 4 \end{smallmatrix}\right]$: $\quad \det(A - rI) = \det\left[\begin{smallmatrix} 5-r & 3 \\ 2 & 4-r \end{smallmatrix}\right] = (5-r)(4-r) - 6 = r^2 - 9r + 14 = (r-7)(r-2) = 0 \quad r = 2, 7.$

$A = \left[\begin{smallmatrix} 4 & 3 \\ 1 & 2 \end{smallmatrix}\right]$: $\quad \det(A - rI) = \det\left[\begin{smallmatrix} 4-r & 3 \\ 1 & 2-r \end{smallmatrix}\right] = (4-r)(2-r) - 3 = r^2 - 6r + 5 = (r-5)(r-1) = 0 \quad r = 1, 5.$